

Chapter 11

Power Series

11.1 Approximating Functions With Polynomials

11.1.1 Let the polynomial be $p(x)$. Then $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$.

11.1.2 It generally increases, because the more derivatives of f are taken into consideration, the better “fit” the polynomial will provide to f .

11.1.3 The approximations are $p_0(0.1) = 1$, $p_1(0.1) = 1 + \frac{0.1}{2} = 1.05$, and $p_2(0.1) = 1 + \frac{0.1}{2} - \frac{0.01}{8} = 1.04875$.

11.1.4 The quadratic approximating polynomial for f is $p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + 2x - \frac{x^2}{2}$. It follows that $f(0.1) \approx p_2(0.1) = 1 + 2(0.1) - \frac{(0.1)^2}{2} = 1.195$.

11.1.5 The third-order Taylor polynomial is $p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x^2 + x^3$. $f(0.2) \approx 1 + (0.2)^2 + (0.2)^3 = 1.048$.

11.1.6 The remainder is the difference between the value of the Taylor polynomial at a point and the true value of the function at that point, $R_n(x) = f(x) - p_n(x)$.

11.1.7 The third-order Taylor polynomial is

$$p_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 = 1 + (x-2) + 2(x-2)^3$$

, so $f(1.9) \approx 1 - 0.1 + 2(-0.1)^3 = 0.898$.

11.1.8 Choose $a = 25$ because it is near 26, and f and its derivatives are easy to evaluate at 25.

11.1.9

- Note that $f(1) = 8$, and $f'(x) = 12\sqrt{x}$, so $f'(1) = 12$. Thus, $p_1(x) = 8 + 12(x-1)$.
- $f''(x) = 6/\sqrt{x}$, so $f''(1) = 6$. Thus $p_2(x) = 8 + 12(x-1) + 3(x-1)^2$.
- $p_1(1.1) = 12(.1) + 8 = 9.2$. $p_2(1.1) = 3(.1)^2 + 12(.1) + 8 = 9.23$.

11.1.10

- Note that $f(1) = 1$, and that $f'(x) = -1/x^2$, so $f'(1) = -1$. Thus, $p_1(x) = 1 - (x-1) = -x + 2$.
- $f''(x) = 2/x^3$, so $f''(1) = 2$. Thus, $p_2(x) = 2 - x + (x-1)^2$.

c. $p_1(1.05) = 0.95$. $p_2(1.05) = (0.05)^2 - 0.05 + 2 = 0.9525$.

11.1.11

a. $f'(x) = -2e^{-2x}$, so $p_1(x) = f(0) + f'(0)(x) = 1 - 2x$.

b. $f''(x) = 4e^{-2x}$, so $p_2(x) = f(0) + f'(0)(x) + \frac{1}{2}f''(0)(x^2) = 1 - 2x + 2x^2$.

c. $p_1(0.2) = 0.6$, and $p_2(0.2) = 1 - 2(0.2) + 2(0.04) = 0.68$.

11.1.12

a. $f'(x) = \frac{1}{2}x^{-1/2}$, so $p_1(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$.

b. $f''(x) = -\frac{1}{4}x^{-3/2}$, so $p_2(x) = f(4) + f'(4)(x - 4) + \frac{1}{2}f''(4)(x - 4)^2 = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2$.

c. $p_1(3.9) = 2 + \frac{1}{4}(-0.1) = 2 - .025 = 1.975$, and $p_2(3.9) = 2 - 0.025 - \frac{1}{64}(0.001) = 1.97484375$.

11.1.13

a. $f'(x) = -\frac{1}{(x+1)^2}$, so $p_1(x) = f(0) + f'(0)(x) = 1 - x$.

b. $f''(x) = \frac{2}{(x+1)^3}$, so $p_2(x) = f(0) + f'(0)(x) + \frac{1}{2}f''(0)x^2 = 1 - x + x^2$.

c. $p_1(0.05) = 0.95$, and $p_2(0.05) = 1 - 0.05 + 0.0025 = 0.9525$.

11.1.14

a. $f'(x) = -\sin x$, so $p_1(x) = \cos(\pi/4) - \sin(\pi/4)(x - \pi/4) = \frac{\sqrt{2}}{2}(1 - (x - \pi/4))$.

b. $f''(x) = -\cos x$, so

$$\begin{aligned} p_2(x) &= \cos(\pi/4) - \sin(\pi/4)(x - \pi/4) - \frac{1}{2}\cos(\pi/4)(x - \pi/4)^2 \\ &= \frac{\sqrt{2}}{2} \left(1 - (x - \pi/4) - \frac{1}{2}(x - \pi/4)^2 \right). \end{aligned}$$

c. $p_1(.24\pi) \approx 0.7293211959$, $p_2(.24\pi) \approx 0.7289722527$.

11.1.15

a. $f'(x) = (1/3)x^{-2/3}$, so $p_1(x) = f(8) + f'(8)(x - 8) = 2 + \frac{1}{12}(x - 8)$.

b. $f''(x) = (-2/9)x^{-5/3}$, so $p_2(x) = f(8) + f'(8)(x - 8) + \frac{1}{2}f''(8)(x - 8)^2 = 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2$.

c. $p_1(7.5) \approx 1.958333333$, $p_2(7.5) \approx 1.957465278$.

11.1.16

a. $f'(x) = \frac{1}{1+x^2}$, so $p_1(x) = f(0) + f'(0)(x) = x$.

b. $f''(x) = \frac{-2x}{(1+x^2)^2}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$.

c. $p_1(0.1) = p_2(0.1) = 0.1$.

11.1.17 $f(0) = 1, f'(0) = -6 \sin 0 = 0, f''(0) = -36 \cos 0 = -36, f'''(0) = 6^3 \sin 0 = 0$, and $f^{(4)}(0) = 6^4 \cos 0 = 1296$, so that $p_1(x) = 1, p_2(x) = p_3(x) = 1 - 18x^2, p_4(x) = 1 - 18x^2 + 54x^4$.

11.1.18 $f(0) = 1, f'(0) = -e^0 = -1, f''(0) = e^0 = 1, f'''(0) = -e^0 = -1, f^{(4)}(0) = e^0 = 1$, and $f^{(5)}(0) = -1$, so that $p_1(x) = 1 - x, p_2(x) = 1 - x + \frac{x^2}{2}, p_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}, p_4(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24},$
 $p_5(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$.

11.1.19 $f(0) = 1, f'(0) = -3(1+0)^{-4} = -3, f''(0) = 12(1+0)^{-5} = 12, f'''(0) = -60(1+0)^{-6} = -60$, and $f^{(4)}(0) = 360(1+0)^{-7} = 360$, so that $p_1(x) = 1 - 3x, p_2(x) = 1 - 3x + 6x^2, p_3(x) = 1 - 3x + 6x^2 - 10x^3,$
 $p_4(x) = 1 - 3x + 6x^2 - 10x^3 + 15x^4$.

11.1.20 If $f(x) = \cos x$, then $f(\pi/6) = \sqrt{3}/2, f'(\pi/6) = -\sin(\pi/6) = -1/2, f''(\pi/6) = -\cos(\pi/6) = -\sqrt{3}/2,$
 $f'''(\pi/6) = \sin(\pi/6) = 1/2, f^{(4)}(\pi/6) = \cos(\pi/6) = \sqrt{3}/2$, and $f^{(5)}(\pi/6) = -\sin(\pi/6) = -1/2$. Then

$$p_4(x) = \frac{\sqrt{3}}{2} - \frac{1}{2} \left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4} \left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12} \left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48} \left(x - \frac{\pi}{6}\right)^4$$

and

$$p_5(x) = \frac{\sqrt{3}}{2} - \frac{1}{2} \left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4} \left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12} \left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48} \left(x - \frac{\pi}{6}\right)^4 - \frac{1}{240} \left(x - \frac{\pi}{6}\right)^5.$$

11.1.21 Note that $f(1) = 1, f'(1) = 3, f''(1) = 6$, and $f'''(1) = 6$. Thus, $p_1(x) = 1 + 3(x - 1)$, and $p_2(x) = 1 + 3(x - 1) + 3(x - 1)^2$, and $p_3(x) = 1 + 3(x - 1) + 3(x - 1)^2 + (x - 1)^3$.

11.1.22 Note that $f(1) = 8, f'(1) = \frac{4}{\sqrt{1}} = 4, f''(1) = \frac{-2}{(1)^{3/2}} = -2, f'''(1) = \frac{3}{1^{5/2}} = 3$, and $f^{(4)}(1) = -\frac{15}{2} \cdot 1^{-7/2} = -\frac{15}{2}$, so

$$p_3(x) = 8 + 4(x - 1) - (x - 1)^2 + \frac{1}{2}(x - 1)^3$$

and

$$p_4(x) = 8 + 4(x - 1) - (x - 1)^2 + \frac{1}{2}(x - 1)^3 - \frac{5}{16}(x - 1)^4.$$

11.1.23 $f(x) = \ln x$ so $f(e) = 1, f'(e) = \frac{1}{e}, f''(e) = -\frac{1}{e^2}$, and $f'''(e) = \frac{2}{e^3}$.

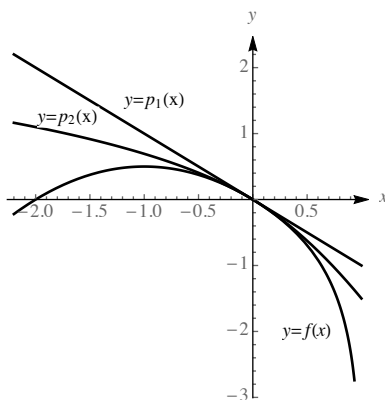
So,

$$p_3(x) = 1 + \frac{1}{e}(x - e) - \frac{1}{2e^2}(x - e)^2 + \frac{1}{3e^3}(x - e)^3.$$

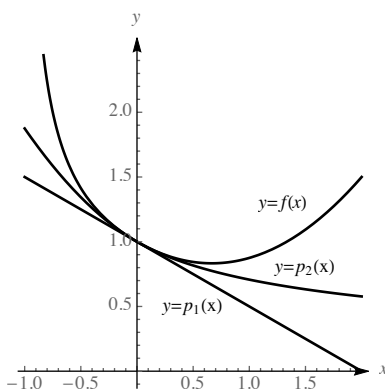
11.1.24 $f(x) = \sqrt[3]{x}$, so $f(8) = 2, f'(8) = \frac{1}{12}, f''(8) = -\frac{2}{9}(8)^{-5/3} = -\frac{1}{144}$ so

$$p_2(x) = x + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2.$$

11.1.25 $f(0) = 0, f'(0) = \frac{-1}{1-0} = -1, f''(0) = \frac{1}{(1-0)^2} = -1$, so that $p_1(x) = -x, p_2(x) = -x - \frac{1}{2}x^2$.



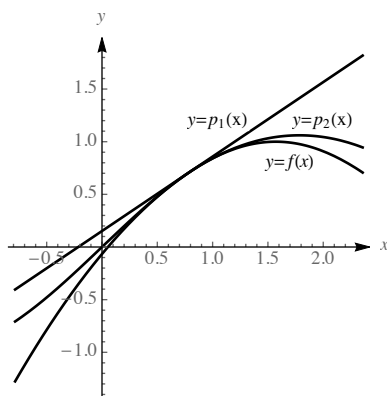
11.1.26 $f(0) = 1$, $f'(0) = (-1/2)(0+1)^{-3/2} = -1/2$, $f''(0) = (3/4)(0+1)^{-5/2} = 3/4$, so that $p_1(x) = 1 - \frac{x}{2}$, $p_2(x) = 1 - \frac{x}{2} + \frac{3}{8}x^2$.



11.1.27

a. $p_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right)$, $p_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4} \left(x - \frac{\pi}{4}\right)^2$.

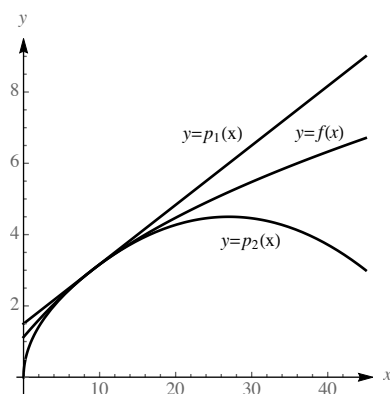
b.



11.1.28

a. $p_1(x) = 3 + \frac{1}{6}(x - 9)$, $p_2(x) = 3 + \frac{1}{6}(x - 9) - \frac{1}{216}(x - 9)^2$.

b.

**11.1.29**

a. $p_2(0.05) = 1.0246875$.

b. The absolute error is $\sqrt{1.05} - p_2(1.05) \approx 1.024695077 - 1.0246875 = 7.58 \times 10^{-6}$.

11.1.30

a. $p_2(0.08) = 0.9624$.

b. The absolute error is $p_2(0.08) - 1/\sqrt{1.08} \approx 0.9624 - 0.9622504482 \approx 1.50 \times 10^{-4}$.

11.1.31

a. $p_2(0.15) = 0.86125$.

b. The absolute error is $p_2(0.15) - e^{-0.15} \approx 0.86125 - 0.8607079764 \approx 5.42 \times 10^{-4}$.

11.1.32

a. $p_2(0.06) = 0.0582$.

b. The absolute error is $\ln(1.06) - p_2(.06) \approx 0.05826890812 - 0.0582 \approx 6.89 \times 10^{-5}$.

11.1.33a. The 3rd-order Taylor polynomial of $f(x) = e^{-x}$ centered at $a = 0$ is

$$p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3,$$

so the approximate value of $e^{0.12}$ is $p(0.12) = 1.127488$.b. The absolute error is $|e^{0.12} - 1.127488| \approx 8.85 \times 10^{-6}$. (Answers may vary if intermediate calculations are rounded.)**11.1.34**a. The 3rd-order Taylor polynomial of $f(x) = \cos x$ centered at $a = 0$ is

$$p(x) = 1 - \frac{1}{2}x^2,$$

so the approximate value of $\cos(-0.2)$ is $p(-0.2) = 0.98$.

- b. The absolute error is $|\cos(-0.2) - 0.98| \approx 6.66 \times 10^{-5}$. (Answers may vary if intermediate calculations are rounded.)

11.1.35

- a. The 3rd-order Taylor polynomial of $f(x) = \tan x$ centered at $a = 0$ is

$$p(x) = x + \frac{1}{3}x^3$$

so the approximate value of $\tan(-0.1)$ is $p(-0.1) \approx -0.10033333$.

- b. The absolute error is $|\tan(-0.1) - (-0.10033333)| \approx 1.34 \times 10^{-6}$. (Answers may vary if intermediate calculations are rounded.)

11.1.36

- a. The 3rd-order Taylor polynomial of $f(x) = \ln x$ centered at $a = 1$ is

$$p(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3,$$

so the approximate value of $\ln(1.05)$ is $p(1.05) \approx 0.04879167$.

- b. The absolute error is $|\ln(1.05) - 0.04879167| \approx 1.50 \times 10^{-6}$. (Answers may vary if intermediate calculations are rounded.)

11.1.37

- a. The 3rd-order Taylor polynomial of $f(x) = \sqrt{x}$ centered at $a = 1$ is

$$p(x) = 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3,$$

so the approximate value of $\sqrt{1.06}$ is $p(1.06) \approx 1.0295635$.

- b. The absolute error is $|\sqrt{1.06} - 1.0295635| \approx 4.86 \times 10^{-7}$. (Answers may vary if intermediate calculations are rounded.)

11.1.38

- a. The 3rd-order Taylor polynomial of $f(x) = \sqrt[4]{x}$ centered at $a = 81$ is

$$p(x) = 3 + \frac{1}{108}(x - 81) - \frac{1}{23328}(x - 81)^2 + \frac{7}{22674816}(x - 81)^3,$$

so the approximate value of $\sqrt[4]{79}$ is $p(79) \approx 2.98130754$.

- b. The absolute error is $|\sqrt[4]{79} - 2.98130754| \approx 4.27 \times 10^{-8}$. (Answers may vary if intermediate calculations are rounded.)

11.1.39

- a. The 3rd-order Taylor polynomial of $f(x) = \sinh x$ centered at $a = 0$ is

$$p(x) = x + \frac{1}{6}x^3,$$

so the approximate value of $\sinh 0.5$ is $p(0.5) \approx 0.52083333$.

- b. The absolute error is $|\sinh(0.5) - 0.52083333| \approx 2.62 \times 10^{-4}$. (Answers may vary if intermediate calculations are rounded.)

11.1.40

a. The 3rd-order Taylor polynomial of $f(x) = \tanh x$ centered at $a = 0$ is

$$p(x) = x - \frac{1}{3}x^3,$$

so the approximate value of $\tanh 0.5$ is $p(0.5) \approx 0.45833333$.

b. The absolute error is $|\tanh(0.5) - 0.45833333| \approx 3.78 \times 10^{-3}$. (Answers may vary if intermediate calculations are rounded.)

$$\mathbf{11.1.41} \quad R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1} \text{ for some } c \text{ between } 0 \text{ and } x.$$

$$\mathbf{11.1.42} \quad R_n(x) = \frac{\cos(2x)^{(n+1)}(c)}{(n+1)!} x^{n+1} \text{ for some } c \text{ between } 0 \text{ and } x.$$

$$\mathbf{11.1.43} \quad R_n(x) = \frac{(-1)^n e^{-c}}{(n+1)!} x^{n+1} \text{ for some } c \text{ between } 0 \text{ and } x.$$

$$\mathbf{11.1.44} \quad R_n(x) = \frac{\cos^{(n+1)}(c)}{(n+1)!} \left(x - \frac{\pi}{2}\right)^{n+1} \text{ for some } c \text{ between } \frac{\pi}{2} \text{ and } x.$$

$$\mathbf{11.1.45} \quad R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} \left(x - \frac{\pi}{2}\right)^{n+1} \text{ for some } c \text{ between } \frac{\pi}{2} \text{ and } x.$$

$$\mathbf{11.1.46} \quad R_n(x) = \frac{1}{(1-c)^{n+2}} (x^{n+1}) \text{ for some } c \text{ between } 0 \text{ and } x.$$

11.1.47 $f(x) = \sin x$, so $f^{(4)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_4(x)| \leq \frac{0.3^5}{5!} \approx 2.03 \times 10^{-5}$.

11.1.48 $f(x) = \cos(x)$, so $f^{(4)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_3(x)| \leq \frac{0.45^4}{4!} \approx 1.7 \times 10^{-3}$.

11.1.49 $f(x) = e^x$, so $f^{(5)}(x) = e^x$. Because $e^{0.25}$ is bounded by 2, $|R_4(x)| \leq 2 \cdot \frac{0.25^5}{5!} \approx 1.63 \times 10^{-5}$.

11.1.50 $f(x) = \tan(x)$, so $f^{(3)}(x) = 2 \sec^2 x (\sec^2 x + 2 \tan^2 x)$ and $f^{(3)}(x) < \frac{16}{3}$ on $[0, 0.3]$. Thus $|R_2(x)| \leq \frac{16}{3} \cdot \frac{0.3^3}{3!} = 2.4 \times 10^{-2}$.

11.1.51 $f(x) = e^{-x}$, so $f^{(5)}(x) = -e^{-x}$. Because $f^{(5)}$ achieves its maximum magnitude in the range at $x = 0$, which has absolute value 1, $|R_4(x)| \leq 1 \cdot \frac{0.5^5}{5!} \approx 2.6 \times 10^{-4}$.

11.1.52 $f(x) = \ln(1+x)$, so $f^{(4)}(x) = \frac{-6}{(x+1)^4}$. On $[0, 0.4]$, the maximum magnitude is 6, so $|R_3(x)| \leq 6 \cdot \frac{0.04^4}{4!} = 6.4 \times 10^{-3}$.

11.1.53 Here $n = 3$ or 4, so use $n = 4$, and $M = 1$ because $f^{(5)}(x) = \cos x$, so that $R_4(x) \leq \frac{(\pi/4)^5}{5!} \approx 2.49 \times 10^{-3}$.

11.1.54 $n = 2$ or 3 , so use $n = 3$, and $M = 1$ because $f^{(4)}(x) = \cos x$, so that $R_3(x) \leq \frac{(\pi/4)^4}{4!} \approx 1.59 \times 10^{-2}$.

11.1.55 $n = 2$ and $M = e^{1/2} < 2$, so $R_2(x) \leq 2 \cdot \frac{(1/2)^3}{3!} \approx 4.17 \times 10^{-2}$.

11.1.56 $n = 1$ or 2 , so use 2 , and $f^{(3)}(x) = 2 \sec^2 x (\sec^2 x + 2 \tan^2 x)$, which is bounded by $\frac{16}{3}$ on the range. Thus $R_2(x) \leq \frac{16}{3} \cdot \frac{(\pi/6)^3}{3!} \approx 1.28 \times 10^{-1}$.

11.1.57 $n = 2$; $f^{(3)}(x) = \frac{2}{(1+x)^3}$, which achieves its maximum at $x = -0.2$: $|f^{(3)}(x)| = \frac{2}{0.8^3} < 4$. Then $R_2(x) \leq 4 \cdot \frac{0.2^3}{3!} \approx 5.4 \times 10^{-3}$.

11.1.58 $n = 1$, $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, which achieves its maximum magnitude at $x = -0.1$, where it is less than 1 . Thus $R_1(x) \leq \frac{0.1^2}{2!} = .005$.

11.1.59 Use the Taylor series for e^x at $x = 0$. The derivatives of e^x are e^x . On $[-0.5, 0]$, the maximum magnitude of any derivative is thus 1 and $x = 0$, so $|R_n(-0.5)| \leq \frac{0.5^{n+1}}{(n+1)!}$, so for $R_n(-0.5) < 10^{-3}$ we need $n = 4$.

11.1.60 Use the Taylor series at $x = 0$ for $\sin x$. The magnitude of any derivative of $\sin x$ is bounded by 1 , so $|R_n(0.2)| \leq \frac{0.2^{n+1}}{(n+1)!}$, so for $R_n(0.2) < 10^{-3}$ we need $n = 3$.

11.1.61 Use the Taylor series for $\cos x$ at $x = 0$. The magnitude of any derivative of $\cos x$ is bounded by 1 , so $|R_n(-0.25)| \leq \frac{0.25^{n+1}}{(n+1)!}$, so for $|R_n(-0.25)| < 10^{-3}$ we need $n = 3$.

11.1.62 Use the Taylor series for $f(x) = \ln(1+x)$ at $x = 0$. Then $|f^{(n+1)}(x)| = \frac{n!}{(1+x)^{n+1}}$, which for $x \in [-0.15, 0]$ achieves its maximum at $x = -0.15$. This maximum is less than $(1.2)^{n+1} \cdot n!$. Thus $|R_n(-0.15)| \leq (1.2)^{n+1} \cdot n! \cdot \frac{.18^{n+1}}{(n+1)!} = \frac{1.2 \cdot (0.15)^{n+1}}{n}$, so for $|R_n(-0.15)| < 10^{-3}$ we need $n = 3$.

11.1.63 Use the Taylor series for $f(x) = \sqrt{x}$ at $x = 1$. Then $|f^{(n+1)}(x)| = \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2^{n+1}} x^{-(2n+1)/2}$, which achieves its maximum on $[1, 1.06]$ at $x = 1$. Then

$$|R_n(1.06)| \leq \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2^{n+1}} \cdot \frac{(1.06-1)^{n+1}}{(n+1)!},$$

and for $|R_n(0.06)| < 10^{-3}$ we need $n = 1$.

11.1.64 Use the Taylor series for $f(x) = \sqrt{1/(1-x)}$ at $x = 0$. Then $|f^{(n+1)}(x)| = \frac{1 \cdot 3 \cdot \dots \cdot (2n+1)}{2^{n+1}} (1-x)^{-(3-2n)/2}$, which achieves its maximum on $[0, 0.15]$ at $x = 0.15$. Thus

$$\begin{aligned} |R_n(0.15)| &\leq \frac{1 \cdot 3 \cdot \dots \cdot (2n+1)}{2^{n+1}} \cdot \left(\frac{1}{1-0.15} \right)^{(2n+3)/2} \cdot \frac{0.15^{n+1}}{(n+1)!} \\ &= \frac{1 \cdot 3 \cdot \dots \cdot (2n+1)}{2^{n+1}(n+1)!} \cdot \left(\frac{0.15^{n+1}}{0.85^{(2n+3)/2}} \right), \end{aligned}$$

and for $|R_n(0.15)| < 10^{-3}$ we need $n = 3$.

11.1.65

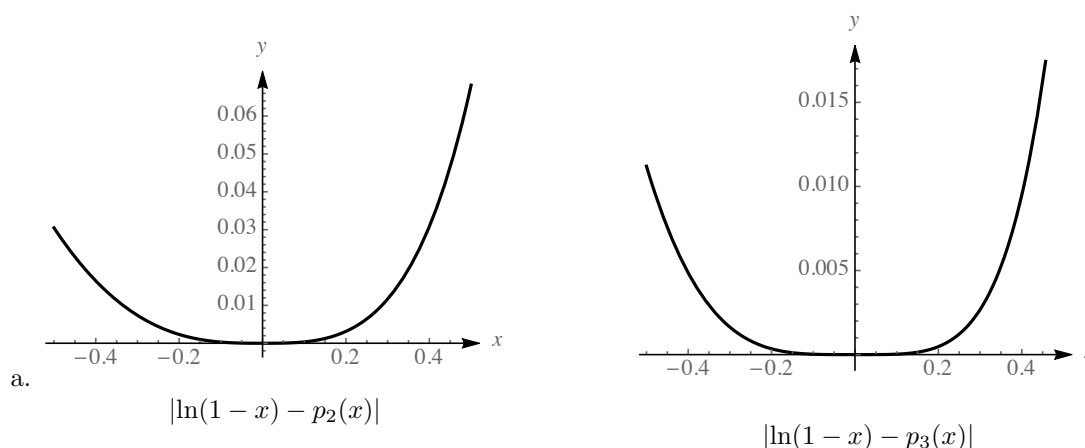
- False. If $f(x) = e^{-2x}$, then $f^{(n)}(x) = (-1)^n 2^n e^{-2x}$, so that $f^{(n)}(0) \neq 0$ and all powers of x are present in the Taylor series.
- True. The constant term of the Taylor series is $f(0) = 1$. Higher-order terms all involve derivatives of $f(x) = x^5 - 1$ evaluated at $x = 0$; clearly for $n < 5$, $f^{(n)}(0) = 0$, and for $n > 5$, the derivative itself vanishes. Only for $n = 5$, where $f^{(5)}(x) = 5!$, is the derivative nonzero, so the coefficient of x^5 in the Taylor series is $f^{(5)}(0)/5! = 1$ and the Taylor polynomial of order 10 is in fact $x^5 - 1$.
- True. The odd derivatives of $\sqrt{1+x^2}$ vanish at $x = 0$, while the even ones do not.
- True. Because then $f''(a) = 0$.

11.1.66 Let $p(x) = \sum_{k=0}^n c_k(x-a)^k$ be the n^{th} polynomial for $f(x)$ at a . Because $f(a) = p(a)$, it follows that $c_0 = f(0)$. Now, the k^{th} derivative of $p(x)$, $1 \leq k \leq n$, is $p^{(k)}(x) = k!c_k + \text{terms involving } (x-a)^i, i > 0$, so that $f^{(k)}(a) = p^{(k)}(a) = k! \cdot c_k$ so that $c_k = \frac{f^{(k)}(a)}{k!}$.

11.1.67

- This matches (C) because for $f(x) = (1+2x)^{1/2}$, $f''(x) = -(1+2x)^{-3/2}$ so $\frac{f''(0)}{2!} = \frac{-1}{2}$.
- This matches (E) because for $f(x) = (1+2x)^{-1/2}$, $f''(x) = 3(1+2x)^{-5/2}$, so $\frac{f''(0)}{2!} = \frac{3}{2}$.
- This matches (A) because $f^{(n)}(x) = 2^n e^{2x}$, so that $f^{(n)}(0) = 2^n$, which is (A)'s pattern.
- This matches (D) because $f''(x) = 8(1+2x)^{-3}$ and $f''(0) = 8$, so that $f''(0)/2! = 4$.
- This matches (B) because $f'(x) = -6(1+2x)^{-4}$ so that $f'(0) = -6$.
- This matches (F) because $f^{(n)}(x) = (-2)^n e^{-2x}$, so $f^{(n)}(0) = (-2)^n$, which is (F)'s pattern.

11.1.68



- The error seems to be largest at $x = \frac{1}{2}$ and smallest at $x = 0$.
- The error bound found in Example 7 for $|\ln(1-x) - p_3(x)|$ was 0.25. The actual error seems much less than that, about 0.02.

11.1.69

a. $p_2(0.1) = 0.1$. The maximum error in the approximation is $1 \cdot \frac{0.1^3}{3!} \approx 1.67 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error in the approximation is $1 \cdot \frac{0.2^3}{3!} \approx 1.33 \times 10^{-3}$.

11.1.70

a. $p_1(0.1) = 0.1$. $f''(x) = 2 \tan(x)(1 + \tan^2(x))$. Because $\tan(0.1) < 0.2$, $|f''(c)| \leq 2(.2)(1 + .2^2) = 0.416$.
Thus the maximum error is $\frac{0.416}{2!} \cdot 0.1^2 = 2.08 \times 10^{-3}$.

b. $p_1(0.2) = 0.2$. The maximum error is $\frac{0.416}{2} \cdot 0.2^2 = 8.32 \times 10^{-3}$.

11.1.71

a. $p_3(0.1) = 1 - .01/2 = 0.995$. The maximum error is $1 \cdot \frac{0.1^4}{4!} \approx 4.17 \times 10^{-6}$.

b. $p_3(0.2) = 1 - .04/2 = 0.98$. The maximum error is $1 \cdot \frac{0.2^4}{4!} \approx 6.67 \times 10^{-5}$.

11.1.72

a. $p_2(0.1) = 0.1$ (we can take $n = 2$ because the coefficient of x^2 is 0). $f^{(3)}(x) = -2 \cdot \frac{1 - 2x^2}{(1 + x^2)^2}$ has a maximum magnitude value of 2, the maximum error is $2 \cdot \frac{0.1^3}{3!} \approx 3.33 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error is $2 \cdot \frac{0.2^3}{3!} \approx 2.67 \times 10^{-3}$.

11.1.73

a. $p_1(0.1) = 1.05$. Because $|f''(x)| = \frac{1}{4}(1 + x)^{-3/2}$ has a maximum value of $1/4$ at $x = 0$, the maximum error is $\frac{1}{4} \cdot \frac{0.1^2}{2} = 1.25 \times 10^{-3}$.

b. $p_1(0.2) = 1.1$. The maximum error is $\frac{1}{4} \cdot \frac{0.2^2}{2} = 5 \times 10^{-3}$.

11.1.74

a. $p_2(0.1) = 0.1 - 0.01/2 = 0.095$. Because $|f^{(3)}(x)| = \frac{2}{(x + 1)^3}$ achieves a maximum of 2 at $x = 0$, the maximum error is $2 \cdot \frac{0.1^3}{3!} \approx 3.33 \times 10^{-4}$.

b. $p_2(0.2) = 0.2 - 0.04/2 = 0.18$. The maximum error is $2 \cdot \frac{0.2^3}{3!} \approx 2.67 \times 10^{-3}$.

11.1.75

a. $p_1(0.1) = 1.1$. Because $f''(x) = e^x$ is less than 2 on $[0, 0.1]$, the maximum error is less than $2 \cdot \frac{0.1^2}{2!} = \frac{1}{100}$.

b. $p_1(0.2) = 1.2$. The maximum error is less than $2 \cdot \frac{0.2^2}{2!} = .04 = \frac{1}{25}$.

11.1.76

a. $p_1(0.1) = 0.1$. Because $f''(x) = \frac{x}{(1-x^2)^{3/2}}$ is less than 1 on $[0, 0.2]$, the maximum error is $1 \cdot \frac{0.1^3}{3!} \approx 1.67 \times 10^{-4}$.

b. $p_1(0.2) = 0.2$. The maximum error is $1 \cdot \frac{0.2^3}{3!} \approx 1.33 \times 10^{-3}$.

11.1.77

	$ \sec x - p_2(x) $	$ \sec x - p_4(x) $
-0.2	3.39×10^{-4}	5.51×10^{-6}
-0.1	2.09×10^{-5}	8.51×10^{-8}
0.0	0	0
0.1	2.09×10^{-5}	8.51×10^{-8}
0.2	3.39×10^{-4}	5.51×10^{-6}

b. The errors are get smaller as $|x|$ decreases.

11.1.78

	$ \cos x - p_2(x) $	$ \cos x - p_4(x) $
-0.2	6.66×10^{-5}	8.88×10^{-8}
-0.1	4.17×10^{-6}	1.39×10^{-9}
0.0	0	0
0.1	4.17×10^{-6}	1.39×10^{-9}
0.2	6.66×10^{-5}	8.88×10^{-8}

b. The errors are equal for positive and negative x . This makes sense, because $\cos(-x) = \cos(x)$ and $p_n(-x) = p_n(x)$ for $n = 2, 4$. The errors appear to get larger as x gets farther from zero.

11.1.79

	$ e^{-x} - p_1(x) $	$ e^{-x} - p_2(x) $
-0.2	2.14×10^{-2}	1.4×10^{-3}
-0.1	5.17×10^{-3}	1.71×10^{-4}
0.0	0	0
0.1	4.84×10^{-3}	1.63×10^{-4}
0.2	1.87×10^{-2}	1.27×10^{-3}

b. The errors get smaller as $|x|$ decreases.

11.1.80 The true value of $\cos \frac{\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}} \approx 0.9659258263$. The 6th-order Taylor polynomial for $\cos x$ centered at $x = 0$ is

$$p_6(x) = 1 - \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$ p_n\left(\frac{\pi}{12}\right) - \cos \frac{\pi}{12} $
1	1.0000000000	3.41×10^{-2}
2	0.9657305403	1.95×10^{-4}
3	0.9657305403	1.95×10^{-4}
4	0.9659262729	4.47×10^{-7}
5	0.9659262729	4.47×10^{-7}
6	0.9659258257	5.47×10^{-10}

The 6th-order Taylor polynomial for $\cos x$ centered at $x = \pi/6$ is

$$p_6(x) = \frac{\sqrt{3}}{2} - \frac{1}{2} \left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4} \left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12} \left(x - \frac{\pi}{6}\right)^3 \\ + \frac{\sqrt{3}}{48} \left(x - \frac{\pi}{6}\right)^4 - \frac{1}{240} \left(x - \frac{\pi}{6}\right)^5 - \frac{\sqrt{3}}{1440} \left(x - \frac{\pi}{6}\right)^6.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$ p_n\left(\frac{\pi}{12}\right) - \cos \frac{\pi}{12} $
1	0.9969250977	3.10×10^{-2}
2	0.9672468750	1.32×10^{-3}
3	0.9657515877	1.74×10^{-4}
4	0.9659210972	4.73×10^{-6}
5	0.9659262214	3.95×10^{-7}
6	0.9659258342	7.88×10^{-9}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate when n is even while using the polynomial centered at $x = \pi/6$ is more accurate when n is odd. To see why, consider the remainder. Let $f(x) = \cos x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at 0 is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!} \left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $0 < c < \frac{\pi}{12}$, while the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at $\pi/6$ is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!} \left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $\frac{\pi}{12} < c < \frac{\pi}{6}$. When n is odd, $|f^{(n+1)}(c)| = |\cos c|$. Because $\cos x$ is a positive and decreasing function over $[0, \pi/6]$, the magnitude of the remainder in using the polynomial centered at $\pi/6$ will be less than the remainder in using the polynomial centered at 0, and the former polynomial will be more accurate. When n is even, $|f^{(n+1)}(c)| = |\sin c|$. Because $\sin x$ is a positive and increasing function over $[0, \pi/6]$, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\pi/6$, and the former polynomial will be more accurate.

11.1.81 The true value of $e^{0.35} \approx 1.419067549$. The 6th-order Taylor polynomial for e^x centered at $x = 0$ is

$$p_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$ p_n(0.35) - e^{0.35} $
1	1.350000000	6.91×10^{-2}
2	1.411250000	7.82×10^{-3}
3	1.418395833	6.72×10^{-4}
4	1.419021094	4.65×10^{-5}
5	1.419064862	2.69×10^{-6}
6	1.419067415	1.33×10^{-7}

The 6th-order Taylor polynomial for e^x centered at $x = \ln 2$ is

$$p_6(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2 + \frac{1}{3}(x - \ln 2)^3 + \frac{1}{12}(x - \ln 2)^4 + \frac{1}{60}(x - \ln 2)^5 + \frac{1}{360}(x - \ln 2)^6.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$ p_n(0.35) - e^{0.35} $
1	1.313705639	1.05×10^{-1}
2	1.431455626	1.24×10^{-2}
3	1.417987101	1.08×10^{-3}
4	1.419142523	7.50×10^{-5}
5	1.419063227	4.32×10^{-6}
6	1.419067762	2.13×10^{-7}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate for all n . To see why, consider the remainder. Let $f(x) = e^x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at 0 is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} (0.35)^{n+1} = \frac{e^c}{(n+1)!} (0.35)^{n+1}$$

for some c with $0 < c < 0.35$ while the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at $\ln 2$ is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |0.35 - \ln 2|^{n+1} = \frac{e^c}{(n+1)!} (\ln 2 - 0.35)^{n+1}$$

for some c with $0.35 < c < \ln 2$. Because $\ln 2 - 0.35 \approx 0.35$, the relative size of the magnitudes of the remainders is determined by e^c in each remainder. Because e^x is an increasing function, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\ln 2$, and the former polynomial will be more accurate.

11.1.82

- Let x be a point in the interval on which the derivatives of f are assumed continuous. Then f' is continuous on $[a, x]$, and the Fundamental Theorem of Calculus implies that because f is an antiderivative of f' , then $\int_a^x f'(t) dt = f(x) - f(a)$, or $f(x) = f(a) + \int_a^x f'(t) dt$.
- Integrate the integral from part (a) by parts using $u = f'(t)$, $dv = dt$ to obtain

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t) dt = f(a) + [tf'(t)] \Big|_{t=a}^x - \int_a^x tf''(t) dt \\ &= f(a) + xf'(x) - af'(a) - \int_a^x tf''(t) dt \end{aligned}$$

Now, $\int_a^x xf''(t) dt = x \int_a^x f''(t) dt = xf'(x) - xf'(a)$. Substituting into the above, we obtain

$$\begin{aligned} f(x) &= f(a) + x \int_a^x f''(t) dt + xf'(a) - af'(a) - \int_a^x tf''(t) dt \\ &= f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t) dt \end{aligned}$$

c. We obtain (using $u = f''(t)$, $dv = (x - t) dt$),

$$\begin{aligned} f(x) &= f(a) + (x - a)f'(a) + \int_a^x (x - t)f''(t) dt \\ &= f(a) + (x - a)f'(a) - \frac{(x - t)^2}{2} f''(t) \Big|_a^x + \frac{1}{2} \int_a^x (x - t)^2 f'''(t) dt \\ &= f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2} f''(t) + \frac{1}{2} \int_a^x (x - t)^2 f'''(t) dt \end{aligned}$$

It is clear that continuing this process will give the desired result, because successive integrals of $x - t$ give $\frac{-1}{k!}(x - t)^k$.

d. **Lemma:** Let g and h be continuous functions on the interval $[a, b]$ with $g(t) \geq 0$. Then there is a number c in $[a, b]$ with

$$\int_a^b h(t)g(t) dt = h(c) \int_a^b g(t) dt$$

Proof: We note first that if $g(t) = 0$ for all t in $[a, b]$, then the result is clearly true. We can thus assume that there is some t in $[a, b]$ for which $g(t) > 0$. Because g is continuous, there must be an interval about this t on which g is strictly positive, so we may assume that

$$\int_a^b g(t) dt > 0$$

Because h is continuous on $[a, b]$, the Extreme Value Theorem shows that h has an absolute minimum value m and an absolute maximum value M on the interval $[a, b]$. Thus

$$m \leq h(t) \leq M$$

for all t in $[a, b]$, so

$$m \int_a^b g(t) dt \leq \int_a^b h(t)g(t) dt \leq M \int_a^b g(t) dt$$

Because $\int_a^b g(t) dt > 0$, we have

$$m \leq \frac{\int_a^b h(t)g(t) dt}{\int_a^b g(t) dt} \leq M.$$

Now there are points in $[a, b]$ at which $h(t)$ equals m and M , so the Intermediate Value Theorem shows that there is a point c in $[a, b]$ at which

$$h(c) = \frac{\int_a^b h(t)g(t) dt}{\int_a^b g(t) dt}$$

or

$$\int_a^b h(t)g(t) dt = h(c) \int_a^b g(t) dt.$$

Applying the lemma with $h(t) = \frac{f^{(n+1)}(t)}{n!}$, $g(t) = (x - t)^n$, we see that $R_n = \frac{f^{(n+1)}(c)}{n!} \int_a^x (x - t)^n dt = \frac{f^{(n+1)}(c)}{n!} \cdot \frac{1}{n+1} (x - a)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$ for some $c \in [a, b]$.

11.1.83

- a. The slope of the tangent line to $f(x)$ at $x = a$ is by definition $f'(a)$; by the point-slope form for the equation of a line, we have $y - f(a) = f'(a)(x - a)$, or $y = f(a) + f'(a)(x - a)$.
- b. The Taylor polynomial centered at a is $p_1(x) = f(a) + f'(a)(x - a)$, which is the tangent line at a .

11.1.84

- a. $p_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$, so that $p'_2(x) = f'(a) + f''(a)(x - a)$ and $p''_2(x) = f''(a)$. If f has a local maximum at a , then $f'(a) = 0$, $f''(a) < 0$, but then $p'_2(a) = 0$ and $p''_2(a) < 0$ by the above, so that $p_2(x)$ also has a local maximum at a .
- b. Similarly, if f has a local minimum at a , then $f'(a) = 0$, $f''(a) > 0$, but then $p'_2(a) = 0$ and $p''_2(a) > 0$ by the above, so that $p_2(x)$ also has a local minimum at a .
- c. Recall that f has an inflection point at a if the second derivative of f changes sign at a . But $p''_2(x)$ is a constant, so p_2 does not have an inflection point at a (or anywhere else).
- d. No. For example, let $f(x) = x^3$. Then $p_2(x) = 0$, so that the second-order Taylor polynomial has a local maximum at $x = 0$, but $f(x)$ does not. It also has a local minimum at $x = 0$, but $f(x)$ does not.

11.1.85

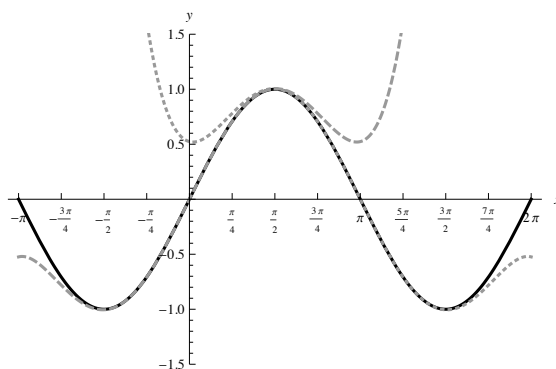
- a. We have

$$\begin{aligned} f(0) &= f^{(4)}(0) = \sin 0 = 0 & f(\pi) &= f^{(4)}(\pi) = \sin \pi = 0 \\ f'(0) &= f^{(5)}(0) = \cos 0 = 1 & f'(\pi) &= f^{(5)}(\pi) = \cos \pi = -1 \\ f''(0) &= -\sin 0 = 0 & f''(\pi) &= -\sin \pi = 0 \\ f'''(0) &= -\cos 0 = -1 & f'''(\pi) &= -\cos \pi = 1. \end{aligned}$$

Thus

$$\begin{aligned} p_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ q_5(x) &= -(x - \pi) + \frac{1}{3!}(x - \pi)^3 - \frac{1}{5!}(x - \pi)^5. \end{aligned}$$

- b. A plot of the three functions, with $\sin x$ the black solid line, $p_5(x)$ the dashed line, and $q_5(x)$ the dotted line is below.



$p_5(x)$ and $\sin x$ are almost indistinguishable on $[-\pi/2, \pi/2]$, after which $p_5(x)$ diverges pretty quickly from $\sin x$. $q_5(x)$ is reasonably close to $\sin x$ over the entire range, but the two are almost indistinguishable on $[\pi/2, 3\pi/2]$. $p_5(x)$ is a better approximation than $q_5(x)$ on about $[-\pi, \pi/2]$, while $q_5(x)$ is better on about $(\pi/2, 2\pi]$.

c. Evaluating the errors gives

x	$ \sin x - p_5(x) $	$ \sin x - q_5(x) $
$\frac{\pi}{4}$	3.6×10^{-5}	7.4×10^{-2}
$\frac{\pi}{2}$	4.5×10^{-3}	4.5×10^{-3}
$\frac{3\pi}{4}$	7.4×10^{-2}	3.6×10^{-5}
$\frac{5\pi}{4}$	2.3	3.6×10^{-5}
$\frac{7\pi}{4}$	20.4	7.4×10^{-2}

d. $p_5(x)$ is a better approximation than $q_5(x)$ only at $x = \frac{\pi}{4}$, in accordance with part (b). The two are equal at $x = \frac{\pi}{2}$, after which $q_5(x)$ is a substantially better approximation than $p_5(x)$.

11.1.86

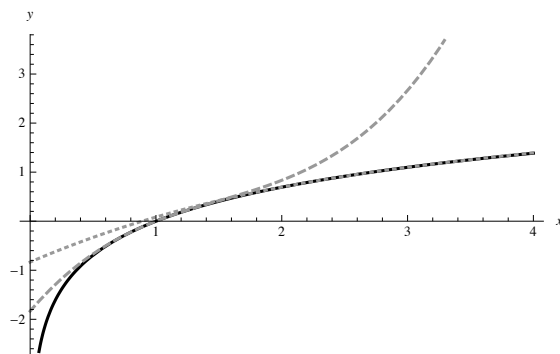
a. We have

$$\begin{aligned} f(1) &= \ln 1 = 0 & f(e) &= \ln e = 1 \\ f'(1) &= 1 & f'(e) &= \frac{1}{e} \\ f''(1) &= -1 & f''(e) &= -\frac{1}{e^2} \\ f'''(1) &= 2 & f'''(e) &= \frac{2}{e^3}. \end{aligned}$$

Thus

$$\begin{aligned} p_3(x) &= (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\ q_3(x) &= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3. \end{aligned}$$

b. A plot of the three functions, with $\ln x$ the black solid line, $p_3(x)$ the dashed line, and $q_3(x)$ the dotted line is below.



c. Evaluating the errors gives

x	$ \ln x - p_3(x) $	$ \ln x - q_3(x) $
0.5	2.6×10^{-2}	3.6×10^{-1}
1.0	0	8.4×10^{-2}
1.5	1.1×10^{-2}	1.6×10^{-2}
2.0	1.4×10^{-1}	1.5×10^{-3}
2.5	5.8×10^{-1}	1.1×10^{-5}
3.0	1.6	2.7×10^{-5}
3.5	3.3	1.4×10^{-3}

- d. $p_3(x)$ is a better approximation than $q_3(x)$ for $x = 0.5, 1.0$, and 1.5 , and $q_3(x)$ is a better approximation for the other points. To see why this is true, note that on $[0.5, 4]$ that $f^{(4)}(x) = -\frac{6}{x^4}$ is bounded in magnitude by $\frac{6}{0.5^4} = 96$, so that (using P_3 for the error term for p_3 and Q_3 as the error term for q_3)

$$P_3(x) \leq 96 \cdot \frac{|x-1|^4}{4!} = 4|x-1|^4, \quad Q_3(x) \leq 96 \cdot \frac{|x-e|^4}{4!} = 4|x-e|^4.$$

Thus the relative sizes of $P_3(x)$ and $Q_3(x)$ are governed by the distance of x from 1 and e . Looking at the different possibilities for x reveals why the results in part (c) hold.

11.1.87

- a. We have

$$\begin{aligned} f(36) &= \sqrt{36} = 6 & f(49) &= \sqrt{49} = 7 \\ f'(36) &= \frac{1}{2} \cdot \frac{1}{\sqrt{36}} = \frac{1}{12} & f'(49) &= \frac{1}{2} \cdot \frac{1}{\sqrt{49}} = \frac{1}{14}. \end{aligned}$$

Thus

$$p_1(x) = 6 + \frac{1}{12}(x-36) \quad q_1(x) = 7 + \frac{1}{14}(x-49).$$

- b. Evaluating the errors gives

x	$ \sqrt{x} - p_1(x) $	$ \sqrt{x} - q_1(x) $
37	5.7×10^{-4}	6.0×10^{-2}
39	5.0×10^{-3}	4.1×10^{-2}
41	1.4×10^{-2}	2.5×10^{-2}
43	2.6×10^{-2}	1.4×10^{-2}
45	4.2×10^{-2}	6.1×10^{-3}
47	6.1×10^{-2}	1.5×10^{-3}

- c. $p_1(x)$ is a better approximation than $q_1(x)$ for $x \leq 41$, and $q_1(x)$ is a better approximation for $x \geq 43$. To see why this is true, note that $f''(x) = -\frac{1}{4}x^{-3/2}$, so that on $[36, 49]$ it is bounded in magnitude by $\frac{1}{4} \cdot 36^{-3/2} = \frac{1}{864}$. Thus (using P_1 for the error term for p_1 and Q_1 for the error term for q_1)

$$P_1(x) \leq \frac{1}{864} \cdot \frac{|x-36|^2}{2!} = \frac{1}{1728}(x-36)^2, \quad Q_1(x) \leq \frac{1}{864} \cdot \frac{|x-49|^2}{2!} = \frac{1}{1728}(x-49)^2.$$

It follows that the relative sizes of $P_1(x)$ and $Q_1(x)$ are governed by the distance of x from 36 and 49. Looking at the different possibilities for x reveals why the results in part (b) hold.

11.1.88

- a. The quadratic Taylor polynomial for $\sin x$ centered at $\frac{\pi}{2}$ is

$$\begin{aligned} p_2(x) &= \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \cdot \left(x - \frac{\pi}{2}\right) - \frac{1}{2} \sin \frac{\pi}{2} \cdot \left(x - \frac{\pi}{2}\right)^2 \\ &= 1 - \frac{1}{2} \left(x - \frac{\pi}{2}\right)^2 \\ &= -\frac{1}{2}x^2 + \frac{\pi}{2}x + 1 - \frac{\pi^2}{8}. \end{aligned}$$

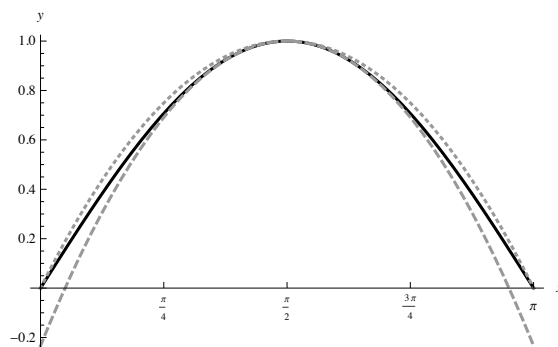
- b. Let $q(x) = ax^2 + bx + c$. Because $q(0) = \sin 0 = 0$, we must have $c = 0$, so that $q(x) = ax^2 + bx$. Then the other two conditions give us a pair of linear equation in a and b :

$$\begin{aligned} \frac{\pi^2}{4}a + \frac{\pi}{2}b &= 1 \\ \pi^2a + \pi b &= 0 \end{aligned}$$

where the first equation comes from the fact that $q(\pi/2) = \sin(\pi/2) = 1$ and the second from the fact that $q(\pi) = \sin \pi = 0$. Solving the linear system of equations gives $b = \frac{4}{\pi}$ and $a = -\frac{4}{\pi^2}$, so that

$$q(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x.$$

- c. A plot of the three function, with $\sin x$ the black solid line, $p_2(x)$ the dashed line, and $q(x)$ the dotted line is below.



- d. Evaluating the errors gives

x	$ \sin x - p_2(x) $	$ \sin x - q(x) $
$\frac{\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
$\frac{\pi}{2}$	0	0
$\frac{3\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
π	2.3×10^{-1}	0

- e. q is a better approximation than p at $x = \pi$, and the two are equal at $x = \frac{\pi}{2}$. At the other two points, however, $p_2(x)$ is a better approximation than $q(x)$. Clearly $q(x)$ will be exact at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$, because it was chosen that way. Also clearly $p_2(x)$ will be exact at $x = \frac{\pi}{2}$ since it is the Taylor polynomial centered at $\frac{\pi}{2}$. The fact that $p_2(x)$ is a better approximation than $q(x)$ at the two intermediate points is a result of the way the polynomials were constructed: the goal of $p_2(x)$ was to be as good an approximation as possible near $x = \frac{\pi}{2}$, while the goal of $q(x)$ was to match $\sin x$ at three given points. Overall, it appears that $q(x)$ does a better job over the full range (the total area between $q(x)$ and $\sin x$ is certainly smaller than the total area between $p_2(x)$ and $\sin x$).

11.2 Properties of Power Series

11.2.1 $c_0 + c_1x + c_2x^2 + c_3x^3$.

11.2.2 Yes. Note that we can write the given series as $\sum_{k=0}^{\infty} 5^k(x-4)^k$, which is a power series with $c_k = 5^k$ and $a = 4$.

11.2.3 Generally the Ratio Test or Root Test is used.

11.2.4 Yes. It is a power series where the center $a = 0$ and $c_k = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$

11.2.5 The radius of convergence does not change, but the interval of convergence may change at the endpoints.

11.2.6 The radius is 4 and the interval is $(-1, 7)$.

11.2.7 The series converges for $|x - 2| \leq 10$, so the radius of convergence is 10 and the interval is $[-8, 12]$.

11.2.8 The radius is $\frac{7 - (-3)}{2} = 5$, and the center is then $7 - 5 = 2$.

11.2.9 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} |2x| = |2x|$. So the radius of convergence is $\frac{1}{2}$. At $x = 1/2$ the series is $\sum_{k=0}^{\infty} 1$ which diverges, and at $x = -1/2$ the series is $\sum_{k=0}^{\infty} (-1)^k$ which also diverges. So the interval of convergence is $(-1/2, 1/2)$

11.2.10 Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(x-1)^{k+1}}{(k+1)!} \cdot \frac{k!}{(x-1)^k} \right| = \lim_{k \rightarrow \infty} \left| \frac{x-1}{k+1} \right| = 0$. Thus the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.11 Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^{k+1}x^{k+1}}{k^k x^k} \right| = \lim_{k \rightarrow \infty} (k+1) \left(\frac{k+1}{k} \right)^k |x| = \infty$ (for $x \neq 0$) because $\lim_{k \rightarrow \infty} \left(\frac{k+1}{k} \right)^k = e$. Thus, the radius of convergence is 0 and the series only converges at $x = 0$.

11.2.12 Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)!(x-10)^{k+1}}{k!(x-10)^k} \right| = \lim_{k \rightarrow \infty} (k+1)|x-10| = \infty$ (for $x \neq 10$). Thus, the radius of convergence is 0 and the series only converges at $x = 10$.

11.2.13 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \sin(1/k)|x| = \sin(0)|x| = 0$. Thus, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.14 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \frac{2|x-3|}{k^{1/k}} = 2|x-3|$. Thus, the radius of convergence is $1/2$. When $x = 7/2$, we have the harmonic series (which diverges), and when $x = 5/2$, we have the alternating harmonic series which converges. The interval of convergence is thus $[5/2, 7/2)$.

11.2.15 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \frac{|x|}{3} = \frac{|x|}{3}$, so the radius of convergence is 3. At $x = -3$, the series is $\sum (-1)^k$, which diverges. At $x = 3$, the series is $\sum 1$, which diverges. So the interval of convergence is $(-3, 3)$.

11.2.16 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \frac{|x|}{5} = \frac{|x|}{5}$, so the radius of convergence is 5. At $x = 5$, we obtain $\sum (-1)^k$ which diverges. At $x = -5$, we have $\sum 1$, which also diverges. So the interval of convergence is $(-5, 5)$.

11.2.17 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \frac{|x|}{k} = 0$, so the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

11.2.18 We use the Ratio Test. The limit of the absolute value of the ratio of successive terms is

$$\lim_{k \rightarrow \infty} |x| \sqrt{\frac{k}{k+1}} = |x|.$$

This is less than 1 for $|x| < 1$, so the radius of convergence is 1. At $x = -1$ we have the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ which is a divergent p -series, and for $x = 1$ we have the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ which is a convergent alternating series. The interval of convergence is $(-1, 1]$.

11.2.19 We use the Ratio Test. The limit of the absolute value of the ratio of successive terms is

$$\lim_{k \rightarrow \infty} |x| \frac{2^k + 1}{2^{k+1} + 1} = |x| \lim_{k \rightarrow \infty} \frac{1 + 1/2^k}{2 + 1/2^k} = \frac{|x|}{2},$$

so the radius of convergence is 2. For $x = 2$, we have the series $\sum_{k=1}^{\infty} \frac{2^k}{2^k + 1}$ which diverges by the Divergence Test because $\lim_{k \rightarrow \infty} \frac{2^k}{2^k + 1} = 1 \neq 0$. Similarly, the series for $x = -2$ fails the divergence test because the k th term does not have limit 0 as $k \rightarrow \infty$. So the interval of convergence is $(-2, 2)$.

11.2.20 Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(2x)^{k+1}}{(k+1)!} \cdot \frac{k!}{(2x)^k} \right| = \lim_{k \rightarrow \infty} \left| \frac{2x}{k+1} \right| = 0$. So the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.21 The given series is $\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k!}$. Note that the limit of the absolute value of successive terms is

$$\lim_{k \rightarrow \infty} \frac{x^2}{k+1} = 0,$$

which is less than 1 for all x . So the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.22 The given series is $\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{k^2}$. Note that the limit of the absolute value of successive terms is

$$\lim_{k \rightarrow \infty} \frac{x^2 k^2}{(k+1)^2} = x^2,$$

which is less than 1 for $-1 < x < 1$. So the radius of convergence is 1. For $x = 1$ we have the convergent alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ and for $x = -1$ we have its opposite (which also converges), so the interval of convergence is $[-1, 1]$.

11.2.23 We use the Root Test. The limit of the k th root of the absolute value of the k th term is

$$\lim_{k \rightarrow \infty} \frac{|x-1|}{k^{1/k}} = |x-1|.$$

This is less than 1 for $0 < x < 2$. The radius of convergence is 1. For $x = 0$, we have -1 times the Harmonic Series (which diverges) and for $x = 2$ we have the Alternating Harmonic series which converges by the Alternating Series Test. Therefore, the interval of convergence is $(0, 2]$.

11.2.24 Using the Ratio Test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \left(\frac{(k+1)(x-4)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{k(x-4)^k} \right) \right| = \lim_{k \rightarrow \infty} \left(\frac{k+1}{k} \cdot \frac{|x-4|}{2} \right) = \frac{|x-4|}{2},$$

so that the radius of convergence is 2. The interval is $(2, 6)$, because at the left endpoint, the series becomes $\sum_{k=1}^{\infty} k$ (which diverges) and at the right endpoint, it becomes $\sum_{k=1}^{\infty} (-1)^k k$ (which diverges).

11.2.25 Using the Ratio Test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} |4x-1| \frac{k^2+4}{k^2+2k+5} = |4x-1|.$$

This is less than 1 for $0 < x < \frac{1}{2}$. The radius of convergence is $\frac{1}{4}$. For $x = \frac{1}{2}$ we obtain the convergent series $\sum_{k=0}^{\infty} \frac{1}{k^2+4}$, which converges because $\frac{1}{k^2+4} < \frac{1}{k^2}$, and $\sum_{k=0}^{\infty} \frac{1}{k^2}$ is a convergent p -series. For $x = 0$, we obtain the alternating version of that same series, which is clearly absolutely convergent and therefore convergent. The interval of convergence is $\left[0, \frac{1}{2}\right)$.

11.2.26 We use the Root Test. The limit of the k th root of the absolute value of the k th term is

$$\lim_{k \rightarrow \infty} |3x+2| \frac{1}{k^{1/k}} = |3x+2|.$$

This is less than 1 for $-1 < 3x+2 < 1$, or $-1 < x < -\frac{1}{3}$. The radius of convergence is $\frac{1}{3}$. At $x = -\frac{1}{3}$, the series is the divergent Harmonic series, while at $x = -1$ the series is the convergent Alternating Harmonic series. The interval of convergence is therefore $\left[-1, -\frac{1}{3}\right)$.

11.2.27 We use the Root Test. The limit of the k th root of the absolute value of the k th term is

$$\lim_{k \rightarrow \infty} |2x-4| \frac{k^{10/k}}{10} = \frac{|2x-4|}{10}.$$

This is less than one for $-10 < 2x-4 < 10$, or $-3 < x < 7$. The radius of convergence is therefore 5. For the endpoints, the series diverges by the Divergence Test. The interval of convergence is thus $(-3, 7)$.

11.2.28 We use the Ratio Test. The limit of the absolute value of the ratio of the $k+1$ st term to the k th term is

$$\lim_{k \rightarrow \infty} |x+3| \frac{k}{k+1} \cdot \frac{\ln^2 k}{\ln^2(k+1)} = |x+3| \cdot 1 \cdot \left(\lim_{k \rightarrow \infty} \frac{\ln^2 k}{\ln^2(k+1)} \right)^2 = |x+3| \left(\lim_{k \rightarrow \infty} \frac{\frac{1}{k}}{\frac{1}{k+1}} \right)^2 = |x+3|.$$

This is less than 1 for $-1 < x+3 < 1$, or $-4 < x < -2$. The radius of convergence is 1. For $x = -2$, the resulting series is $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$ which can be shown to converge using the Integral Test. For $x = -4$, the series is the alternating version of the one for $x = -2$, so it converges absolutely. The interval of convergence is therefore $[-4, -2]$.

11.2.29 Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{(k+1)^2 x^{2k+2}}{(k+1)!} \cdot \frac{k!}{k^2 x^{2k}} \right| = \lim_{k \rightarrow \infty} \frac{k+1}{k^2} x^2 = 0$, so the radius of convergence is infinite, and the interval of convergence is $(-\infty, \infty)$.

11.2.30 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} k^{1/k} |x-1| = |x-1|$. The radius of convergence is therefore 1. At both $x = 2$ and $x = 0$ the series diverges by the Divergence Test. The interval of convergence is therefore $(0, 2)$.

11.2.31 Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{2k+3}}{3^k} \cdot \frac{3^{k-1}}{x^{2k+1}} \right| = \frac{x^2}{3}$ so that the radius of convergence is $\sqrt{3}$.

At $x = \sqrt{3}$, the series is $\sum_{k=1}^{\infty} 3\sqrt{3}$, which diverges. At $x = -\sqrt{3}$, the series is $\sum_{k=1}^{\infty} (-3\sqrt{3})$, which also diverges, so the interval of convergence is $(-\sqrt{3}, \sqrt{3})$.

11.2.32 $\sum_{k=0}^{\infty} \left(\frac{-x}{10} \right)^{2k} = \sum_{k=0}^{\infty} \left(\frac{x^2}{100} \right)^k$. Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \frac{x^2}{100} = \frac{x^2}{100}$, so that the radius of convergence is 10. At $x = \pm 10$, the series is then $\sum_{k=0}^{\infty} 1$, which diverges, so the interval of convergence is $(-10, 10)$.

11.2.33 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \frac{(|x-1|)k}{k+1} = |x-1|$, so the series converges when $|x-1| < 1$, so for $0 < x < 2$. The radius of convergence is 1. At $x = 2$, the series diverges by the Divergence Test. At $x = 0$, the series diverges as well by the Divergence Test. Thus the interval of convergence is $(0, 2)$.

11.2.34 Using the Ratio Test:

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \left| \frac{(-2)^{k+1}(x+3)^{k+1}}{3^{k+2}} \cdot \frac{3^{k+1}}{(-2)^k(x+3)^k} \right| = \frac{2}{3}|x+3|.$$

Thus the series converges when $\frac{2}{3}|x+3| < 1$, or $-\frac{9}{2} < x < -\frac{3}{2}$. At $x = -\frac{9}{2}$, the series diverges by the Divergence Test. At $x = -\frac{3}{2}$, the series diverges by the Divergence Test. Thus the interval of convergence is $\left(-\frac{9}{2}, -\frac{3}{2}\right)$.

11.2.35 Using the Ratio Test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{(k+1)^{20} x^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^k k^{20}} \right| = \lim_{k \rightarrow \infty} \left(\frac{k+1}{k} \right)^{20} \frac{|x|}{(2k+2)(2k+3)} = 0,$$

so the radius of convergence is infinite, and the interval of convergence is $(-\infty, \infty)$.

11.2.36 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \frac{|x^3|}{27} = \frac{|x^3|}{27}$, so the radius of convergence is 3. The series is divergent by the Divergence Test for $x = \pm 3$, so the interval of convergence is $(-3, 3)$.

11.2.37 Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)! x^{k+1}}{(k+1)^{k+1}} \cdot \frac{k^k}{k! x^k} \right| = \lim_{k \rightarrow \infty} \left(\frac{k}{k+1} \right)^k |x| = \frac{1}{e} |x|$. The radius of convergence is therefore e .

11.2.38 Using the Root Test: $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k |x| = ex$. Thus, the radius of convergence is $\frac{1}{e}$.

11.2.39 Using the Root Test:

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = |x| \lim_{k \rightarrow \infty} \left(\frac{k}{k+4} \right)^k = |x| \lim_{k \rightarrow \infty} \frac{1}{\left(1 + \frac{4}{k}\right)^k} = \frac{|x|}{e^4}.$$

This is less than 1 for $|x| < e^4$, so the radius of convergence is e^4 .

11.2.40 We use the Ratio Test. We have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} &= |x| \lim_{k \rightarrow \infty} \frac{1 - \cos \frac{1}{2^{k+1}}}{1 - \cos \frac{1}{2^k}} \\ &= |x| \lim_{k \rightarrow \infty} \frac{(\sin 2^{-k-1})2^{-k-1}(-1) \ln 2}{(\sin 2^{-k})2^{-k}(-1) \ln 2} \\ &= |x| \lim_{k \rightarrow \infty} \frac{\frac{\sin 2^{-k-1}}{2^{-k-1}} 2^{-2k-2}}{\frac{\sin 2^{-k}}{2^{-k}} 2^{-2k}} \\ &= \frac{|x|}{4}, \end{aligned}$$

where we used the fact that $\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$. The expression $\frac{|x|}{4}$ is less than 1 for $|x| < 4$, so the radius of convergence is 4.

11.2.41 $f(3x) = \frac{1}{1-3x} = \sum_{k=0}^{\infty} 3^k x^k$, which converges for $|x| < 1/3$, and diverges at the endpoints.

11.2.42 $g(x) = \frac{x^3}{1-x} = \sum_{k=0}^{\infty} x^{k+3}$, which converges for $|x| < 1$ and is divergent at the endpoints.

11.2.43 $h(x) = \frac{2x^3}{1-x} = \sum_{k=0}^{\infty} 2x^{k+3}$, which converges for $|x| < 1$ and is divergent at the endpoints.

11.2.44 $f(x^3) = \frac{1}{1-x^3} = \sum_{k=0}^{\infty} x^{3k}$. By the Root Test, $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = |x^3|$, so this series also converges for $|x| < 1$. It is divergent at the endpoints.

11.2.45 $p(x) = \frac{4x^{12}}{1-x} = \sum_{k=0}^{\infty} 4x^{k+12} = 4 \sum_{k=0}^{\infty} x^{k+12}$, which converges for $|x| < 1$. It is divergent at the endpoints.

11.2.46 $f(-4x) = \frac{1}{1+4x} = \sum_{k=0}^{\infty} (-4x)^k = \sum_{k=0}^{\infty} (-1)^k 4^k x^k$, which converges for $|x| < 1/4$ and is divergent at the endpoints.

11.2.47 $f(3x) = \ln(1-3x) = -\sum_{k=1}^{\infty} \frac{(3x)^k}{k} = -\sum_{k=1}^{\infty} \frac{3^k}{k} x^k$ for $-1 \leq 3x < 1$, or $-\frac{1}{3} \leq x < \frac{1}{3}$. The radius of convergence is $\frac{1}{3}$. The series diverges at $\frac{1}{3}$ (harmonic series), and converges at $-\frac{1}{3}$ (alternating harmonic series). The interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right)$.

11.2.48 $g(x) = x^3 \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^{k+3}}{k}$ for $-1 \leq x < 1$, so the radius of convergence is 1. The series diverges at 1 and converges at -1 . The interval of convergence is $[-1, 1)$.

11.2.49 $p(x) = 2x^6 \ln(1-x) = -2 \sum_{k=1}^{\infty} \frac{x^{k+6}}{k}$ for $-1 \leq x < 1$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series), so the interval of convergence is $[-1, 1)$.

11.2.50 $f(x^3) = \ln(1-x^3) = - \sum_{k=1}^{\infty} \frac{x^{3k}}{k}$ for $-1 \leq x^3 < 1$ or $-1 \leq x < 1$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series), so the interval of convergence is $[-1, 1)$.

11.2.51 The power series for $f(x)$ is $\sum_{k=0}^{\infty} (2x)^k$, which is convergent for $-1 < 2x < 1$, so for $-1/2 < x < 1/2$. The power series for $g(x) = f'(x)$ is $\sum_{k=1}^{\infty} k(2x)^{k-1} \cdot 2 = 2 \sum_{k=1}^{\infty} k(2x)^{k-1}$, also convergent on $|x| < 1/2$. At the endpoints the series diverges by the Divergence Test. The interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

11.2.52 The power series for $f(x)$ is $\sum_{k=0}^{\infty} x^k$, which is convergent for $-1 < x < 1$, so the power series for $g(x) = \frac{1}{2}f''(x)$ is $\frac{1}{2} \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{1}{2} \sum_{k=0}^{\infty} (k+1)(k+2)x^k$, which is also convergent on $|x| < 1$. The series diverges at the endpoints by the Divergence Test. The interval of convergence is $(-1, 1)$.

11.2.53 $f(x) = \frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k$ for $-1 < -x < 1$, or $-1 < x < 1$. So

$$g(x) = -\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} -k(-x)^{k-1} = \sum_{k=1}^{\infty} (-1)^k k x^{k-1},$$

for $-1 < x < 1$. At each endpoint $x = \pm 1$, the series diverges by the Divergence Test. So the interval of convergence is $(-1, 1)$.

11.2.54 The power series for $f(x)$ is $\sum_{k=0}^{\infty} (-1)^k x^{2k}$, which is convergent on $|x| < 1$. Because $g(x) = -\frac{1}{2}f'(x)$, the power series for g is $-\frac{1}{2} \sum_{k=1}^{\infty} (-1)^k 2k x^{2k-1} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^{2k-1}$, which is also convergent on $|x| < 1$. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-1, 1)$.

11.2.55 The power series for $\frac{1}{1-3x}$ is $\sum_{k=0}^{\infty} (3x)^k$, which is convergent on $|x| < 1/3$. Because $g(x) = \ln(1-3x) = -3 \int \frac{1}{1-3x} dx$ and because $g(0) = 0$, the power series for $g(x)$ is $-3 \sum_{k=0}^{\infty} 3^k \frac{1}{k+1} x^{k+1} = - \sum_{k=1}^{\infty} \frac{3^k}{k} x^k$. The series converges at $x = -\frac{1}{3}$ (alternating harmonic) and diverges at $x = \frac{1}{3}$ (harmonic series). The interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right)$.

11.2.56 The power series for $\frac{x}{1+x^2}$ is $x \sum_{k=0}^{\infty} (-1)^k x^{2k} = \sum_{k=0}^{\infty} (-1)^k x^{2k+1}$, which is convergent on $|x| < 1$. Because $g(x) = 2 \int f(x) dx$, and because $g(0) = 0$, the power series for $g(x)$ is $2 \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+2} x^{2k+2} =$

$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} x^{2k+2}$. This can be written as $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} x^{2k}$. At both endpoints, we have the alternating harmonic series which converges, so the interval of convergence is $[-1, 1]$.

11.2.57 Recall that $\frac{d}{dx} \left(\frac{1}{1+x^2} \right) = -\frac{2x}{(1+x^2)^2}$, and by Example 3c, $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$. So

$$-\frac{2x}{(1+x^2)^2} = -\frac{d}{dx} (1 - x^2 + x^4 - x^6 + \dots) = 2x - 4x^3 + 6x^5 - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} 2kx^{2k-1},$$

for $|x| < 1$. Checking the endpoints for $x = \pm 1$ reveals that the series diverges by the Divergence Test.

11.2.58 Start with $g(x) = \frac{1}{1-x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty} x^k$. Because $f(x) = g(x^4)$, its power series is $\sum_{k=0}^{\infty} x^{4k}$. The radius of convergence is still 1, and the series is divergent at both endpoints. The interval of convergence is $(-1, 1)$.

11.2.59 Note that $f(x) = \frac{3}{3+x} = \frac{1}{1+(1/3)x}$. Let $g(x) = \frac{1}{1+x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty} (-1)^k x^k$, so the power series for $f(x) = g((1/3)x)$ is $\sum_{k=0}^{\infty} (-1)^k 3^{-k} x^k = \sum_{k=0}^{\infty} \left(\frac{-x}{3} \right)^k$. Using the Ratio Test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{3^{-(k+1)} x^{k+1}}{3^{-k} x^k} \right| = \frac{|x|}{3},$$

so the radius of convergence is 3. The series diverges at both endpoints by the Divergence test. The interval of convergence is $(-3, 3)$.

11.2.60 Note that $f(x) = \frac{1}{2} \ln(1-x^2)$. The power series for $g(x) = \ln(1-x)$ is $-\sum_{k=1}^{\infty} \frac{1}{k} x^k$, so the power series for $f(x) = \frac{1}{2} g(x^2)$ is $-\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} x^{2k}$. The radius of convergence is still 1. The series diverges at both 1 and -1 , its interval of convergence is $(-1, 1)$.

11.2.61 Note that $f(x) = \ln \sqrt{4-x^2} = \frac{1}{2} \ln(4-x^2) = \frac{1}{2} \left(\ln 4 + \ln \left(1 - \frac{x^2}{4} \right) \right) = \ln 2 + \frac{1}{2} \ln \left(1 - \frac{x^2}{4} \right)$. Now, the power series for $g(x) = \ln(1-x)$ is $-\sum_{k=1}^{\infty} \frac{1}{k} x^k$, so the power series for $f(x)$ is $\ln 2 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{x^{2k}}{4^k} = \ln 2 - \sum_{k=1}^{\infty} \frac{x^{2k}}{k 2^{2k+1}}$. Now, $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{x^{2k+2}}{(k+1)2^{2k+3}} \cdot \frac{k 2^{2k+1}}{x^{2k}} \right| = \lim_{k \rightarrow \infty} \frac{k}{4(k+1)} x^2 = \frac{x^2}{4}$, so that the radius of convergence is 2. The series diverges at both endpoints, so its interval of convergence is $(-2, 2)$.

11.2.62 By Example 5, the Taylor series for $g(x) = \tan^{-1} x$ is $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, so that $f(x) = g((2x)^2)$ has Taylor series $\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{4k+2}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k+1}}{2k+1} x^{4k+2}$. Using the Ratio Test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{4^{2k+3} x^{4k+6}}{2k+3} \cdot \frac{2k+1}{4^{2k+1} x^{4k+2}} \right| = \lim_{k \rightarrow \infty} \frac{16(2k+1)}{2k+3} x^4 = 16x^4,$$

so that the radius of convergence is $1/2$. The interval of convergence is $(-1/2, 1/2)$.

11.2.63

- True. This power series is centered at $x = 3$, so its interval of convergence will be symmetric about 3.
- True. Use the Root Test. At the endpoints, the series diverges by the Divergence Test.
- True. Substitute x^2 for x in the series.
- True. Because the power series is zero on the interval, all its derivatives are as well, which implies (differentiating the power series) that all the c_k are zero.

11.2.64 The power series for $f(ax)$ is $\sum c_k(ax)^k$. Then $\sum c_k(ax)^k$ converges if and only if $|ax| < R$ (because $\sum c_k x^k$ converges for $|x| < R$), which happens if and only if $|x| < \frac{R}{|a|}$.

11.2.65 The power series for $f(x - a)$ is $\sum c_k(x - a)^k$. Then $\sum c_k(x - a)^k$ converges if and only if $|x - a| < R$, which happens if and only if $a - R < x < a + R$, so the radius of convergence is the same.

11.2.66 Replacing x by $x - 1$ gives $\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}$. Using the Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(x-1)^{k+1}}{k+1} \cdot \frac{k}{(x-1)^k} \right| = \lim_{k \rightarrow \infty} \frac{k}{k+1} |x-1| = |x-1|$, so that the series converges for $|x-1| < 1$. Checking the endpoints, the interval of convergence is $(0, 2]$.

11.2.67 This is a geometric series with ratio $\sqrt{x} - 2$, so its sum is $\frac{1}{1 - (\sqrt{x} - 2)} = \frac{1}{3 - \sqrt{x}}$, for $-1 < \sqrt{x} - 2 < 1$. Simplifying the inequality, we have $1 < \sqrt{x} < 3$, or $1 < x < 9$.

11.2.68 Note that $\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k} = \sum_{k=1}^{\infty} \frac{(x^2)^k}{4^k} = \sum_{k=1}^{\infty} \left(\frac{x^2}{4}\right)^k$, which implies that $\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k}$ is a geometric series with $a = \frac{x^2}{4}$ and $r = \frac{x^2}{4}$. It follows that

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k} = \frac{a}{1-r} = \frac{x^2/4}{1-x^2/4} = \frac{x^2}{4-x^2},$$

for $\frac{x^2}{4} < 1$, or $-2 < x < 2$.

11.2.69 This is a geometric series with ratio e^{-x} , so its sum is $\frac{1}{1 - e^{-x}}$, for $e^{-x} < 1$, or $e^x > 1$ or $x > 0$.

11.2.70 This is a geometric series with ratio $\frac{x-2}{9}$, so its sum is $\frac{(x-2)/9}{1 - (x-2)/9} = \frac{x-2}{9 - (x-2)} = \frac{x-2}{11-x}$, for $-1 < \frac{x-2}{9} < 1$, or $-7 < x < 11$. It diverges at both endpoints.

11.2.71 This is a geometric series with ratio $(x^2 - 1)/3$, so its sum is $\frac{1}{1 - \frac{x^2-1}{3}} = \frac{3}{3 - (x^2 - 1)} = \frac{3}{4 - x^2}$, convergent for $|x^2 - 1| < 3$, so that $-2 < x^2 < 4$ or $-2 < x < 2$. It diverges at both endpoints.

11.2.72 Substitute $2x$ for x in the power series for e^x to get $e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$. The series converges for all x .

11.2.73 Substitute $-3x$ for x in the power series for e^x to get $e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{3^k}{k!} x^k$. The series converges for all x .

11.2.74 Multiply the power series for e^x by x^2 to get $x^2 e^x = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}$, which converges for all x .

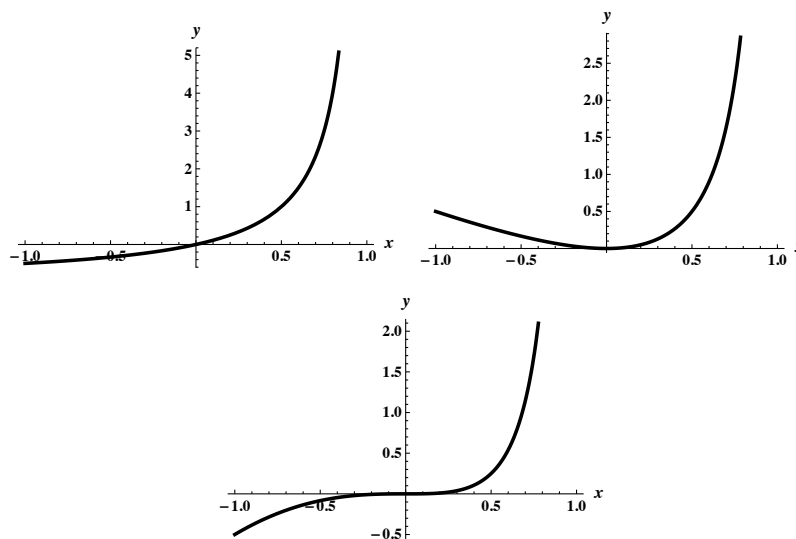
11.2.75 The power series for $x^m f(x)$ is $\sum c_k x^{k+m}$. The radius of convergence of this power series is determined by the limit

$$\lim_{k \rightarrow \infty} \left| \frac{c_{k+1} x^{k+1+m}}{c_k x^{k+m}} \right| = \lim_{k \rightarrow \infty} \left| \frac{c_{k+1} x^{k+1}}{c_k x^k} \right|,$$

and the right-hand side is the limit used to determine the radius of convergence for the power series for $f(x)$. Thus the two have the same radius of convergence.

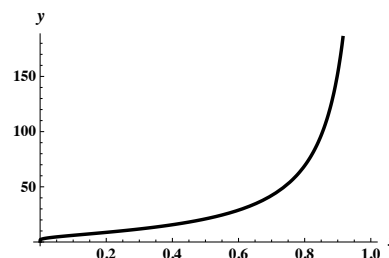
11.2.76

- a. $R_n = f(x) - S_n(x) = \sum_{k=n}^{\infty} x^k$. This is a geometric series with ratio x . Its sum is then $R_n = \frac{x^n}{1-x}$ as desired.
- b. $R_n(x)$ increases without bound as x approaches 1, and its absolute value smallest at $x = 0$ (where it is zero). In general, for $x > 0$, $R_n(x) < R_{n-1}(x)$, so the approximations get better the more terms of the series are included.



- c. To minimize $|R_n(x)|$, set its derivative to zero. Assuming $n > 1$, we have $R'_n(x) = \frac{n(1-x)x^{n-1} + x^n}{(1-x)^2}$, which is zero for $x = 0$. There is a minimum at this critical point.

- d. The following is a plot that shows, for each $x \in (0, 1)$, the n required so that $R_n(x) < 10^{-6}$. The closer x gets to 1, the more terms are required in order for the estimate given by the power series to be accurate. The number of terms increases rapidly as $x \rightarrow 1$.



11.2.77

a. $f(x)g(x) = c_0d_0 + (c_0d_1 + c_1d_0)x + (c_0d_2 + c_1d_1 + c_2d_0)x^2 + \dots$

b. The coefficient of x^n in $f(x)g(x)$ is $\sum_{i=0}^n c_i d_{n-i}$.

11.2.78 The function $\frac{1}{\sqrt{1-x^2}}$ is the derivative of the inverse sine function, and $\sin^{-1}(0) = 0$, so the power series for $\sin^{-1} x$ is the integral of the given power series, or $x + \frac{1}{6}x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}x^7 + \dots$. This can also be written $x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots 2k \cdot (2k+1)} x^{2k+1}$.

11.3 Taylor Series

11.3.1 The n th Taylor Polynomial is the n th sum of the corresponding Taylor Series.

11.3.2 In order to have a Taylor series centered at a , a function f must have derivatives of all orders on some interval containing a .

11.3.3 $\sum_{k=0}^{\infty} \frac{(x-2)^k}{k!}$.

11.3.4 $\sum_{k=0}^{\infty} (k+1)x^k$.

11.3.5 Substitute x^2 for x in the Taylor series. By theorems proved in the previous section about power series, the interval of convergence does not change except perhaps at the endpoints of the interval.

11.3.6 The Taylor series terminates if $f^{(n)}(0) = 0$ for $n > N$ for some N . For $(1+x)^p$, this occurs if and only if p is an integer ≥ 0 .

11.3.7 It means that the limit of the remainder term is zero.

11.3.8 For $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, while $f''(x) = f(x) = e^{-x}$. So $f(0) = 1$, $f'(0) = -1$, $f''(0) = 1$, etc. Thus, $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$. Using Table 11.5, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, so $e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$.

11.3.9

a. $f(x) = \frac{1}{x^2}$, so $f(1) = 1$. $f'(x) = -\frac{2}{x^3}$, so $f'(1) = -2$. $f''(x) = \frac{6}{x^4}$, so $f''(1) = 6$. $f'''(x) = -\frac{24}{x^5}$, so $f'''(1) = -24$. So the Taylor series is

$$1 - 2(x-1) + 6\frac{(x-1)^2}{2!} - 24\frac{(x-1)^3}{3!} + \dots = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots$$

b. $\sum_{k=0}^{\infty} (-1)^k (k+1)(x-1)^k$.

c. $r = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1}(k+2)(x-1)^{k+1}}{(-1)^k(k+1)(x-1)^k} \right| = |x-1|$. This is less than 1 for $-1 < x-1 < 1$, or $0 < x < 2$.

The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(0, 2)$.

11.3.10

- a. $f(x) = \frac{1}{x^2}$, so $f(-1) = 1$. $f'(x) = -\frac{2}{x^3}$, so $f'(-1) = 2$. $f''(x) = \frac{6}{x^4}$, so $f''(-1) = 6$. $f'''(x) = -\frac{24}{x^5}$, so $f'''(-1) = 24$. So the Taylor series is

$$1 + 2(x+1) + 6\frac{(x+1)^2}{2!} + 24\frac{(x+1)^3}{3!} + \cdots = 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + \cdots$$

b. $\sum_{k=0}^{\infty} (k+1)(x+1)^k$.

- c. $r = \lim_{k \rightarrow \infty} \left| \frac{(k+2)(x+1)^{k+1}}{(k+1)(x+1)^k} \right| = |x+1|$. This is less than 1 for $-1 < x+1 < 1$, or $-2 < x < 0$. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-2, 0)$.

11.3.11

- a. Note that $f(0) = 1$, $f'(0) = -1$, $f''(0) = 1$, and $f'''(0) = -1$. So the Maclaurin series is $1 - x + x^2/2 - x^3/6 + \cdots$.

b. $\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$.

- c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

11.3.12

- a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -4$, $f'''(0) = 0$, $f^{(4)}(0) = 16$, $\dots$. Thus the Maclaurin series is

$$1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \cdots$$

b. $\sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!}$

- c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

11.3.13

- a. Note that $f(0) = 2$, $f'(x) = \frac{6}{(1-x)^4}$, so $f'(0) = 6$, $f''(x) = \frac{24}{(1-x)^5}$, so $f''(0) = 24$, and $f'''(x) = \frac{120}{(1-x)^6}$, so $f'''(0) = 120$. Thus the Maclaurin series is

$$2 + 6x + 12x^2 + 20x^3 + \cdots$$

b. $\sum_{k=0}^{\infty} (k+1)(k+2)x^k$.

- c. Using the Ratio Test, we see that the series converges on $(-1, 1)$. It does not converge at the endpoints by the Divergence Test.

11.3.14

- a. $f(0) = 0$, $f'(x) = x \cos x + \sin x$, so $f'(0) = 0$. $f''(x) = -x \sin x + \cos x + \cos x$, so $f''(0) = 2$. $f'''(x) = -x \cos x - \sin x - 2 \sin x$, so $f'''(0) = 0$. $f^{(4)}(x) = -4 \cos x + x \sin x$, so $f^{(4)}(0) = -4$. Similarly, the 5th and 7th derivatives of f at 0 have value 0, while the 6th has value 6 and the 8th has value -8 . So the series is

$$x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^8}{5040} + \cdots$$

b. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k}}{(2k-1)!}.$

c. The limit of the ratio of successive terms is 0, so the interval of convergence is $(-\infty, \infty)$.

11.3.15

a. For $f(x) = \frac{1}{1+x^2}$, we have $f(0) = 1$, $f'(x) = -\frac{2x}{(1+x^2)^2}$ so $f'(0) = 0$. $f''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$, so $f''(0) = -2$. Similarly, $f'''(0) = 0$ and $f^{(4)}(0) = 24$, $f^{(5)}(0) = 0$, and $f^{(6)}(0) = 6!$. Thus the series is $1 - x^2 + x^4 - x^6 + \dots$.

b. $\sum_{k=0}^{\infty} (-1)^k x^{2k}.$

c. The absolute value of the ratio of consecutive terms is x^2 , so by the Ratio Test, the radius of convergence is 1. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-1, 1)$.

11.3.16

a. Note that $f(0) = 0$, $f'(0) = 4$, $f''(0) = -16$, $f'''(0) = 128$, and $f^{(4)}(0) = -1536$. Thus, the series is given by $4x - \frac{16x^2}{2} + \frac{128x^3}{6} - \frac{1536x^4}{24} + \dots$.

b. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-1)!(4x)^k}{k!} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(4x)^k}{k}.$

c. The absolute value of the ratio of consecutive terms is $\frac{4|x|k}{k+1}$, which has limit $4|x|$ as $k \rightarrow \infty$, so the interval of convergence is $(-1/4, 1/4]$. Note that for $x = 1/4$ we have the alternating harmonic series, while for $x = -1/4$ we have negative 1 times the harmonic series, which diverges.

11.3.17

a. Note that $f(0) = 1$, and that $f^{(n)}(0) = 2^n$. Thus, the series is given by $1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \dots$.

b. $\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}.$

c. The absolute value of the ratio of consecutive terms is $\frac{2|x|}{n}$, which has limit 0 as $n \rightarrow \infty$. So by the Ratio Test, the interval of convergence is $(-\infty, \infty)$.

11.3.18

a. We have $f(0) = 1$, while $f'(x) = -2(1+2x)^{-2}$, so $f'(0) = -2$. Then $f''(x) = 8(1+2x)^{-3}$, so $f''(0) = 8$. $f'''(x) = -48(1+2x)^{-4}$, so $f'''(0) = -48$. The series is given by $1 - 2x + 4x^2 - 8x^3 + \dots$.

b. $\sum_{k=0}^{\infty} (-1)^k (2x)^k.$

c. The Root Test shows that the series converges absolutely for $|2x| < 1$, or $|x| < 1/2$. The interval of convergence is $(-1/2, 1/2)$, because the series at both endpoints diverge by the Divergence Test.

11.3.19

- a. We have $f(0) = 0$, while $f'(x) = \frac{2}{x^2 + 4}$, so $f'(0) = \frac{1}{2}$. $f''(x) = -\frac{4x}{(x^2 + 4)^2}$, so $f''(0) = 0$. Continuing, we have $f'''(x) = \frac{4(3x^2 - 4)}{(x^2 + 4)^3}$, so $f'''(0) = -\frac{1}{4}$. In a similar manner, $f^{(4)}(0) = 0$, $f^{(5)}(0) = \frac{3}{4}$, $f^{(6)}(0) = 0$, and $f^{(7)}(0) = -\frac{45}{8}$. So the series is $\frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{160} - \frac{x^7}{896} + \cdots$.
- b. $\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1) \cdot 2^{2k+1}} x^{2k+1}$.
- c. The ratio of consecutive terms has limit $\frac{x^2}{4}$, so by the Ratio Test, the radius of convergence is $|x| < 2$. Also, at the endpoints we have convergence by the Alternating Series Test, so the interval of convergence is $[-2, 2]$.

11.3.20

- a. $f(0) = 0$, $f'(x) = 3 \cos 3x$, so $f'(0) = 3$. $f''(0) = 0$, but $f'''(x) = -27 \cos 3x$, so $f'''(0) = -27$. Similarly, $f^{(4)}(0) = 0$, but $f^{(5)}(0) = 243$ and $f^{(7)}(0) = -3^7$. So the series is $3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \frac{243x^7}{560} + \cdots$.
- b. $\sum_{k=0}^{\infty} (-1)^k \frac{3^{2k+1}}{(2k+1)!} x^{2k+1}$.
- c. The ratio of successive terms is $\frac{9}{2n(2n+1)} x^2$, which has limit zero as $n \rightarrow \infty$, so the interval of convergence is $(-\infty, \infty)$.

11.3.21

- a. Note that $f(0) = 1$, $f'(0) = \ln 3$, $f''(0) = \ln^2 3$, $f'''(0) = \ln^3 3$. So the first four terms of the desired series are $1 + (\ln 3)x + \frac{\ln^2 3}{2}x^2 + \frac{\ln^3 3}{6}x^3 + \cdots$.
- b. $\sum_{k=0}^{\infty} \frac{(\ln^k 3)x^k}{k!}$.
- c. The ratio of successive terms is $\frac{(\ln^{k+1} 3)x^{k+1}}{(k+1)!} \cdot \frac{k!}{(\ln^k 3)x^k} = \frac{\ln 3}{k+1}x$, and the limit as $k \rightarrow \infty$ of this quantity is 0, so the interval of convergence is $(-\infty, \infty)$.

11.3.22

- a. Note that $f(0) = 0$, $f'(0) = \frac{1}{\ln 3}$, $f''(0) = -\frac{1}{\ln 3}$, $f'''(0) = \frac{2}{\ln 3}$, $f^{(4)}(0) = -\frac{6}{\ln 3}$. So the first terms of the desired series are $0 + \frac{x}{\ln 3} - \frac{x^2}{2 \ln 3} + \frac{x^3}{3 \ln 3} - \frac{x^4}{4 \ln 3} + \cdots$.
- b. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k \ln 3}$.
- c. The absolute value of the ratio of successive terms is $\left| \frac{x^{k+1}}{(k+1) \ln 3} \cdot \frac{k \ln 3}{x^k} \right| = \frac{k}{k+1} |x|$, which has limit $|x|$ as $k \rightarrow \infty$. Thus the radius of convergence is 1. At $x = -1$ we have a multiple of the harmonic series (which diverges) and at $x = 1$ we have a multiple of the alternating harmonic series (which converges) so the interval of convergence is $(-1, 1]$.

11.3.23

- a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = 9$, $f'''(0) = 0$, etc. The first terms of the series are $1 + 9x^2/2 + 81x^4/4! + 3^6x^6/6! + \dots$.
- b.
$$\sum_{k=0}^{\infty} \frac{(3x)^{2k}}{(2k)!}.$$
- c. The absolute value of the ratio of successive terms is $\left| \frac{(3x)^{2k+2}}{(2k+2)!} \cdot \frac{(2k)!}{(3x)^{2k}} \right| = \frac{1}{(2k+2)(2k+1)} \cdot 9x^2$, which has limit 0 as $x \rightarrow \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

11.3.24

- a. Note that $f(0) = 0$, $f'(0) = 2$, $f''(0) = 0$, $f'''(0) = 8$, etc. The first terms of the series are $2x + 8x^3/6 + 32x^5/5! + 128x^7/7! + \dots$, or $2x + \frac{4x^3}{3} + \frac{4x^5}{15} + \frac{8x^7}{315} + \dots$.
- b.
$$\sum_{k=0}^{\infty} \frac{2^{2k+1}x^{2k+1}}{(2k+1)!}.$$
- c. The absolute value of the ratio of successive terms is $\left| \frac{2^{2k+3}x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{2^{2k+1}x^{2k+1}} \right| = \frac{4}{(2k+3)(2k+2)}x^2$, which has limit 0 as $x \rightarrow \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

11.3.25

- a. $f(3) = 0$, $f'(x) = \frac{1}{x-2}$, so $f'(3) = 1$. $f''(x) = -\frac{1}{(x-2)^2}$, so $f''(3) = -1$. $f'''(x) = \frac{2}{(x-2)^3}$, so $f'''(3) = 2$, and $f^{(4)}(x) = -\frac{6}{(x-2)^4}$, so $f^{(4)}(3) = -6$. The series is

$$(x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 - \frac{1}{4}(x-3)^4 + \dots$$

- b.
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-3)^k}{k}.$$

- c. The limit of the absolute value of the ratio of successive terms is $x-3$. So the series converges for $-1 < x-3 < 1$, or $2 < x < 4$. At the left endpoint it diverges (it is the opposite of the harmonic series) and at the right endpoint it converges (it is the alternating harmonic series), so the interval of convergence is $(2, 4]$.

11.3.26

- a. Because all the derivatives of e^x are e^x , we have $f^{(n)}(2) = e^2$ for all n . Thus the series is given by

$$e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \dots$$

- b.
$$\sum_{k=0}^{\infty} \frac{e^2(x-2)^k}{k!}.$$

- c. The limit of the ratio of the absolute value of successive terms is 0, so the series converges on $(-\infty, \infty)$.

11.3.27

a. Note that $f(\pi/2) = 1$, $f'(\pi/2) = \cos(\pi/2) = 0$, $f''(\pi/2) = -\sin(\pi/2) = -1$, $f'''(\pi/2) = -\cos(\pi/2) = 0$, and so on. Thus the series is given by $1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6 + \cdots$.

b.
$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k}.$$

11.3.28

a. Note that $f(\pi) = -1$, $f'(\pi) = -\sin \pi = 0$, $f''(\pi) = -\cos \pi = 1$, $f'''(\pi) = -\sin \pi = 0$, and so on. Thus the series is given by $-1 + \frac{1}{2}(x - \pi)^2 - \frac{1}{24}(x - \pi)^4 + \frac{1}{720}(x - \pi)^6 + \cdots$.

b.
$$\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{(2k)!} (x - \pi)^{2k}.$$

11.3.29

a. Note that $f^{(k)}(1) = (-1)^k \frac{k!}{1^{k+1}} = (-1)^k \cdot k!$. Thus the series is given by $1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$.

b.
$$\sum_{k=0}^{\infty} (-1)^k (x-1)^k.$$

11.3.30

a. Note that $f^{(k)}(2) = (-1)^k \frac{k!}{2^{k+1}}$. Thus the series is given by $\frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 + \cdots$.

b.
$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{k+1}} (x-2)^k.$$

11.3.31

a. Note that $f^{(k)}(3) = (-1)^{k-1} \frac{(k-1)!}{3^k}$. Thus the series is given by $\ln(3) + \frac{x-3}{3} - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3 + \cdots$.

b.
$$\ln 3 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k \cdot 3^k} (x-3)^k.$$

11.3.32

a. Note that $f^{(k)}(\ln 2) = 2$. Thus the series is given by $2 + 2(x - \ln(2)) + (x - \ln(2))^2 + \frac{1}{3}(x - \ln(2))^3 + \frac{1}{12}(x - \ln(2))^4 + \cdots$.

b.
$$\sum_{k=0}^{\infty} \frac{2}{k!} (x - \ln(2))^k.$$

11.3.33

a. Note that $f(1) = 2$, $f'(1) = 2 \ln 2$, $f''(1) = 2 \ln^2 2$, $f'''(1) = 2 \ln^3 2$. The first terms of the series are $2 + (2 \ln 2)(x-1) + (\ln^2 2)(x-1)^2 + \frac{(\ln^3 2)(x-1)^3}{3} + \cdots$.

b.
$$\sum_{k=0}^{\infty} \frac{2(x-1)^k \ln^k 2}{k!}.$$

11.3.34

a. Note that $f(1) = 0$, and that $f'(x) = \ln x$, so $f'(1) = 0$. Then $f''(x) = \frac{1}{x}$, so $f''(1) = 1$, $f'''(x) = -\frac{1}{x^2}$, so $f'''(1) = -1$. $f^{(4)}(x) = \frac{2}{x^3}$, so $f^{(4)}(1) = 2$, and $f^{(5)}(x) = -\frac{6}{x^4}$, so $f^{(5)}(1) = -6$. So the series is

$$\frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \frac{2(x-1)^4}{4!} - \frac{6(x-1)^5}{5!} + \cdots$$

b.
$$\sum_{k=2}^{\infty} \frac{(-1)^k (x-1)^k}{k(k-1)}.$$

11.3.35 Because the Taylor series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, the first four terms of the Taylor series for $\ln(1+x^2)$ are $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$, obtained by substituting x^2 for x .

11.3.36 Because the Taylor series for $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, the first four terms of the Taylor series for $\sin x^2$ are $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$, obtained by substituting x^2 for x .

11.3.37 Because the Taylor series for $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, the first four terms of the Taylor series for $\frac{1}{1-2x}$ are $1 + 2x + 4x^2 + 8x^3 + \cdots$ obtained by substituting $2x$ for x .

11.3.38 Because the Taylor series for $\ln(1+x)$ is $x - x^2/2 + x^3/3 - x^4/4 + \cdots$, the first four terms of the Taylor series for $2x - 2x^2 + 8x^3/3 - 4x^4 + \cdots$ obtained by substituting $2x$ for x .

11.3.39 The Taylor series for $e^x - 1$ is the Taylor series for e^x , less the constant term of 1, so it is $x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Thus, the first four terms of the Taylor series for $\frac{e^x-1}{x}$ are $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$, obtained by dividing the terms of the first series by x .

11.3.40 Because the Taylor series for $\cos x$ is $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, the first four terms of the Taylor series for $\cos x^3$ are $1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$, obtained by substituting x^3 for x .

11.3.41 Because the Taylor series for $(1+x)^{-1}$ is $1 - x + x^2 - x^3 + \cdots$, if we substitute x^4 for x , we obtain $1 - x^4 + x^8 - x^{12} + \cdots$.

11.3.42 The Taylor series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. Thus, the Taylor series for $\tan^{-1} x^2$ is $x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots$ and, multiplying by x , the Taylor series for $x \tan^{-1} x^2$ is $x^3 - \frac{x^7}{3} + \frac{x^{11}}{5} - \frac{x^{15}}{7} + \cdots$.

11.3.43 The Taylor series for $\sinh x$ is $x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$. Thus, the Taylor series for $\sinh x^2$ is $x^2 + \frac{x^6}{6} + \frac{x^{10}}{120} + \frac{x^{14}}{5040} + \cdots$ obtained by substituting x^2 for x .

11.3.44 The Taylor series for $\cosh x$ is $1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$. Thus, the Taylor series for $\cosh 3x$ is $1 + \frac{9x^2}{2} + \frac{81x^4}{24} + \frac{729x^6}{720} + \cdots$, obtained by substituting $3x$ for x .

11.3.45

- a. The binomial coefficients are $\binom{-2}{0} = 1$, $\binom{-2}{1} = \frac{-2}{1!} = -2$, $\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$, $\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4$.

Thus the first four terms of the series are $1 - 2x + 3x^2 - 4x^3 + \dots$.

- b. $1 - 2 \cdot 0.1 + 3 \cdot 0.01 - 4 \cdot 0.001 = 0.826$

11.3.46

- a. The binomial coefficients are $\binom{1/2}{0} = 1$, $\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2}$, $\binom{1/2}{2} = \frac{(1/2)(-1/2)}{2!} = -\frac{1}{8}$, $\binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16}$, so the first four terms of the series are $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$.

- b. $1 + \frac{1}{2} \cdot 0.06 - \frac{1}{8} \cdot 0.06^2 + \frac{1}{16} \cdot 0.06^3 \approx 1.030$

11.3.47

- a. The binomial coefficients are $\binom{1/4}{0} = 1$, $\binom{1/4}{1} = \frac{1/4}{1} = \frac{1}{4}$, $\binom{1/4}{2} = \frac{(1/4)(-3/4)}{2!} = -\frac{3}{32}$, $\binom{1/4}{3} = \frac{(1/4)(-3/4)(-7/4)}{3!} = \frac{7}{128}$, so the first four terms of the series are $1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \dots$.

- b. Substitute $x = 0.12$ to get approximately 1.029.

11.3.48

- a. The binomial coefficients are $\binom{-3}{0} = 1$, $\binom{-3}{1} = -3$, $\binom{-3}{2} = \frac{(-3)(-4)}{2!} = 6$, $\binom{-3}{3} = \frac{(-3)(-4)(-5)}{3!} = -10$, so the first four terms of the series are $1 - 3x + 6x^2 - 10x^3 + \dots$.

- b. Substitute $x = 0.1$ to get 0.750.

11.3.49

- a. The binomial coefficients are $\binom{-2/3}{0} = 1$, $\binom{-2/3}{1} = -\frac{2}{3}$, $\binom{-2/3}{2} = \frac{(-2/3)(-5/3)}{2!} = \frac{5}{9}$, $\binom{-2/3}{3} = \frac{(-2/3)(-5/3)(-8/3)}{3!} = -\frac{40}{81}$, so the first four terms of the series are $1 - \frac{2}{3}x + \frac{5}{9}x^2 - \frac{40}{81}x^3 + \dots$.

- b. Substitute $x = 0.18$ to get 0.89512.

11.3.50

- a. The binomial coefficients are $\binom{2/3}{0} = 1$, $\binom{2/3}{1} = \frac{2}{3}$, $\binom{2/3}{2} = \frac{(2/3)(-1/3)}{2!} = -\frac{1}{9}$, $\binom{2/3}{3} = \frac{(2/3)(-1/3)(-4/3)}{3!} = \frac{4}{81}$, so the first four terms of the series are $1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 + \dots$.

- b. Substitute $x = 0.02$ to get ≈ 1.013289284 .

11.3.51 $\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \dots$. By the Ratio Test, the radius of convergence is 1. At the endpoints, the series obtained are convergent by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1]$.

11.3.52 $\sqrt{4+x} = 2\sqrt{1+x/4} = 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \dots$. The interval of convergence is $(-4, 4]$.

11.3.53 $\sqrt{9-9x} = 3\sqrt{1-x} = 3 - \frac{3}{2}x - \frac{3}{8}x^2 - \frac{3}{16}x^3 - \dots$. The interval of convergence is $[-1, 1]$.

11.3.54 $\sqrt{1-4x} = 1 - 2x - 2x^2 - 4x^3 - \dots$, obtained by substituting $-4x$ for x in the original series. The interval of convergence of $[-1/4, 1/4]$.

11.3.55 $\sqrt{a^2+x^2} = a\sqrt{1+\frac{x^2}{a^2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \dots$. The series converges when $\frac{x^2}{a^2}$ is less than 1 in magnitude, so the radius of convergence is a . The series given by the endpoints is convergent by the Alternating Series Test, so the interval of convergence is $[-a, a]$.

11.3.56 $\sqrt{4-16x^2} = 2\sqrt{1-(2x)^2} = 2 - 4x^2 - 4x^4 - 8x^6 - \dots$. Because $2x$ was substituted for x to produce this series, this series converges when $-1 < 2x < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$. Because only even powers of x appear in the series, the series at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ are identical, and are convergent. Thus the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

11.3.57 $(1+4x)^{-2} = 1 - 2(4x) + 3(4x)^2 - 4(4x)^3 + \dots = 1 - 8x + 48x^2 - 256x^3 + \dots$.

11.3.58 $\frac{1}{(1-4x)^2} = (1-4x)^{-2} = 1 - 2(-4x) + 3(-4x)^2 - 4(-4x)^3 + \dots = 1 + 8x + 48x^2 + 256x^3 + \dots$.

11.3.59 $\frac{1}{(4+x^2)^2} = (4+x^2)^{-2} = \frac{1}{16}(1+(x^2/4))^{-2} = \frac{1}{16}\left(1 - 2 \cdot \frac{x^2}{4} + 3 \cdot \frac{x^4}{16} - 4 \cdot \frac{x^6}{64} + \dots\right) = \frac{1}{16} - \frac{1}{32}x^2 + \frac{3}{256}x^4 - \frac{1}{256}x^6 + \dots$

11.3.60 Note that $x^2 - 4x + 5 = 1 + (x-2)^2$, so $(1+(x-2)^2)^{-2} = 1 - 2(x-2)^2 + 3(x-2)^4 - 4(x-2)^6 + \dots$.

11.3.61 $(3+4x)^{-2} = \frac{1}{9}\left(1+\frac{4x}{3}\right)^{-2} = \frac{1}{9} - \frac{2}{9}\left(\frac{4x}{3}\right) + \frac{3}{9}\left(\frac{4x}{3}\right)^2 - \frac{4}{9}\left(\frac{4x}{3}\right)^3 + \dots$.

11.3.62 $(1+4x^2)^{-2} = (1+(2x)^2)^{-2} = 1 - 2(2x)^2 + 3(2x)^4 - 4(2x)^6 + \dots = 1 - 8x^2 + 48x^4 - 256x^6 + \dots$.

11.3.63 The interval of convergence for the Taylor series for $f(x) = \sin x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c . Because $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we have

$$\lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} \frac{1}{(n+1)!} |x^{n+1}| = 0$$

for any x .

11.3.64 The interval of convergence for the Taylor series for $f(x) = \cos 2x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c . The n th derivative of $\cos 2x$ is 2^n times either $\pm \sin x$ or $\pm \cos x$, so that $f^{(n+1)}$ is bounded by 2^{n+1} in magnitude. Thus $\lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} |x^{n+1}| = \lim_{n \rightarrow \infty} \frac{(2|x|)^{n+1}}{(n+1)!} = 0$ for any x .

11.3.65 The interval of convergence for the Taylor series for e^{-x} is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for some c . Thus $\lim_{n \rightarrow \infty} |R_n(x)| = 0$ for any x .

11.3.66 The interval of convergence for the Taylor series for $f(x) = \cos x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - \pi/2)^{n+1}$ for some c . Because $f^{n+1}(x)$ is $\pm \cos x$ or $\pm \sin x$, we have

$$\lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} \frac{1}{(n+1)!} |(x - \pi/2)^{n+1}| = 0$$

for any x .

11.3.67

- False. Not all of its derivatives are defined at zero - in fact, none of them are.
- True. The derivatives of $\csc x$ involve positive powers of $\csc x$ and $\cot x$, both of which are defined at $\pi/2$, so that $\csc x$ has continuous derivatives at $\pi/2$.
- False. For example, the Taylor series for $f(x^2)$ doesn't converge at $x = 1.9$, because the Taylor series for $f(x)$ doesn't converge at $1.9^2 = 3.61$.
- False. The Taylor series centered at 1 involves derivatives of f evaluated at 1, not at 0.
- True. The follows because the Taylor series must itself be an even function.

11.3.68

- The relevant Taylor series are: $\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$, and $2 \sin x = 2x - \frac{1}{3}x^3 + \frac{1}{60}x^5 - \dots$.
Thus, the first four terms of the resulting series are $\cos 2x + 2 \sin x = 1 + 2x - 2x^2 - \frac{1}{3}x^3 + \frac{2}{3}x^4 + \dots$.
- Because each series converges (absolutely) on $(-\infty, \infty)$, so does their sum. The radius of convergence is ∞ .

11.3.69

- The relevant Taylor series are: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \dots$. Thus the first four terms of the resulting series are $\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$.
- Because each series converges (absolutely) on $(-\infty, \infty)$, so does their sum. The radius of convergence is ∞ .

11.3.70

- The first four terms of the Taylor series for $\sin x$ are $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$, so the first four terms for $\frac{\sin x}{x}$ are $1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}$.
- The radius of convergence is the same as that for $\sin x$, namely ∞ .

11.3.71

- Use the binomial theorem. The binomial coefficients are $\binom{-2/3}{0} = 1$, $\binom{-2/3}{1} = -\frac{2}{3}$, $\binom{-2/3}{2} = \frac{(-2/3)(-5/3)}{2!} = \frac{5}{9}$, $\binom{-2/3}{3} = \frac{(-2/3)(-5/3)(-8/3)}{3!} = -\frac{40}{81}$ and then, substituting x^2 for x , we obtain $1 - \frac{2}{3}x^2 + \frac{5}{9}x^4 - \frac{40}{81}x^6 + \dots$.

- b. The radius of convergence is determined from $|x^2| < 1$, so it is 1.

11.3.72

- a. The first four terms of $\cos x$ are $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$, so the first four terms of $\cos x^2$ are $1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720}$, and thus the first four terms of $x^2 \cos x^2$ are $x^2 - \frac{x^6}{2} + \frac{x^{10}}{24} - \frac{x^{14}}{720}$.
- b. The radius of convergence is ∞ .

11.3.73

- a. From the binomial formula, the Taylor series for $(1-x)^p$ is $\sum \binom{p}{k} (-1)^k x^k$, so the Taylor series for $(1-x^2)^p$ is $\sum \binom{p}{k} (-1)^k x^{2k}$. Here $p = 1/2$, and the binomial coefficients are $\binom{1/2}{0} = 1$, $\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2}$, $\binom{1/2}{2} = \frac{(1/2)(-1/2)}{2!} = -\frac{1}{8}$, $\binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16}$ so that $(1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \dots$.
- b. The radius of convergence is determined from $|x^2| < 1$, so it is 1.

11.3.74

- a. Because $b^x = e^{x \ln b}$, the Taylor series is $1 + x \ln b + \frac{1}{2!}(x \ln b)^2 + \frac{1}{3!}(x \ln b)^3 + \dots$.
- b. Because the series for e^x converges on $(-\infty, \infty)$, the radius of convergence for the series in part a is ∞ .

11.3.75

- a. $f(x) = (1+x^2)^{-2}$; using the binomial series and substituting x^2 for x we obtain $1 - 2x^2 + 3x^4 - 4x^6 + \dots$.
- b. The radius of convergence is determined from $|x^2| < 1$, so it is 1.

11.3.76 Because $f(36) = 6$, and $f'(x) = \frac{1}{2}x^{-1/2}$, $f'(36) = \frac{1}{12}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f''(36) = -\frac{1}{864}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, and $f'''(36) = \frac{3}{62208}$, the first four terms of the Taylor series are $6 + \frac{1}{12}(x-36) - \frac{1}{864 \cdot 2!}(x-36)^2 + \frac{3}{62208 \cdot 3!}(x-36)^3$. Evaluating at $x = 39$ we obtain 6.245008681.

11.3.77 Because $f(64) = 4$, and $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(64) = \frac{1}{48}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, $f''(64) = -\frac{1}{4608}$, $f'''(x) = \frac{10}{27}x^{-8/3}$, and $f'''(64) = \frac{10}{1769472} = \frac{5}{884736}$, the first four terms of the Taylor series are $4 + \frac{1}{48}(x-64) - \frac{1}{4608 \cdot 2!}(x-64)^2 + \frac{5}{884736 \cdot 3!}(x-64)^3$. Evaluating at $x = 60$, we obtain 3.914870274.

11.3.78 Because $f(4) = \frac{1}{2}$, and $f'(x) = -\frac{1}{2}x^{-3/2}$, $f'(4) = -\frac{1}{16}$, $f''(x) = \frac{3}{4}x^{-5/2}$, $f''(4) = \frac{3}{128}$, $f'''(x) = -\frac{15}{8}x^{-7/2}$, and $f'''(4) = -\frac{15}{1024}$, the first four terms of the Taylor series are $\frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{128 \cdot 2!}(x-4)^2 - \frac{15}{1024 \cdot 3!}(x-4)^3$. Evaluating at $x = 3$, we get 0.5766601563.

11.3.79 Because $f(16) = 2$, and $f'(x) = \frac{1}{4}x^{-3/4}$, $f'(16) = \frac{1}{32}$, $f''(x) = -\frac{3}{16}x^{-7/4}$, $f''(16) = -\frac{3}{2048}$, $f'''(x) = \frac{21}{64}x^{-11/4}$, and $f'''(16) = \frac{21}{131072}$, the first four terms of the Taylor series are $2 + \frac{1}{32}(x-16) - \frac{3}{2048 \cdot 2!}(x-16)^2 + \frac{21}{131072 \cdot 3!}(x-16)^3$. Evaluating at $x = 13$, we get 1.898937225.

11.3.80 Evaluate the binomial coefficient $\binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = (-1)^k$, so that the binomial expansion for $(1+x)^{-1}$ is $\sum_{k=0}^{\infty} (-1)^k x^k$. Substituting $-x$ for x , we obtain $(1-x)^{-1} = \sum_{k=0}^{\infty} (-1)^k (-x)^k = \sum_{k=0}^{\infty} x^k$.

11.3.81 Evaluate the binomial coefficient

$$\begin{aligned}\binom{1/2}{k} &= \frac{(1/2)(-1/2)(-3/2)\cdots(1/2-k+1)}{k!} = \frac{(1/2)(-1/2)\cdots((3-2k)/2)}{k!} \\ &= (-1)^{k-1} 2^{-k} \frac{1 \cdot 3 \cdots (2k-3)}{k!} = (-1)^{k-1} 2^{-k} \frac{(2k-2)!}{2^{k-1} \cdot (k-1)! \cdot k!} = (-1)^{k-1} 2^{1-2k} \cdot \frac{1}{k} \binom{2k-2}{k-1}.\end{aligned}$$

This is the coefficient of x^k in the Taylor series for $\sqrt{1+x}$. Substituting $4x$ for x , the Taylor series becomes

$$\sum_{k=0}^{\infty} (-1)^{k-1} 2^{1-2k} \cdot \frac{1}{k} \binom{2k-2}{k-1} (4x)^k = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{2}{k} \binom{2k-2}{k-1} x^k.$$

If we can show that k divides $\binom{2k-2}{k-1}$, we will be done, for then the coefficient of x^k will be an integer. But

$$\begin{aligned}\binom{2k-2}{k-1} - \binom{2k-2}{k-2} &= \frac{(2k-2)!}{(k-1)!(k-1)!} - \frac{(2k-2)!}{(k-2)!k!} = \frac{(2k-2)!}{(k-1)!(k-1)!} - \frac{(2k-2)!(k-1)}{(k-1)!(k-1)!k} \\ &= \frac{k(2k-2)! - (k-1)(2k-2)!}{k(k-1)!(k-1)!} = \frac{1}{k} \frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{1}{k} \binom{2k-2}{k-1}\end{aligned}$$

and thus we have shown that k divides $\binom{2k-2}{k-1}$.

11.3.82 The two Taylor series are:

$$\begin{aligned}8 + \frac{1}{16}(x-64) - \frac{1}{4096}(x-64)^2 + \frac{1}{524288}(x-64)^3 - \frac{5}{268435456}(x-64)^4 + \cdots \\ 9 + \frac{1}{18}(x-81) - \frac{1}{5832}(x-81)^2 + \frac{1}{944784}(x-81)^3 - \frac{5}{612220032}(x-81)^4 + \cdots\end{aligned}$$

Evaluating these Taylor series at $n = 2, 3, 4$ (after the quadratic, cubic, and quartic terms) we obtain the errors:

n	64	81
2	9.064×10^{-4}	-8.297×10^{-4}
3	-7.019×10^{-5}	-5.813×10^{-5}
4	6.106×10^{-6}	-4.550×10^{-6}

The errors using the Taylor series centered at 81 are consistently smaller.

11.3.83

a. The Maclaurin series for $\sin x$ is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$. Squaring the first four terms yields

$$\begin{aligned}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right)^2 \\ = x^2 - \frac{2}{3!}x^4 + \left(\frac{2}{5!} + \frac{1}{3!3!}\right)x^6 + \left(-2 \cdot \frac{1}{7!} - 2 \cdot \frac{1}{3!5!}\right)x^8 \\ = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8.\end{aligned}$$

- b. The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1 - \cos 2x)/2$, we obtain

$$\begin{aligned} & (1 - (1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \frac{1}{8!}(2x)^8)/2) \\ &= (2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \frac{2}{315}x^8)/2 \\ &= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8, \end{aligned}$$

and the two are the same.

- c. If $f(x) = \sin^2 x$, then $f(0) = 0$, $f'(x) = \sin 2x$, so $f'(0) = 0$. $f''(x) = 2 \cos 2x$, so $f''(0) = 2$, $f'''(x) = -4 \sin 2x$, so $f'''(0) = 0$. Note that from this point $f^{(n)}(0) = 0$ if n is odd and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n . Thus, the series for $\sin^2 x$ is

$$2x^2/2 - 8x^4/4! + 32x^6/6! - 128x^8/8! + \dots = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \dots$$

11.3.84

- a. The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$. Squaring the first four terms yields

$$\begin{aligned} & \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\right)^2 \\ &= 1 - \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{4!} + \frac{1}{4!} + \frac{1}{4}\right)x^4 + \left(-\frac{1}{6!} - \frac{1}{6!} - \frac{1}{2 \cdot 4!} - \frac{1}{2 \cdot 4!}\right)x^6 \\ &= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6. \end{aligned}$$

- b. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1 + \cos 2x)/2$, we obtain

$$\begin{aligned} & \left(1 + 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6\right)/2 \\ &= \left(2 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right)/2 \\ &= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6, \end{aligned}$$

and the two are the same.

- c. If $f(x) = \cos^2 x$, then $f(0) = 1$. Also, $f'(x) = -2 \cos x \sin x = -\sin 2x$. So $f'(0) = 0$. $f''(x) = -2 \cos 2x$, so $f''(0) = -2$. $f'''(x) = 8 \sin 2x$, so $f'''(0) = 0$. Note that from this point on, $f^{(n)}(0) = 0$ if n is odd, and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n . Thus, the series for $\cos^2 x$ is

$$1 - 2x^2/2 + 8x^4/4! - 32x^6/6! + \dots = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots$$

11.3.85 There are many solutions. For example, first find a series that has $(-1, 1)$ as an interval of convergence, say $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. Then the series $\frac{1}{1-x/2} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$ has $(-2, 2)$ as its interval of convergence.

Now shift the series up so that it is centered at 4. We have $\sum_{k=0}^{\infty} \left(\frac{x-4}{2}\right)^k$, which has interval of convergence $(2, 6)$.

11.3.86

a. The Maclaurin series in question are

$$\begin{aligned}\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\ e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots,\end{aligned}$$

so substituting the series for $\sin x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\sin x} = 1 + (x - \frac{1}{3!}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

b. The Maclaurin series in question are

$$\begin{aligned}\tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\ e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots,\end{aligned}$$

so substituting the series for $\tan x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\tan x} = 1 + (x + \frac{1}{3}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

c. The Maclaurin series in question are

$$\begin{aligned}\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\ \sqrt{1+x^2} &= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots,\end{aligned}$$

so substituting the series for $\sin x$ for x in the series for $\sqrt{1+x^2}$ (and considering only those terms that will give us an exponent at most 4), we obtain $\sqrt{1+\sin^2 x} = 1 + \frac{1}{2}(x - \frac{1}{3!}x^3)^2 - \frac{1}{8}x^4 + \cdots = 1 + \frac{1}{2}x^2 - \frac{7}{24}x^4 + \cdots$.

11.3.87 Use the Taylor series for $\cos x$ centered at $\pi/4$:

$$\frac{\sqrt{2}}{2}(1 - (x - \pi/4) - \frac{1}{2}(x - \pi/4)^2 + \frac{1}{6}(x - \pi/4)^3 + \cdots).$$

The remainder after n terms (because the derivatives of $\cos x$ are bounded by 1 in magnitude) is

$$|R_n(x)| \leq \frac{1}{(n+1)!} \cdot \left(\frac{\pi}{4} - \frac{2\pi}{9}\right)^{n+1}.$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 3$. Evaluating the first four terms (through $n = 3$) of the series we get 0.7660427050. The true value is ≈ 0.7660444431 .

11.3.88 Use the Taylor series for $\sin x$ centered at π :

$$-(x - \pi) + \frac{1}{6}(x - \pi)^3 - \frac{1}{120}(x - \pi)^5 + \cdots.$$

The remainder after n terms (because the derivatives of $\sin x$ are bounded by 1 in magnitude) is

$$|R_n(x)| \leq \frac{1}{(n+1)!} \cdot (\pi - 0.98\pi)^{n+1}.$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 2$. Evaluating the first term of the series gives 0.06283185307. The true value is ≈ 0.06279051953 .

11.3.89

- Use the Taylor series for $(125 + x)^{1/3}$ centered at $x = 0$. Using the first four terms and evaluating at $x = 3$ gives a result (5.03968) accurate to within 10^{-4} .
- Use the Taylor series for $x^{1/3}$ centered at $x = 125$. Note that this gives the identical Taylor series except that the exponential terms are $(x - 125)^n$ rather than x^n . Thus we need terms up through $(x - 125)^3$, just as before, evaluated at $x = 128$, and we obtain the identical result.
- Because the two Taylor series are the same except for the shifting, the results are equivalent.

11.3.90

- To show that $f'(0) = 0$, we compute the limits of the left and right difference quotients and show that they are both zero:

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2} - 0}{x} = \lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x} \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{e^{-1/x^2} - 0}{x} = \lim_{x \rightarrow 0^-} \frac{e^{-1/x^2}}{x}.$$

For the limit from the right, use the substitution $x = \frac{1}{\sqrt{y}}$; then $y = x^2$ and the limit becomes

$$\lim_{y \rightarrow \infty} e^{-y} \sqrt{y} = \lim_{y \rightarrow \infty} \frac{\sqrt{y}}{e^y} = 0,$$

because exponentials dominate power functions. Similarly, for the limit from the left, use the substitution $x = -\frac{1}{\sqrt{y}}$; then again $y = x^2$ and the limit becomes

$$\lim_{y \rightarrow \infty} (-e^{-y} \sqrt{y}) = - \lim_{y \rightarrow \infty} \frac{\sqrt{y}}{e^y} = 0.$$

Since the left and right limits are both zero, it follows that f is differentiable at $x = 0$, and its derivative is zero.

- Because $f^{(k)}(0) = 0$, the Taylor series centered at 0 has only one term: $f(x) = f(0) = 0$, so the Taylor series is zero.
- It does not converge to $f(x)$ because $f(x) \neq 0$ for all $x \neq 0$.

11.3.91 Consider the remainder after the first term of the Taylor series. Taylor's Theorem indicates that

$$R_1(x) = \frac{f''(c)}{2}(x-a)^2 \quad \text{for some } c \text{ between } x \text{ and } a,$$

so that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$. But $f'(a) = 0$, so that for every x in an interval containing a , there is a c between x and a such that $f(x) = f(a) + \frac{f''(c)}{2}(x-a)^2$.

- If $f''(x) > 0$ on the interval containing a , then for every x in that interval, we have $f(x) = f(a) + \frac{f''(c)}{2}(x-a)^2$ for some c between x and a . But $f''(c) > 0$ and $(x-a)^2 > 0$, so that $f(x) > f(a)$ and a is a local minimum.
- If $f''(x) < 0$ on the interval containing a , then for every x in that interval, we have $f(x) = f(a) + \frac{f''(c)}{2}(x-a)^2$ for some c between x and a . But $f''(c) < 0$ and $(x-a)^2 > 0$, so that $f(x) < f(a)$ and a is a local maximum.

11.4 Working with Taylor Series

11.4.1 Replace f and g by their Taylor series centered at a , and evaluate the limit.

11.4.2 Integrate the Taylor series for $f(x)$ centered at a , and evaluate it at the endpoints.

11.4.3 Substitute -0.6 for x in the Taylor series for e^x centered at 0 . Note that this series is an alternating series, so the error can easily be estimated by looking at the magnitude of the first neglected term.

11.4.4

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \frac{1 - (1 - x^2/2! + x^4/4! - \cdots)}{x} = \lim_{x \rightarrow 0} (x/2! - x^3/4! + \cdots) = 0.$$

11.4.5

$$\begin{aligned} \frac{d}{dx} \sinh x &= \frac{d}{dx} \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \\ &= 1 + \frac{3x^2}{3!} + \frac{5x^4}{5!} + \cdots = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \frac{x^{2k}}{(2k)!} + \cdots. \end{aligned}$$

11.4.6 It must have derivatives of all orders on some interval containing a .

11.4.7 Because $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, we have $\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \cdots$, so $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$.

11.4.8 Because $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, we have $\frac{\tan^{-1} x - x}{x^3} = -\frac{1}{3} + \frac{x^2}{5} - \cdots$.

$$\text{So } \lim_{x \rightarrow 0} \frac{\tan^{-1} x - x}{x^3} = -\frac{1}{3}.$$

11.4.9 Because $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, we have $\frac{-x - \ln(1-x)}{x^2} = \frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \cdots$, so $\lim_{x \rightarrow 0} \frac{-x - \ln(1-x)}{x^2} = \frac{1}{2}$.

11.4.10 Because $\sin 2x = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} + \cdots$, we have $\frac{\sin 2x}{x} = 2 - \frac{4x^2}{3} + \frac{4x^4}{15} + \cdots$, so $\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2$.

11.4.11 We compute that

$$\begin{aligned} \frac{e^x - e^{-x}}{x} &= \frac{1}{x} \left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots \right) \right) \\ &= \frac{1}{x} \left(2x + \frac{x^3}{3} + \cdots \right) = 2 + \frac{x^2}{3} + \cdots \end{aligned}$$

so the limit of $\frac{e^x - e^{-x}}{x}$ as $x \rightarrow 0$ is 2 .

11.4.12 Because $-e^x = -1 - x - \frac{x^2}{2} - \frac{x^3}{6} + \cdots$, we have $\frac{1+x-e^x}{4x^2} = -\frac{1}{8} - \frac{x}{24} + \cdots$, so $\lim_{x \rightarrow 0} \frac{1+x-e^x}{4x^2} = -\frac{1}{8}$.

11.4.13 We compute that

$$\begin{aligned} \frac{2 \cos 2x - 2 + 4x^2}{2x^4} &= \frac{1}{2x^4} \left(2 \left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \cdots \right) - 2 + 4x^2 \right) \\ &= \frac{1}{2x^4} \left(\frac{(2x)^4}{12} - \frac{(2x)^6}{360} + \cdots \right) = \frac{2}{3} - \frac{4x^2}{45} + \cdots \end{aligned}$$

so the limit of $\frac{2 \cos 2x - 2 + 4x^2}{2x^4}$ as $x \rightarrow 0$ is $\frac{2}{3}$.

11.4.14 We substitute $t = \frac{1}{x}$ and find $\lim_{t \rightarrow 0} \frac{\sin t}{t}$. We compute that

$$\frac{\sin t}{t} = \frac{1}{t} \left(t - \frac{t^3}{6} + \cdots \right) = 1 - \frac{t^2}{6} + \cdots$$

so the limit of $x \sin \left(\frac{1}{x} \right)$ as $x \rightarrow \infty$ is 1.

11.4.15 We have $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, so that

$$\frac{\ln(1+x) - x + x^2/2}{x^3} = \frac{x^3/3 - x^4/4 + \cdots}{x^3} = \frac{1}{3} - \frac{x}{4} + \cdots$$

so that $\lim_{x \rightarrow 0} \frac{\ln(1+x) - x + x^2/2}{x^3} = \frac{1}{3}$.

11.4.16 The Taylor series for $\ln(x-3)$ centered at $x=4$ is

$$(x-4) - \frac{1}{2}(x-4)^2 + \cdots$$

We compute that

$$\begin{aligned} \frac{x^2-16}{\ln(x-3)} &= \frac{x^2-16}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} = \frac{(x-4)(x+4)}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} \\ &= \frac{x+4}{1 - \frac{1}{2}(x-4) + \cdots} \end{aligned}$$

so the limit of $\frac{x^2-16}{\ln(x-3)}$ as $x \rightarrow 4$ is 8.

11.4.17 We compute that

$$\begin{aligned} \frac{3 \tan^{-1} x - 3x + x^3}{x^5} &= \frac{1}{x^5} \left(3 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) - 3x + x^3 \right) \\ &= \frac{1}{x^5} \left(\frac{3x^5}{5} - \frac{3x^7}{7} + \cdots \right) = \frac{3}{5} - \frac{3x^2}{7} + \cdots \end{aligned}$$

so the limit of $\frac{3 \tan^{-1} x - 3x + x^3}{x^5}$ as $x \rightarrow 0$ is $\frac{3}{5}$.

11.4.18 The Taylor series for $\sqrt{1+x}$ centered at 0 is

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

We compute that

$$\begin{aligned} \frac{\sqrt{1+x} - 1 - (x/2)}{4x^2} &= \frac{1}{4x^2} \left(\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots \right) - 1 - \frac{x}{2} \right) \\ &= \frac{1}{4x^2} \left(-\frac{x^2}{8} + \frac{x^3}{16} + \cdots \right) = -\frac{1}{32} + \frac{x}{64} + \cdots \end{aligned}$$

so the limit of $\frac{\sqrt{1+x} - 1 - (x/2)}{4x^2}$ as $x \rightarrow 0$ is $-\frac{1}{32}$.

11.4.19 The Taylor series for $\sin 2x$ centered at 0 is

$$\sin 2x = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots.$$

Thus

$$\begin{aligned} \frac{12x - 8x^3 - 6 \sin 2x}{x^5} &= \frac{12 - 8x^3 - (12x - 8x^3 + \frac{8}{5}x^5 - \frac{16}{105}x^7 + \cdots)}{x^5} \\ &= -\frac{8}{5} + \frac{16}{105}x^2 - \cdots, \end{aligned}$$

$$\text{so } \lim_{x \rightarrow 0} \frac{12x - 8x^3 - 6 \sin 2x}{x^5} = -\frac{8}{5}.$$

11.4.20 The Taylor series for $\ln x$ centered at 1 is

$$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \cdots.$$

We compute that

$$\frac{x - 1}{\ln x} = \frac{x - 1}{(x - 1) - \frac{1}{2}(x - 1)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x - 1) + \cdots}$$

so the limit of $\frac{x - 1}{\ln x}$ as $x \rightarrow 1$ is 1.

11.4.21 The Taylor series for $\ln(x - 1)$ centered at 2 is

$$\ln(x - 1) = (x - 2) - \frac{1}{2}(x - 2)^2 + \cdots.$$

We compute that

$$\frac{x - 2}{\ln(x - 1)} = \frac{x - 2}{(x - 2) - \frac{1}{2}(x - 2)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x - 2) + \cdots}$$

so the limit of $\frac{x - 2}{\ln(x - 1)}$ as $x \rightarrow 2$ is 1.

11.4.22 Because $e^{1/x} = 1 + (1/x) + 1/(2x^2) + \cdots$, we have

$$x(e^{1/x} - 1) = 1 + 1/(2x) + \cdots.$$

Thus, $\lim_{x \rightarrow \infty} x(e^{1/x} - 1) = 1$.

11.4.23 Computing Taylor series centers at 0 gives

$$\begin{aligned} e^{-2x} &= 1 - 2x + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \cdots = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots \\ e^{-x/2} &= 1 - \frac{x}{2} + \frac{1}{2!}\left(-\frac{x}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x}{2}\right)^3 + \cdots = 1 - \frac{x}{2} + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \cdots. \end{aligned}$$

Thus

$$\begin{aligned} \frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} &= \frac{1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots - (4 - 2x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \cdots) + 3}{2x^2} \\ &= \frac{\frac{3}{2}x^2 - \frac{5}{4}x^3 + \cdots}{2x^2} \\ &= \frac{3}{4} - \frac{5}{8}x + \cdots \end{aligned}$$

$$\text{so } \lim_{x \rightarrow 0} \frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} = \frac{3}{4}.$$

11.4.24 The Taylor series for $(1 - 2x)^{-1/2}$ centered at 0 is

$$(1 - 2x)^{-1/2} = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots.$$

We compute that

$$\begin{aligned} \frac{(1 - 2x)^{-1/2} - e^x}{8x^2} &= \frac{1}{8x^2} \left(\left(1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots \right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \right) \right) \\ &= \frac{1}{8x^2} \left(x^2 + \frac{7x^3}{3} + \cdots \right) = \frac{1}{8} + \frac{7x}{24} + \cdots \end{aligned}$$

so the limit of $\frac{(1 - 2x)^{-1/2} - e^x}{8x^2}$ as $x \rightarrow 0$ is $\frac{1}{8}$.

11.4.25

a. $f'(x) = \frac{d}{dx} \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) = \sum_{k=1}^{\infty} k \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x).$

b. $f'(x) = e^x$ as well.

c. The series converges on $(-\infty, \infty)$.

11.4.26

a. $f'(x) = \frac{d}{dx} \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \right) = \sum_{k=1}^{\infty} (-1)^k (2k) \frac{x^{2k-1}}{(2k)!} = \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k-1}}{(2k-1)!} = - \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$

b. $f'(x) = -\sin x.$

c. The series converges on $(-\infty, \infty)$, because the series for $\cos x$ does.

11.4.27

a. $f'(x) = \frac{d}{dx} (\ln(1 + x)) = \frac{d}{dx} \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} x^k \right) = \sum_{k=1}^{\infty} (-1)^{k+1} x^{k-1} = \sum_{k=0}^{\infty} (-1)^k x^k.$

b. This is the power series for $\frac{1}{1+x}.$

c. The Taylor series for $\ln(1 + x)$ converges on $(-1, 1)$, as does the Taylor series for $\frac{1}{1+x}.$

11.4.28

$$\begin{aligned} \text{a. } f'(x) &= \frac{d}{dx}(\sin x^2) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}\right) = \sum_{k=0}^{\infty} (-1)^k \cdot 2(2k+1) \frac{x^{4k+1}}{(2k+1)!} = 2 \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{(2k)!} = \\ &= 2x \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!}. \end{aligned}$$

b. This is the power series for $2x \cos x^2$.

c. Because the Taylor series for $\sin x^2$ converges everywhere, the Taylor series for $2x \cos x^2$ does as well.

11.4.29

a.

$$f'(x) = \frac{d}{dx}(e^{-2x}) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-2x)^k}{k!}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-2)^k \frac{x^k}{k!}\right) = -2 \sum_{k=1}^{\infty} (-2)^{k-1} \frac{x^{k-1}}{(k-1)!} = -2 \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!}.$$

b. This is the Taylor series for $-2e^{-2x}$.

c. Because the Taylor series for e^{-2x} converges on $(-\infty, \infty)$, so does this one.

11.4.30

a. We have

$$f'(x) = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \left(\sum_{k=0}^{\infty} x^k \right) = \frac{d}{dx} \left(1 + \sum_{k=1}^{\infty} x^k \right) = \sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (k+1)x^k.$$

b. From the formula for $(1+x)^p$ in the text, we see that the Taylor series for $\frac{1}{(1-x)^2}$ is

$$\sum_{k=0}^{\infty} \frac{(-2)(-3) \cdots (-2-k+1)}{k!} (-x)^k = \sum_{k=0}^{\infty} (-1)^k (-1)^k \frac{(k+1)!}{k!} x^k = \sum_{k=0}^{\infty} (k+1)x^k,$$

so that $f'(x)$ is simply $\frac{1}{(1-x)^2}$ as expected.

c. Since the Taylor series for $\frac{1}{1-x}$ converges on $(-1, 1)$, so does the series for $\frac{1}{(1-x)^2}$. Checking the endpoints, we see that the series diverges at both endpoints by the Divergence test, so that the interval of convergence for $f'(x)$ is also $(-1, 1)$.

11.4.31

$$\text{a. } \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \text{ so } \frac{d}{dx} \tan^{-1} x^2 = 1 - x^2 + x^4 - x^6 + \cdots.$$

b. This is the series for $\frac{1}{1+x^2}$.

c. Because the series for $\tan^{-1} x$ has a radius of convergence of 1, this series does too. Checking the endpoints shows that the interval of convergence is $(-1, 1)$.

11.4.32

$$\text{a. } -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots, \text{ so } \frac{d}{dx} [-\ln(1-x)] = 1 + x + x^2 + x^3 + \cdots.$$

b. This is the series for $\frac{1}{1-x}$.

c. The interval of convergence for $\frac{1}{1-x}$ is $(-1, 1)$.

11.4.33

a. Because $y(0) = 2$, we have $0 = y'(0) - y(0) = y'(0) - 2$ so that $y'(0) = 2$. Differentiating the equation gives $y''(0) = y'(0)$, so that $y''(0) = 2$. Successive derivatives also have the value 2 at 0, so the Taylor series is $2 \sum_{k=0}^{\infty} \frac{t^k}{k!}$.

b. $2 \sum_{k=0}^{\infty} \frac{t^k}{k!} = 2e^t$.

11.4.34

a. Because $y(0) = 0$, we see that $y'(0) = 8$. Differentiating the equation gives $y''(0) + 4y'(0) = 0$, so $y''(0) + 4 \cdot 8 = 0$, $y''(0) = -4 \cdot 8$. Continuing, $y'''(0) + 4 \cdot (-4 \cdot 8) = 0$, so $y'''(0) = 4 \cdot 4 \cdot 8$, and in general $y^{(k)}(0) = (-1)^{k+1} 2 \cdot 4^k$ for $k \geq 1$, so the Taylor series is $2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(4t)^k}{k!}$.

b. $2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(4t)^k}{k!} = 2(1 - e^{-4t})$.

11.4.35

a. $y(0) = 2$, so that $y'(0) = 16$. Differentiating, $y''(t) - 3y'(t) = 0$, so that $y''(0) = 48$, and in general $y^{(k)}(0) = 3y^{(k-1)}(0) = 3^{k-1} \cdot 16$. Thus the power series is $2 + \frac{16}{3} \sum_{k=1}^{\infty} \frac{(3t)^k}{k!} = 2 + \sum_{k=1}^{\infty} \frac{3^{k-1} 16}{k!} t^k$.

b. $2 + \frac{16}{3} \sum_{k=1}^{\infty} \frac{(3t)^k}{k!} = 2 + \frac{16}{3} (e^{3t} - 1) = \frac{16}{3} e^{3t} - \frac{10}{3}$.

11.4.36

a. $y(0) = 2$, so $y'(0) = 12 + 9 = 21$. Differentiating, $y^{(n)}(0) = 6y^{(n-1)}(0)$ for $n > 1$, so that $y^{(n)}(0) = 6^{n-1} \cdot 21$ for $n \geq 1$. Thus the power series is $2 + \sum_{k=1}^{\infty} 21 \cdot 6^{k-1} \frac{t^k}{k!} = 2 + \frac{7}{2} \sum_{k=1}^{\infty} \frac{(6t)^k}{k!}$.

b. $2 + \frac{7}{2} \sum_{k=1}^{\infty} \frac{(6t)^k}{k!} = 2 + \frac{7}{2} (e^{6t} - 1) = \frac{7}{2} e^{6t} - \frac{3}{2}$.

11.4.37 The Taylor series for e^{-x^2} is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$. Thus, the desired integral is

$$\int_0^{0.25} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!} dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)k!} \Big|_0^{0.25} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)k! 4^{2k+1}}.$$

Because this is an alternating series, to approximate it to within 10^{-4} , we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{1}{(2n+3)(n+1)! \cdot 4^{2n+3}} < 10^{-4}$. This occurs for $n = 1$, so $\sum_{k=0}^1 (-1)^k \frac{1}{(2k+1) \cdot k! \cdot 4^{2k+1}} = \frac{1}{4} - \frac{1}{192} \approx 0.245$.

11.4.38 The Taylor series for $\sin x^2$ is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}$. Thus the desired integral is

$$\int_0^{0.2} \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!} dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+3}}{(4k+3)(2k+1)!} \Big|_0^{0.2} = \sum_{k=0}^{\infty} (-1)^k \frac{0.2^{4k+3}}{(4k+3)(2k+1)!}.$$

Because this is an alternating series, to approximate it to within 10^{-4} , we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{0.2^{4n+7}}{(4n+7)(2n+3)!} < 10^{-4}$. This occurs first for $n = 0$, so we obtain $\frac{0.2^3}{3 \cdot 1!} \approx 2.67 \times 10^{-3}$.

11.4.39 The Taylor series for $\cos 2x^2$ is $\sum_{k=0}^{\infty} (-1)^k \frac{(2x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{4^k x^{4k}}{(2k)!}$. Note that $\cos x$ is an even function, so we compute the integral from 0 to 0.35 and double it:

$$2 \int_0^{0.35} \sum_{k=0}^{\infty} (-1)^k \frac{4^k x^{4k}}{(2k)!} dx = 2 \left(\sum_{k=0}^{\infty} (-1)^k \frac{4^k x^{4k+1}}{(4k+1)(2k)!} \right) \Big|_0^{0.35} = 2 \left(\sum_{k=0}^{\infty} (-1)^k \frac{4^k (0.35)^{4k+1}}{(4k+1)(2k)!} \right).$$

Because this is an alternating series, to approximate it to within $\frac{1}{2} \cdot 10^{-4}$, we must find n such that $a_{n+1} < \frac{1}{2} \cdot 10^{-4}$, or $\frac{4^{n+1} (0.35)^{4n+5}}{(4n+3)(2n+2)!} < \frac{1}{2} \cdot 10^{-4}$. This occurs first for $n = 1$, and we have $2 \left(0.35 - \frac{4 \cdot (0.35)^5}{5 \cdot 2!} \right) \approx 0.696$.

11.4.40 The Taylor series for $(1+x^4)^{1/2}$ is $\sum_{k=0}^{\infty} \binom{1/2}{k} x^{4k}$, so the desired integral is

$$\int_0^{0.2} \sum_{k=0}^{\infty} \binom{1/2}{k} x^{4k} dx = \sum_{k=0}^{\infty} \frac{1}{4k+1} \binom{1/2}{k} x^{4k+1} \Big|_0^{0.2} = \sum_{k=0}^{\infty} \frac{1}{4k+1} \binom{1/2}{k} (0.2)^{4k+1}.$$

This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4} , we must find n such that $a_{n+1} < 10^{-4}$, or $\left| \frac{1}{4n+5} \binom{1/2}{n+1} (0.2)^{4n+5} \right| < 10^{-4}$. This happens first for $n = 0$, so the approximation is $\binom{1/2}{0} \cdot 0.2 = 0.2$.

11.4.41 $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \dots$, so $\int \tan^{-1} x dx = \int (x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \dots) dx = C + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \dots$. Thus,

$$\int_0^{0.35} \tan^{-1} x dx = \frac{(0.35)^2}{2} - \frac{(0.35)^4}{12} + \frac{(0.35)^6}{30} - \frac{(0.35)^8}{56} + \dots$$

Note that this series is alternating, and $\frac{(0.35)^6}{30} < 10^{-4}$, so we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.060.

11.4.42 $\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots$, so $\int \ln(1+x^2) dx = \int (x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots) dx = C + \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} - \frac{x^9}{36} + \frac{x^{11}}{55} + \dots$. Thus,

$$\int_0^{0.4} \ln(1+x^2) dx = \frac{(0.4)^3}{3} - \frac{(0.4)^5}{10} + \frac{(0.4)^7}{21} - \frac{(0.4)^9}{36} + \dots$$

Because $\frac{(0.4)^7}{21} < 10^{-4}$, we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.020.

11.4.43 The Taylor series for $(1+x^6)^{-1/2}$ is $\sum_{k=0}^{\infty} \binom{-1/2}{k} x^{6k}$, so the desired integral is

$$\int_0^{0.5} \sum_{k=0}^{\infty} \binom{-1/2}{k} x^{6k} dx = \sum_{k=0}^{\infty} \frac{1}{6k+1} \binom{-1/2}{k} x^{6k+1} \Big|_0^{0.5} = \sum_{k=0}^{\infty} \frac{1}{6k+1} \binom{-1/2}{k} (0.5)^{6k+1}.$$

This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4} , we must find n such that $a_{n+1} < 10^{-4}$, or $\left| \frac{1}{6n+7} \binom{-1/2}{n+1} (0.5)^{6n+7} \right| < 10^{-4}$. This occurs first for $n = 1$, so we have $\binom{-1/2}{0} 0.5 + \frac{1}{7} \binom{-1/2}{1} (0.5)^7 \approx 0.499$.

11.4.44 The Taylor series for $\frac{\ln(1+t)}{t}$ centered at 0 is $\sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k+1}$. The desired integral is thus

$$\int_0^{0.2} \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k+1} dt = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{(k+1)^2} \Big|_0^{0.2} = \sum_{k=0}^{\infty} (-1)^k \frac{(0.2)^{k+1}}{(k+1)^2}.$$

This is an alternating series, so to approximate it to within 10^{-4} , we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{(0.2)^{n+2}}{(n+2)^2} < 10^{-4}$. This occurs first

for $n = 3$, so we have $\sum_{k=0}^3 (-1)^k \frac{(0.2)^{k+1}}{(k+1)^2} \approx 0.191$.

11.4.45 Use the Taylor series for e^x at 0: $1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}$.

11.4.46 Use the Taylor series for e^x at 0: $1 + \frac{1/2}{1!} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{8 \cdot 3!}$.

11.4.47 Use the Taylor series for $\cos x$ at 0: $1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!}$.

11.4.48 Use the Taylor series for $\sin x$ at 0: $1 - \frac{1^3}{3!} + \frac{1^5}{5!} - \frac{1^7}{7!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$.

11.4.49 Use the Taylor series for $\ln(1+x)$ evaluated at $x = 1/2$: $\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{8} - \frac{1}{4} \cdot \frac{1}{16}$.

11.4.50 Use the Taylor series for $\tan^{-1} x$ evaluated at $1/2$: $\frac{1}{2} - \frac{1}{3} \cdot \frac{1}{8} + \frac{1}{5} \cdot \frac{1}{32} - \frac{1}{7} \cdot \frac{1}{128}$.

11.4.51 The Taylor series for f centered at 0 is $\frac{-1 + \sum_{k=0}^{\infty} \frac{x^k}{k!}}{x} = \frac{\sum_{k=1}^{\infty} \frac{x^k}{k!}}{x} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$.

Evaluating both sides at $x = 1$, we have $e - 1 = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}$.

11.4.52 The Taylor series for f centered at 0 is

$$\frac{-1 + \sum_{k=0}^{\infty} \frac{x^k}{k!}}{x} = \frac{\sum_{k=1}^{\infty} \frac{x^k}{k!}}{x} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}.$$

Differentiating, the Taylor series for $f'(x)$ is $f'(x) = \frac{(x-1)e^x + 1}{x^2} = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{(k+1)!}$. Evaluating both sides at

2 gives. $\frac{e^2 + 1}{4} = \sum_{k=1}^{\infty} \frac{k \cdot 2^{k-1}}{(k+1)!}$.

11.4.53 The Maclaurin series for $\ln(1+x)$ is

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}.$$

By the Ratio Test, $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{x^{k+1}k}{x^k(k+1)} \right| = |x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1 , so the interval of convergence is $(-1, 1]$. Evaluating at 1 gives $\ln 2 = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

11.4.54 The Taylor series for $\ln(1+x)$ at 0 is $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$. By the Ratio

Test, $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{x^{k+1}k}{x^k(k+1)} \right| = |x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1 , so the interval of convergence is $(-1, 1]$. Evaluate both sides at $-1/2$ to get

$$f\left(-\frac{1}{2}\right) = \ln(1/2) = -\ln 2 = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(-1/2)^k}{k} = -\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}.$$

so that $\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}$.

$$\mathbf{11.4.55} \quad \sum_{k=0}^{\infty} \frac{x^k}{2^k} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2-x}.$$

$$\mathbf{11.4.56} \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{-x}{3}\right)^k = \frac{1}{1 + \frac{x}{3}} = \frac{3}{3+x}.$$

$$\mathbf{11.4.57} \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k} = \sum_{k=0}^{\infty} \left(\frac{-x^2}{4}\right)^k = \frac{1}{1 + \frac{x^2}{4}} = \frac{4}{4+x^2}.$$

$$\mathbf{11.4.58} \quad \sum_{k=0}^{\infty} 2^k x^{2k+1} = x \sum_{k=0}^{\infty} (2x^2)^k = \frac{x}{1-2x^2}.$$

$$\mathbf{11.4.59} \quad \ln(1+x) = -\sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k}, \text{ so } \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ and finally } -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}.$$

$$\mathbf{11.4.60} \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{4^k} = -4 \sum_{k=0}^{\infty} \left(\frac{-x}{4}\right)^{k+1} = -4(-1 + \sum_{k=0}^{\infty} \left(\frac{-x}{4}\right)^k) = 4 - \frac{4}{1 + \frac{x}{4}} = 4 - \frac{16}{4+x} = \frac{4x}{4+x}$$

11.4.61

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^k \frac{kx^{k+1}}{3^k} &= \sum_{k=1}^{\infty} (-1)^k \frac{k}{3^k} x^{k+1} = \sum_{k=1}^{\infty} k \left(-\frac{1}{3}\right)^k x^{k+1} \\ &= x^2 \sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k kx^{k-1} = x^2 \sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k \frac{d}{dx}(x^k) \\ &= x^2 \frac{d}{dx} \left(\sum_{k=1}^{\infty} \left(-\frac{x}{3}\right)^k \right) = x^2 \frac{d}{dx} \left(\frac{1}{1 + \frac{x}{3}} \right) = -\frac{3x^2}{(x+3)^2}. \end{aligned}$$

11.4.62 By Exercise 53, $\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$, so $\sum_{k=1}^{\infty} \frac{x^{2k}}{k} = \sum_{k=1}^{\infty} \frac{(x^2)^k}{k} = -\ln(1-x^2)$.

$$\begin{aligned} \mathbf{11.4.63} \quad \sum_{k=2}^{\infty} \frac{k(k-1)x^k}{3^k} &= x^2 \sum_{k=2}^{\infty} \frac{k(k-1)x^{k-2}}{3^k} = x^2 \frac{d^2}{dx^2} \left(\sum_{k=2}^{\infty} \frac{x^k}{3^k} \right) \\ &= x^2 \frac{d^2}{dx^2} \left(\sum_{k=2}^{\infty} \left(\frac{x}{3} \right)^k \right) = x^2 \frac{d^2}{dx^2} \left(\frac{x^2}{9} \cdot \frac{1}{1 - \frac{x}{3}} \right) = x^2 \frac{d^2}{dx^2} \left(\frac{x^2}{9-3x} \right) = x^2 \frac{-6}{(x-3)^3} = \frac{-6x^2}{(x-3)^3}. \end{aligned}$$

$$\mathbf{11.4.64} \quad \sum_{k=2}^{\infty} \frac{x^k}{k(k-1)} = \sum_{k=2}^{\infty} \frac{x^k}{k-1} - \sum_{k=2}^{\infty} \frac{x^k}{k} = x \sum_{k=1}^{\infty} \frac{x^k}{k} - \sum_{k=1}^{\infty} \frac{x^k}{k} + x, = -x \ln(1-x) + \ln(1-x) + x = x + (1-x) \ln(1-x).$$

11.4.65

- a. False. This is because $\frac{1}{1-x}$ is not continuous at 1, which is in the interval of integration.
- b. False. The Ratio Test shows that the radius of convergence for the Taylor series for $\tan^{-1} x$ centered at 0 is 1.
- c. True. $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$. Substitute $x = \ln 2$.

11.4.66 The Taylor series for e^{ax} centered at 0 is

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots.$$

We compute that

$$\begin{aligned} \frac{e^{ax} - 1}{x} &= \frac{1}{x} \left(\left(1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots \right) - 1 \right) \\ &= \frac{1}{x} \left(ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots \right) = a + \frac{a^2 x}{2} + \frac{a^3 x^2}{6} + \cdots \end{aligned}$$

so the limit of $\frac{e^{ax} - 1}{x}$ as $x \rightarrow 0$ is a .

11.4.67 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots.$$

We compute that

$$\begin{aligned} \frac{\sin ax}{\sin bx} &= \frac{ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots}{bx - \frac{(bx)^3}{6} + \frac{(bx)^5}{120} - \cdots} \\ &= \frac{a - \frac{a^3 x^2}{6} + \frac{a^5 x^4}{120} - \cdots}{b - \frac{b^3 x^2}{6} + \frac{b^5 x^4}{120} - \cdots} \end{aligned}$$

so the limit of $\frac{\sin ax}{\sin bx}$ as $x \rightarrow 0$ is $\frac{a}{b}$.

11.4.68 The Taylor series for $\sin ax$ centered at 0 is

$$\sin ax = ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \dots$$

and the Taylor series for $\tan^{-1} ax$ centered at 0 is

$$\tan^{-1} ax = ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \dots$$

We compute that

$$\begin{aligned} \frac{\sin ax - \tan^{-1} ax}{bx^3} &= \frac{1}{bx^3} \left(\left(ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \dots \right) - \left(ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \dots \right) \right) \\ &= \frac{1}{bx^3} \left(\frac{(ax)^3}{6} - \frac{23(ax)^5}{120} + \dots \right) = \frac{a^3}{6b} - \frac{23a^5}{120b}x^2 + \dots \end{aligned}$$

so the limit of $\frac{\sin ax - \tan^{-1} ax}{bx^3}$ as $x \rightarrow 0$ is $\frac{a^3}{6b}$.

11.4.69 Compute instead the limit of the log of this expression, $\lim_{x \rightarrow 0} \frac{\ln(\sin x/x)}{x^2}$. If the Taylor expansion

of $\ln(\sin x/x)$ is $\sum_{k=0}^{\infty} c_k x^k$, then $\lim_{x \rightarrow 0} \frac{\ln(\sin x/x)}{x^2} = \lim_{x \rightarrow 0} \sum_{k=0}^{\infty} c_k x^{k-2} = \lim_{x \rightarrow 0} c_0 x^{-2} + c_1 x^{-1} + c_2$, because the higher-order terms have positive powers of x and thus approach zero as x does. So compute the terms of the Taylor series of $\ln\left(\frac{\sin x}{x}\right)$ up through the quadratic term. The relevant Taylor series are: $\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \dots$, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$ and we substitute the Taylor series for $\frac{\sin x}{x} - 1$ for x in the Taylor series for $\ln(1+x)$. Because the lowest power of x in the first Taylor series is 2, it follows that only the linear term in the series for $\ln(1+x)$ will give any powers of x that are at most quadratic. The only term that results is $-\frac{1}{6}x^2$. Thus $c_0 = c_1 = 0$ in the above, and $c_2 = -\frac{1}{6}$, so that $\lim_{x \rightarrow 0} \frac{\ln(\sin x/x)}{x^2} = -\frac{1}{6}$

and thus $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

11.4.70 We can find the Taylor series for $\ln(x + \sqrt{1+x^2})$ by substituting into $\ln(1+t)$ the Taylor series for $x + \sqrt{x^2+1} - 1$. The Taylor series in question are: $x + \sqrt{x^2+1} - 1 = x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \dots$, $\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 - \frac{1}{6}t^6 + \frac{1}{7}t^7 - \dots$. Substituting the former into the latter and simplifying (not a simple task!), we obtain $\ln(x + \sqrt{x^2+1}) = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \dots$. Using the second definition, start with the Taylor series for $(1+t^2)^{-1/2}$, which is $1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \dots$, and integrate it: $\int_0^x \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \dots\right) dt = \left(t - \frac{1}{6}t^3 + \frac{3}{40}t^5 - \frac{5}{112}t^7 + \dots\right) \Big|_0^x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \dots$

11.4.71 The Taylor series we need are $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$, and $e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \dots$. We are looking for powers of x^3 and x^4 that occur when the first series is substituted for t in the second series. Clearly there will be no odd powers of x , because $\cos x$ has only even powers. Thus the coefficient of x^3 is zero, so that $f^{(3)}(0) = 0$. The coefficient of x^4 comes from the expansion of $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ in each term of e^t . Higher powers of x clearly cannot contribute to the coefficient of x^4 . Thus consider $\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)^k$. The term $-\frac{1}{2}x^2$ generates $\binom{k}{2}$ terms of value $\frac{1}{4}x^4$ for $k \geq 2$, while the other term generates k terms of value

$\frac{1}{24}x^4$ for $k \geq 1$. These terms all have to be divided by the $k!$ appearing in the series for e^t . So the total coefficient of x^4 is $\frac{1}{24} \sum_{k=1}^{\infty} \frac{k}{k!} + \frac{1}{4} \sum_{k=2}^{\infty} \binom{k}{2} \frac{1}{k!} = \frac{1}{24} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} + \frac{1}{4} \sum_{k=2}^{\infty} \frac{1}{2 \cdot (k-2)!} = \frac{1}{24} \sum_{k=0}^{\infty} \frac{1}{k!} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{k!}$
 $= \frac{1}{24}e + \frac{1}{8}e = \frac{e}{6}$ Thus $f^{(4)}(0) = \frac{e}{6} \cdot 4! = 4e$.

11.4.72 The Taylor series for $(1+x)^{-1/3}$ is $(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 - \dots$, so we want the coefficients of x^3 and x^4 in $(x^2+1) \left(1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4\right)$. The coefficient of x^3 is $-\frac{1}{3} - \frac{14}{81} = -\frac{41}{81}$, and the coefficient of x^4 is $\frac{2}{9} + \frac{35}{243} = \frac{89}{243}$. Thus $f^{(3)}(0) = 6 \cdot \frac{-41}{81} = \frac{-82}{27}$, and $f^{(4)}(0) = 24 \cdot \frac{89}{243} = \frac{712}{81}$.

11.4.73 The Taylor series for $\sin t^2$ is $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \dots$, so that $\int_0^x \sin t^2 dt = \frac{1}{3}t^3 - \frac{1}{7 \cdot 3!}t^7 + \dots \Big|_0^x = \frac{1}{3}x^3 - \frac{1}{7 \cdot 3!}x^7 + \dots$. Thus $f^{(3)}(0) = \frac{3!}{3} = 2$ and $f^{(4)}(0) = 0$.

11.4.74 $\frac{1}{1+t^4} = 1 - t^4 + t^8 - \dots$, so that $\int_0^x \frac{1}{1+t^4} dt = t - \frac{1}{5}t^5 + \frac{1}{9}t^9 - \dots \Big|_0^x = x - \frac{1}{5}x^5 + \dots$ so that both $f^{(3)}(0)$ and $f^{(4)}(0)$ are zero.

11.4.75 Consider the series $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{x} \sum_{k=0}^{\infty} kx^k$ so that $\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$. Evaluate both sides at $x = 1/2$ to see that the sum of the series is $\frac{1/2}{(1-1/2)^2} = 2$. Thus the expected number of tosses is 2.

11.4.76

a. $\sum_{k=0}^{\infty} \frac{1}{6} \left(\frac{5}{6}\right)^{2k} = \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{25}{36}\right)^k = \frac{1}{6} \cdot \frac{1}{1-25/36} = \frac{6}{11}$.

b. Consider the series $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$. Evaluating at $x = 5/6$ and multiplying the result by $1/6$, we get $\frac{1}{6} \cdot \frac{1}{(1-5/6)^2} = 6$.

11.4.77 We look first for a Taylor series for $(1-k^2 \sin^2 \theta)^{-1/2}$. Because $(1-k^2 x^2)^{-1/2} = (1-(kx)^2)^{-1/2} = \sum_{i=0}^{\infty} \binom{-1/2}{i} (kx)^{2i}$, and $\sin \theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots$, substituting the second series into the first gives

$$\begin{aligned} \frac{1}{\sqrt{1-k^2 \sin^2 \theta}} &= 1 + \frac{1}{2}k^2\theta^2 + \left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\theta^4 \\ &+ \left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\theta^6 + \left(\frac{-1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\theta^8 + \dots \end{aligned}$$

Integrating with respect to θ and evaluating at $\pi/2$ (the value of the antiderivative is 0 at 0) gives

$$\begin{aligned} \frac{1}{2}\pi + \frac{1}{48}k^2\pi^3 + \frac{1}{160}\left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\pi^5 + \frac{1}{896}\left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\pi^7 \\ + \frac{1}{4608}\left(-\frac{1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\pi^9. \end{aligned}$$

Evaluating these terms for $k = 0.1$ gives $F(0.1) \approx 1.574749680$. The true value is approximately 1.574745562.

11.4.78

- a. $\frac{\sin t}{t} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k+1)!} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots$
- b. $\int_0^x \frac{\sin t}{t} dt = \sum_{k=0}^{\infty} \int_0^x (-1)^k \frac{t^{2k}}{(2k+1)!} dt = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)(2k+1)!}$
- c. This is an alternating series, so we want n such that $a_{n+1} < 10^{-3}$, or $\frac{0.5^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$ (resp. $\frac{1^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$), which gives $n = 1$ (resp. $n = 2$). Thus $\text{Si}(0.5) \approx \frac{0.5}{1} - \frac{0.5^3}{3 \cdot 3!} \approx 0.4930555556$, $\text{Si}(1.0) \approx 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} \approx 0.9461111111$.

11.4.79

- a. By the Fundamental Theorem, $S'(x) = \sin x^2$, $C'(x) = \cos x^2$.
- b. The relevant Taylor series are $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \frac{1}{7!}t^{14} + \dots$, and $\cos t^2 = 1 - \frac{1}{2!}t^4 + \frac{1}{4!}t^8 - \frac{1}{6!}t^{12} + \dots$. Integrating, we have $S(x) = \frac{1}{3}x^3 - \frac{1}{7 \cdot 3!}x^7 + \frac{1}{11 \cdot 5!}x^{11} - \frac{1}{15 \cdot 7!}x^{15} + \dots$, and $C(x) = x - \frac{1}{5 \cdot 2!}x^5 + \frac{1}{9 \cdot 4!}x^9 - \frac{1}{13 \cdot 6!}x^{13} + \dots$
- c. $S(0.05) \approx \frac{1}{3}(0.05)^3 - \frac{1}{42}(0.05)^7 + \frac{1}{1320}(0.05)^{11} - \frac{1}{75600}(0.05)^{15} \approx 4.166664807 \times 10^{-5}$. $C(-0.25) \approx (-0.25) - \frac{1}{10}(-0.25)^5 + \frac{1}{216}(-0.25)^9 - \frac{1}{9360}(-0.25)^{13} \approx -0.2499023616$.
- d. The series is alternating. Because $a_{n+1} = \frac{1}{(4n+7)(2n+3)!}(0.05)^{4n+7}$, and this is less than 10^{-4} for $n = 0$, only one term is required.
- e. The series is alternating. Because $a_{n+1} = \frac{1}{(4n+5)(2n+2)!}(0.25)^{4n+5}$, and this is less than 10^{-6} for $n = 1$, two terms are required.

11.4.80

- a. $\frac{d}{dx} \text{erf}(x) = \frac{2}{\sqrt{\pi}}(e^{-x^2})$.
- b. $e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{k!}$, so that the Maclaurin series for the error function is $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \right)$.
- c. $\text{erf}(0.15) \approx \frac{2}{\sqrt{\pi}} \left(0.15 - \frac{0.15^3}{3} + \frac{0.15^5}{10} - \frac{0.15^7}{42} \right) \approx 0.1679959712$.
 $\text{erf}(-0.09) \approx \frac{2}{\sqrt{\pi}} \left(-0.09 + \frac{0.09^3}{3} - \frac{0.09^5}{10} + \frac{0.09^7}{42} \right) \approx -0.1012805939$.
- d. The first omitted term in each case is $\frac{x^9}{9 \cdot 5!} = \frac{x^9}{1080}$. For $x = 0.15$, this is $\approx 3.56 \times 10^{-11}$. For $x = -0.09$, this is (in absolute value) $\approx 3.59 \times 10^{-13}$.

11.4.81

- a. $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{16 \cdot 2!^2}x^4 - \frac{1}{2^6 \cdot 3!^2}x^6 + \dots$
- b. Using the Ratio Test: $\left| \frac{a_{k+1}}{a_k} \right| = \frac{x^{2k+2}}{2^{2k+2}((k+1)!)^2} \cdot \frac{2^{2k}(k!)^2}{x^{2k}} = \frac{x^2}{4(k+1)^2}$, which has limit 0 as $k \rightarrow \infty$ for any x . Thus the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.
- c. Starting only with terms up through x^8 , we have $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 + \dots$, $J'_0(x) = -\frac{1}{2}x + \frac{1}{16}x^3 - \frac{1}{384}x^5 + \frac{1}{18432}x^7 + \dots$, $J''_0(x) = -\frac{1}{2} + \frac{3}{16}x^2 - \frac{5}{384}x^4 + \frac{7}{18432}x^6 + \dots$ so that $x^2 J_0(x) = x^2 - \frac{1}{4}x^4 + \frac{1}{64}x^6 - \frac{1}{2304}x^8 + \frac{1}{147456}x^{10} + \dots$, $x J'_0(x) = -\frac{1}{2}x^2 + \frac{1}{16}x^4 - \frac{1}{384}x^6 + \frac{1}{18432}x^8 + \dots$, $x^2 J''_0(x) = -\frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{5}{384}x^6 + \frac{7}{18432}x^8 + \dots$, and $x^2 J''_0(x) + x J'_0(x) + x^2 J_0(x) = 0$.

11.4.82

- a. Clearly $x = \sin s$ because BE , of length x , is the side opposite the angle measured by s in a right triangle with unit length hypotenuse.
- b. In the formula $\frac{1}{2}r^2\theta$ for the formula for the area of a circular sector, we have $r = 1$, and $\theta = s$, so that the area is in fact $\frac{s}{2}$. But the area can also be expressed as an integral as follows: the area of the sector is the area under the circle between P and F (i.e. the area of the region $PAEF$), minus the area of the right triangle PEF . The area of the right triangle is $\frac{1}{2}x\sqrt{1-x^2}$ by the Pythagorean theorem and the formula for the area of a triangle. Equating these two formulae for the area of the sector, we have $\frac{s}{2} = \int_0^x \sqrt{1-t^2} dt - \frac{1}{2}x\sqrt{1-x^2}$, so $s = 2 \int_0^x \sqrt{1-t^2} dt - x\sqrt{1-x^2}$.
- c. The Taylor series for $\sqrt{1-t^2}$ is $1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{16}t^6 - \frac{5}{128}t^8 - \dots$. Integrating and evaluating at x we have $s = \sin^{-1} x = 2 \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \frac{5}{1152}x^9 \right) - x \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 \right) + \dots = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots$.
- d. Suppose $x = \sin s = a_0 + a_1 s + a_2 s^2 + \dots$. Then $x = \sin(\sin^{-1}(x)) = a_0 + a_1 \left(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots \right) + a_2 \left(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots \right)^2 + \dots$. Equating coefficients yields $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{6}$, and so on.

11.4.83

- a. Because $f(a) = g(a) = 0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots}{g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \dots} \\ &= \lim_{x \rightarrow a} \frac{f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots}{g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \dots} \\ &= \lim_{x \rightarrow a} \frac{f'(a) + \frac{1}{2}f''(a)(x-a) + \dots}{g'(a) + \frac{1}{2}g''(a)(x-a) + \dots} = \frac{f'(a)}{g'(a)}. \end{aligned}$$

Because $f'(x)$ and $g'(x)$ are assumed to be continuous at a and $g'(a) \neq 0$,

$$\frac{f'(a)}{g'(a)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

and we have that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

which is one form of L'Hôpital's Rule.

- b. Because $f(a) = g(a) = f'(a) = g'(a) = 0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots}{g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \frac{1}{6}g'''(a)(x-a)^3 + \cdots} \\ &= \lim_{x \rightarrow a} \frac{\frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots}{\frac{1}{2}g''(a)(x-a)^2 + \frac{1}{6}g'''(a)(x-a)^3 + \cdots} \\ &= \lim_{x \rightarrow a} \frac{\frac{1}{2}f''(a) + \frac{1}{6}f'''(a)(x-a) + \cdots}{\frac{1}{2}g''(a) + \frac{1}{6}g'''(a)(x-a) + \cdots} = \frac{f''(a)}{g''(a)}. \end{aligned}$$

Because $f''(x)$ and $g''(x)$ are assumed to be continuous at a and $g''(a) \neq 0$,

$$\frac{f''(a)}{g''(a)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$$

and we have that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$$

which is consistent with two applications of L'Hôpital's Rule.

11.4.84

- The power series for $\cos x$ has only even powers of x , so that the power series has the same value evaluated at $-x$ as it does at x .
- The power series for $\sin x$ has only odd powers of x , so that evaluating it at $-x$ gives the opposite of its value at x .

Chapter Eleven Review

1

- True. The approximations tend to get better as n increases in size, and also when the value being approximated is closer to the center of the series. Because 2.1 is closer to 2 than 2.2 is, and because $3 > 2$, we should have $|p_3(2.1) - f(2.1)| < |p_2(2.2) - f(2.2)|$.
- False. The interval of convergence may or may not include the endpoints.
- True. The interval of convergence is an interval centered at 0, and the endpoints may or may not be included.
- True. Because $f(x)$ is a polynomial, all its derivatives vanish after a certain point (in this case, $f^{(12)}(x)$ is the last nonzero derivative).
- True.

2 $p_3(x) = 2x - \frac{(2x)^3}{3!}.$

3 $p_2(x) = 1 - \frac{3}{2}x^2.$

$$4 \quad p_2(x) = \frac{\pi}{3} - \frac{2\left(x - \frac{1}{2}\right)}{\sqrt{3}} - \frac{2\left(x - \frac{1}{2}\right)^2}{3\sqrt{3}}.$$

$$5 \quad p_2(x) = 1 - (x - 1) + \frac{3}{2}(x - 1)^2.$$

$$6 \quad p_2(x) = 1 + x + \frac{x^2}{2}.$$

$$7 \quad p_2(x) = 1 - \frac{1}{2}(x - 1)^2.$$

$$8 \quad \sinh x = x + \frac{x^3}{6} = \cdots, \text{ so for } \sinh(-3x) \text{ we have } p_3(x) = -3x - \frac{9}{2}x^3.$$

$$9 \quad p_3(x) = \frac{5}{4} + \frac{3(x - \ln 2)}{4} + \frac{5(x - \ln 2)^2}{8} + \frac{(x - \ln 2)^3}{8}.$$

10

a. $p_1(x) = 1$, and $p_2(x) = 1 - \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$ p_n(-0.08) - \cos(-0.08) $
1	1	3.2×10^{-3}
2	0.997	1.7×10^{-6}

11

a. $p_1(x) = 1 + x$, and $p_2(x) = 1 + x + \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$ p_n(-0.08) - e^{-0.08} $
1	0.92	3.1×10^{-3}
2	0.923	8.4×10^{-5}

12

a. $p_1(x) = 1 + \frac{1}{2}x$, and $p_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$.

b.

n	$p_n(0.08)$	$ p_n(0.08) - \sqrt{1 + 0.08} $
1	1.04	7.7×10^{-4}
2	1.039	3.0×10^{-5}

13

a. $p_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$, and $p_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$.

b.

n	$p_n(\pi/5)$	$ p_n(\pi/5) - \sin(\pi/5) $
1	0.596	8.2×10^{-3}
2	0.587	4.7×10^{-4}

14 The bound is $|R_n(x)| \leq M \frac{|x|^{n+1}}{(n+1)!}$, where M is a bound for $|e^x|$ (because e^x is its own derivative) on $[-1, 1]$. Thus take $M = 3$ so that $|R_3(x)| \leq \frac{3x^4}{4!} = \frac{x^4}{8}$. But $|x| < 1$, so this is at most $\frac{1}{8}$.

15 The derivatives of $\sin x$ are bounded in magnitude by 1, so $|R_n(x)| \leq M \frac{|x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!}$. But $|x| < \pi$, so $|R_3(x)| \leq \frac{\pi^4}{24}$.

16 The third derivative of $\ln(1-x)$ is $\frac{-2}{(x-1)^3}$, which is bounded in magnitude by 16 on $|x| < 1/2$ (at $x = 1/2$). Thus $|R_3(x)| \leq 16 \frac{|x|^4}{4!} \leq 16 \frac{1}{2^4 4!} = \frac{1}{4!}$.

17 Using the Ratio Test, $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2 x^{k+1}}{(k+1)!} \cdot \frac{k!}{k^2 x^k} \right| = \lim_{k \rightarrow \infty} \left(\frac{k+1}{k} \right)^2 \frac{|x|}{k+1} = 0$, so the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

18 Using the Ratio Test, $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{x^{4k+4}}{(k+1)^2} \cdot \frac{k^2}{x^{4k}} \right| = \lim_{k \rightarrow \infty} \left(\frac{k}{k+1} \right)^2 x^4 = x^4$, so that the radius of convergence is 1. Because $\sum \frac{1}{k^2}$ converges, the given power series converges at both endpoints, so its interval of convergence is $[-1, 1]$.

19 Using the Ratio Test, $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \left| \frac{(x+1)^{2k+2}}{(k+1)!} \cdot \frac{k!}{(x+1)^{2k}} \right| = \lim_{k \rightarrow \infty} \frac{1}{k+1} (x+1)^2 = 0$, so the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

20 Using the Ratio Test, $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(x-1)^{k+1}}{(k+1)5^{k+1}} \cdot \frac{k5^k}{(x-1)^k} \right| = \lim_{k \rightarrow \infty} \frac{k}{5k+5} |x-1| = \frac{1}{5}(|x-1|)$, so the series converges when $|1/5(x-1)| < 1$, or $-5 < x-1 < 5$, so that $-4 < x < 6$. The radius of convergence is 5. At $x = -4$, the series is the alternating harmonic series. At $x = 6$, it is the harmonic series, so the interval of convergence is $[-4, 6)$.

21 By the Root Test, $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \left(\frac{|x|}{9} \right)^3 = \frac{|x^3|}{729}$, so the series converges for $|x| < 9$. The radius of convergence is 9. The series given by letting $x = \pm 9$ are both divergent by the Divergence Test. Thus, $(-9, 9)$ is the interval of convergence.

22 By the Ratio Test, $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(x+2)^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{(x+2)^k} \right| = \lim_{k \rightarrow \infty} \sqrt{\frac{k}{k+1}} (|x+2|) = |x+2|$, so that the series converges for $|x+2| < 1$, so $-3 < x < -1$. The radius of convergence is 1. At $x = -3$, we have a series which converges by the Alternating Series Test. At $x = -1$, we have the divergent p -series with $p = 1/2$. Thus, $[-3, -1)$ is the interval of convergence.

23 By the Ratio Test, $\lim_{k \rightarrow \infty} \left| \frac{(x+2)^{k+1}}{2^{k+1} \ln(k+1)} \cdot \frac{2^k \ln k}{(x+2)^k} \right| = \lim_{k \rightarrow \infty} \frac{\ln k}{2 \ln(k+1)} |x+2| = \frac{|x+2|}{2}$. The radius of convergence is thus 2, and a check of the endpoints gives the divergent series $\sum \frac{1}{\ln k}$ at $x = 0$ and the convergent alternating series $\sum \frac{(-1)^k}{\ln k}$ at $x = -4$. The interval of convergence is therefore $[-4, 0)$.

24

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{(2k+2)!(x-2)^{k+1}}{(k+1)!}}{\frac{(2k)!(x-2)^k}{k!}} \right| = \lim_{k \rightarrow \infty} \frac{(2k+2)(2k+1)|x-2|}{k+1} = \begin{cases} 0 & \text{if } x = 2 \\ \infty & \text{if } x \neq 2 \end{cases}.$$

So the radius of convergence is 0 and the series only converges in the case that $x = 2$, which implies that the interval of convergence is $\{2\}$.

25

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{(-1)^{k+1}(2x+1)^{k+1}}{(k+1)^2 3^{k+1}}}{\frac{(-1)^k(2x+1)^k}{k^2 3^k}} \right| = \lim_{k \rightarrow \infty} \frac{k^2 |2x+1|}{(k+1)^2 3} = \frac{|2x+1|}{3}.$$

This is less than 1 for $-3 < 2x+1 < 3$, so for $-2 < x < 1$. So the radius of convergence is $\frac{3}{2}$, and the interval of convergence is $[-2, 1]$, because at the endpoints the series converges absolutely as the corresponding series of all positive terms is a convergent p -series.

26 We use the Ratio Test. $\lim_{k \rightarrow \infty} \left| \frac{x^{2k+3}}{2k+3} \cdot \frac{2k+1}{x^{2k+1}} \right| = x^2$. The radius of convergence is thus 1. At each endpoint we have a divergent series, so the interval of convergence is $(-1, 1)$.

27

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{(3k+3)!x^{k+1}}{((k+1)!)^3}}{\frac{(3k)!x^k}{(k!)^3}} \right| = \lim_{k \rightarrow \infty} \frac{(3k+3)(3k+2)(3k+1)|x|}{(k+1)^3} = 27|x|.$$

This is less than 1 for $|x| < \frac{1}{27}$. The radius of convergence is $R = \frac{1}{27}$.

28

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{2^{k+1}(k+1)!(x-5)^{k+1}}{(k+1)^{k+1}}}{\frac{2^k k!(x-5)^k}{k^k}} \right| = \lim_{k \rightarrow \infty} \frac{2(k+1)k^k |x-5|}{(k+1)^{(k+1)}} = \lim_{k \rightarrow \infty} \frac{2k^k |x-5|}{(k+1)^k} = \lim_{k \rightarrow \infty} \frac{2|x-5|}{(1+1/k)^k} = \frac{2|x-5|}{e}.$$

This is less than 1 for $|x-5| < \frac{e}{2}$, so the radius of convergence is $R = \frac{e}{2}$.

29 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty} x^{2k}$. By the Root Test, this converges for $|x^2| < 1$, so $-1 < x < 1$. It diverges at both endpoints, so the interval of convergence is $(-1, 1)$.

30 The Maclaurin series for $f(x)$ is determined by replacing x by $(-x)^3$ in the power series for $\frac{1}{1-x}$, so it is $\sum_{k=0}^{\infty} (-1)^k x^{3k}$. The radius of convergence is still 1. The series diverges at both endpoints, so the interval of convergence is $(-1, 1)$.

31 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty} (-5x)^k = \sum_{k=0}^{\infty} (-5)^k x^k$. By the Root Test, this has radius of convergence $1/5$. Checking the endpoints, we obtain an interval of convergence of $(-1/5, 1/5)$.

32 Replace x by $-x$ in the original power series, and multiply the result by $10x$, to get the Maclaurin series for $f(x)$, which is $\sum_{k=0}^{\infty} (-1)^k 10x^{k+1}$. By the Ratio Test, the radius of convergence is 1. Checking the endpoints, we obtain an interval of convergence of $(-1, 1)$.

33 Note that $\frac{1}{1-10x} = \sum_{k=0}^{\infty} (10x)^k$, so $\frac{1}{10} \cdot \frac{1}{1-10x} = \frac{1}{10} \sum_{k=0}^{\infty} (10x)^k$. Taking the derivative of $\frac{1}{10} \cdot \frac{1}{1-10x}$ gives $f(x)$. Thus, the Maclaurin series for $f(x)$ is $\frac{1}{10} \sum_{k=1}^{\infty} 10k(10x)^{k-1} = \sum_{k=1}^{\infty} k(10x)^{k-1}$. Using the Ratio Test, we see that the radius of convergence is $1/10$, and checking endpoints we obtain an interval of convergence of $(-1/10, 1/10)$.

34 Integrating $\frac{1}{1-x}$ and then replacing x by $4x$ gives $-f(x)$, so the series for $f(x)$ is $-\sum_{k=0}^{\infty} \frac{1}{k+1} (4x)^{k+1}$. The Ratio Test shows that the series has a radius of convergence of $1/4$; checking the endpoints, we obtain an interval of convergence of $[-1/4, 1/4)$.

35 The first three terms are $1 + 3x + \frac{9x^2}{2}$. The series is $\sum_{k=0}^{\infty} \frac{(3x)^k}{k!}$.

36 The first three terms are $\frac{1}{3} - \frac{2(x-1)}{9} + \frac{4}{27}(x-1)^2$. The series is $\sum_{k=0}^{\infty} \frac{(-1)^k 2^k (x-1)^k}{3^{k+1}}$.

37 The first three terms are $-(x - \pi/2) + \frac{1}{6}(x - \pi/2)^3 - \frac{1}{120}(x - \pi/2)^5$. The series is

$$\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{(2k+1)!} \left(x - \frac{\pi}{2}\right)^{2k+1}.$$

38 The first three terms for $\frac{1}{1+x}$ are $1 - x + x^2$, so the first three terms of $x^2 \cdot \frac{1}{1+x}$ are $x^2 - x^3 + x^4$. The series is $\sum_{k=0}^{\infty} (-1)^k x^{k+2}$.

39 The first three terms are $4x - \frac{1}{3}(4x)^3 + \frac{1}{5}(4x)^5$. The series is $\sum_{k=0}^{\infty} (-1)^k \frac{(4x)^{2k+1}}{2k+1}$.

40 The n th derivative of $f(x) = \sin(2x)$ is $\pm 2^n$ times either $\sin 2x$ or $\cos 2x$. Evaluated at $-\frac{\pi}{2}$, the even derivatives are therefore zero, and the $(2n+1)^{\text{st}}$ derivative is $(-1)^{n+1} 2^{2n+1}$. The Taylor series for $\sin 2x$ around $x = -\frac{\pi}{2}$ is thus $-2 \left(x + \frac{\pi}{2}\right) + \frac{2^3}{3!} \left(x + \frac{\pi}{2}\right)^3 - \frac{2^5}{5!} \left(x + \frac{\pi}{2}\right)^5 + \cdots$, and the general series is $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{2^{2k+1}}{(2k+1)!} \left(x + \frac{\pi}{2}\right)^{2k+1}$.

41 The n th derivative of $\cosh(2x-2)$ at $x=1$ is 0 if n is odd and is 2^n if n is even. The first 3 terms of the series are thus $1 + 2(x-1)^2 + \frac{2}{3}(x-1)^4$. The whole series can be written as $\sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} (x-1)^{2k}$.

42 $f(0) = \frac{1}{4}$, $f'(x) = \frac{-2x}{(x^2+4)^2}$, so $f'(0) = 0$. $f''(x) = \frac{6x^2-8}{(x^2+4)^3}$, so $f''(0) = -\frac{1}{8}$. $f'''(0) = 0$, and $f''''(0) = \frac{3}{8}$. The first three terms are $\frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64}$. The series is given by $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^{k+1}}$.

$$43 \quad f(x) = \binom{1/3}{0} + \binom{1/3}{1}x + \binom{1/3}{2}x^2 + \cdots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \cdots.$$

$$44 \quad f(x) = \binom{-1/2}{0} + \binom{-1/2}{1}x + \binom{-1/2}{2}x^2 + \cdots = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots.$$

$$45 \quad f(x) = \binom{-3}{0} + \binom{-3}{1}\frac{x}{2} + \binom{-3}{2}\frac{x^2}{4} + \cdots = 1 - \frac{3}{2}x + \frac{3}{2}x^2 + \cdots.$$

$$46 \quad f(x) = \binom{-5}{0} + \binom{-5}{1}(2x) + \binom{-5}{2}(2x)^2 + \cdots = 1 - 10x + 60x^2 + \cdots.$$

$$47 \quad \text{Note that } f(x) = \sinh x + \cosh x = e^x. \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c}{(n+1)!}x^{n+1} \text{ for some } c.$$

$$\lim_{n \rightarrow \infty} \frac{e^c}{(n+1)!}x^{n+1} = 0 \text{ for all } x.$$

$$48 \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \text{ for some } c \text{ in } (-1/2, 1/2). \text{ Now, } |f^{(n+1)}(c)| = \frac{n!}{(1+c)^{n+1}}, \text{ so } \lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} (2|x|)^{n+1} \cdot \frac{1}{n+1} \leq \lim_{n \rightarrow \infty} 1^{n+1} \frac{1}{n+1} = 0.$$

49 The Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots.$$

We compute that

$$\begin{aligned} \frac{x^2/2 - 1 + \cos x}{x^4} &= \frac{1}{x^4} \left(x^2/2 - 1 + \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \right) \\ &= \frac{1}{x^4} \left(\frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) = \frac{1}{24} - \frac{x^2}{720} + \cdots \end{aligned}$$

so the limit of $\frac{x^2/2 - 1 + \cos x}{x^4}$ as $x \rightarrow 0$ is $\frac{1}{24}$.

50 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

and the Taylor series for $\tan^{-1} x$ centered at 0 is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

We compute that

$$\begin{aligned} &\frac{2 \sin x - \tan^{-1} x - x}{2x^5} \\ &= \frac{1}{2x^5} \left(2 \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) - x \right) \\ &= \frac{1}{2x^5} \left(\frac{11x^5}{60} + \frac{359x^7}{2520} - \cdots \right) = -\frac{11}{120} + \frac{359x^2}{5040} - \cdots \end{aligned}$$

so the limit of $\frac{2 \sin x - \tan^{-1} x - x}{2x^5}$ as $x \rightarrow 0$ is $-\frac{11}{120}$.

51 The Taylor series for $\ln(x-3)$ centered at 4 is

$$\ln(x-3) = (x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3 - \cdots.$$

We compute that

$$\begin{aligned} \frac{\ln(x-3)}{x^2-16} &= \frac{1}{(x-4)(x+4)} \left((x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3 - \cdots \right) \\ &= \frac{1}{(x-4)(x+4)} \left((x-4) \left(1 - \frac{1}{2}(x-4) + \frac{1}{3}(x-4)^2 - \cdots \right) \right) \\ &= \frac{1}{x+4} \left(1 - \frac{1}{2}(x-4) + \frac{1}{3}(x-4)^2 - \cdots \right) \end{aligned}$$

so the limit of $\frac{\ln(x-3)}{x^2-16}$ as $x \rightarrow 4$ is $\frac{1}{8}$.

52 The Taylor series for $\sqrt{1+2x}$ centered at 0 is

$$\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \cdots.$$

We compute that

$$\begin{aligned} \frac{\sqrt{1+2x}-1-x}{x^2} &= \frac{1}{x^2} \left(\left(1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \cdots \right) - 1 - x \right) \\ &= \frac{1}{x^2} \left(-\frac{x^2}{2} + \frac{x^3}{2} - \cdots \right) = -\frac{1}{2} + \frac{x}{2} - \cdots \end{aligned}$$

so the limit of $\frac{\sqrt{1+2x}-1-x}{x^2}$ as $x \rightarrow 0$ is $-\frac{1}{2}$.

53 The Taylor series for $\sec x$ centered at 0 is

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

and the Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots.$$

We compute that

$$\begin{aligned} \frac{\sec x - \cos x - x^2}{x^4} &= \frac{1}{x^4} \left(\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots \right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) - x^2 \right) \\ &= \frac{1}{x^4} \left(\frac{x^4}{6} + \frac{31x^6}{360} + \cdots \right) = \frac{1}{6} + \frac{31x^2}{360} + \cdots \end{aligned}$$

so the limit of $\frac{\sec x - \cos x - x^2}{x^4}$ as $x \rightarrow 0$ is $\frac{1}{6}$.

54 The Taylor series for $(1+x)^{-2}$ centered at 0 is

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

and the Taylor series for $\sqrt[3]{1-6x}$ centered at 0 is

$$\sqrt[3]{1-6x} = 1 - 2x - 4x^2 - \frac{40x^3}{3} - \dots$$

We compute that

$$\begin{aligned} & \frac{(1+x)^{-2} - \sqrt[3]{1-6x}}{2x^2} \\ &= \frac{1}{2x^2} \left((1-2x+3x^2-4x^3+\dots) - \left(1-2x-4x^2-\frac{40x^3}{3}-\dots \right) \right) \\ &= \frac{1}{2x^2} \left(7x^2 + \frac{28x^3}{3} + \dots \right) = \frac{7}{2} + \frac{14x}{3} + \dots \end{aligned}$$

so the limit of $\frac{(1+x)^{-2} - \sqrt[3]{1-6x}}{2x^2}$ as $x \rightarrow 0$ is $\frac{7}{2}$.

55 We have $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \dots$, so $\int e^{-x^2} dx = \int (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \dots) dx = C + x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots$. Thus, $\int_0^{1/2} e^{-x^2} dx = (0.5) - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{10} - \frac{(0.5)^7}{42} + \dots$. Because $(0.5)^7/42 < 0.001$, we can calculate the approximation using the first three numbers shown, arriving at approximately 0.461.

56 $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \dots$, so $\int \tan^{-1}(x) dx = \int (x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \dots) dx = C + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \dots$. Thus, $\int_0^{0.5} \tan^{-1} x dx = \frac{(0.5)^2}{2} - \frac{(0.5)^4}{12} + \frac{(0.5)^6}{30} - \frac{(0.5)^8}{56} + \dots$. Note that this series is alternating, and $\frac{(0.5)^6}{30} < 0.001$, so we add the first two terms showing to approximate the integral to the desired accuracy. Calculating gives approximately 0.120.

57 $x \cos x = x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \dots$, so $\int x \cos x dx = \int (x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \dots) dx = C + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{144} - \frac{x^8}{5760} + \frac{x^{10}}{403200} - \dots$. Thus $\int_0^1 x \cos x dx = \frac{1}{2} - \frac{1}{8} + \frac{1}{144} - \frac{1}{5760} + \dots$. Because $\frac{1}{5760} < 0.001$, we add the first three terms to approximate to the desired accuracy. Calculating gives $\int_0^1 x \cos x dx \approx 0.382$.

58 $x^2 \tan^{-1} x = x^3 - x^5/3 + x^7/5 - x^9/7 + x^{11}/9 + \dots$, so $\int x^2 \tan^{-1}(x) dx = \int (x^3 - x^5/3 + x^7/5 - x^9/7 + x^{11}/9 - \dots) dx = C + \frac{x^4}{4} - \frac{x^6}{18} + \frac{x^8}{40} - \frac{x^{10}}{70} + \dots$. Thus, $\int_0^{0.5} x^2 \tan^{-1} x dx = \frac{(0.5)^4}{4} - \frac{(0.5)^6}{18} + \frac{(0.5)^8}{40} - \frac{(0.5)^{10}}{70} + \dots$. Note that this series is alternating, and $\frac{(0.5)^6}{18} < 0.001$, so we use the first term showing to approximate the integral to the desired accuracy. Calculating gives approximately 0.015.

59 The series for $f(x) = \sqrt{x}$ centered at $a = 121$ is $11 + \frac{x-121}{22} - \frac{(x-121)^2}{10648} + \frac{(x-121)^3}{2576816} + \dots$. Letting $x = 119$ gives $\sqrt{119} \approx 11 - \frac{1}{11} - \frac{1}{2 \cdot 11^3} - \frac{1}{2 \cdot 11^5}$.

60 Because 20 degrees corresponds to $\frac{\pi}{9}$ radians, we consider the series for $\sin x$ centered at 0. We have $\sin x \approx x - x^3/3! + x^5/5! - x^7/7! + \dots$, so $\sin \pi/9 \approx \frac{\pi}{9} - \frac{(\pi/9)^3}{3!} + \frac{(\pi/9)^5}{5!} - \frac{(\pi/9)^7}{7!}$.

61 $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 + \cdots$, so $\tan^{-1}(-1/3) \approx \frac{-1}{3} + \frac{1}{3 \cdot 3^3} - \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7}$.

62 $\sinh x = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$, so $\sinh(-1) \approx (-1) + \frac{(-1)^3}{6} + \frac{(-1)^5}{120} + \frac{(-1)^7}{5040}$.

63 Because $y(0) = 4$, we have $y'(0) - 16 + 12 = 0$, so $y'(0) = 4$. Differentiating the equation $n - 1$ times and evaluating at 0 we obtain $y^{(n)}(0) = 4y^{(n-1)}(0)$, so that $y^{(n)}(0) = 4^n$. The Taylor series for $y(x)$ is thus $y(x) = 4 + 4x + \frac{4^2 x^2}{2!} + \frac{4^3 x^3}{3!} + \cdots$, or $y(x) = 3 + e^{4x}$.

64 We begin with $e^{-102x^2} = 1 - 102x^2 + \frac{102^2 x^4}{2!} + \cdots$. For $n = 2$, we have $11.4 \int_0^{0.14} (1 - 102x^2) dx = 11.4(x - 34x^3) \Big|_0^{0.14} = 0.5324256$.

65

a. The Taylor series for $\ln(1+x)$ is $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$. Evaluating at $x = 1$ gives $\ln 2 = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$.

b. The Taylor series for $\ln(1-x)$ is $-\sum_{k=1}^{\infty} \frac{x^k}{k}$. Evaluating at $x = 1/2$ gives $\ln(1/2) = -\sum_{k=1}^{\infty} \frac{1}{k2^k}$, so that $\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k}$.

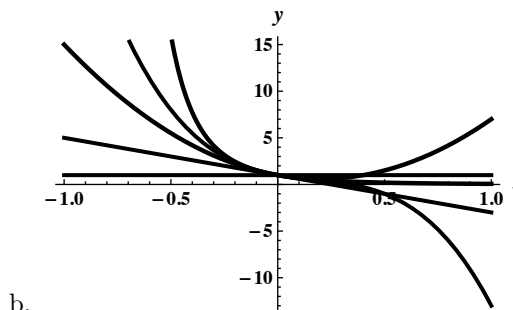
c. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$. Using the two Taylor series above we have $f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} - \left(-\sum_{k=1}^{\infty} \frac{x^k}{k}\right) = \sum_{k=1}^{\infty} (1 + (-1)^{k+1}) \frac{x^k}{k} = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$.

d. Because $\frac{1+x}{1-x} = 2$ when $x = \frac{1}{3}$, the resulting infinite series for $\ln 2$ is $2 \sum_{k=0}^{\infty} \frac{1}{3^{2k+1}(2k+1)}$.

e. The first four terms of each series are: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx 0.5833333333$, $\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} \approx 0.6822916667$, $\frac{2}{3} + \frac{2}{81} + \frac{2}{1215} + \frac{2}{15309} \approx 0.6931347573$. The true value is $\ln 2 \approx 0.6931471806$. The third series converges the fastest, because it has 3^{k+1} in the denominator as opposed to 2^k , so its terms get small faster.

66

a. $p_3(x) = 1 - 4x + 10x^2 - 20x^3$.



b.

c. The constant polynomial looks like $f(x)$ only at 0. The linear polynomial looks like $f(x)$ on about $(-0.1, 0.1)$. The quadratic approximation looks like $f(x)$ on about $(-0.1, 0.1)$ as well, and the cubic approximation looks like $f(x)$ on about $(-0.2, 0.2)$.

Chapter 12

Parametric and Polar Curves

12.1 Parametric Equations

12.1.1 Given an input value of t , the point $(x(t), y(t))$ can be plotted in the xy -plane, generating a curve.

12.1.2 $x = 6 \cos t$ and $y = 6 \sin t$ for $0 \leq t \leq 2\pi$ generates the circle, because $x^2 + y^2 = 36 \cos^2 t + 36 \sin^2 t = 36$. Similarly, $x = 6 \sin t$ and $y = 6 \cos t$ for $0 \leq t \leq 2\pi$ generates the same curve.

12.1.3 Let $x = R \cos(\pi t/5)$ and $y = -R \sin(\pi t/5)$. Note that as t ranges from 0 to 10, $\pi t/5$ ranges from 0 to 2π . Because $x^2 + y^2 = R^2$, this curve represents a circle of radius R . Note also that for $t = 0$ the initial point is $(R, 0)$, and for small values of t the plotted points are in the third quadrant — so the curve is being traced with clockwise orientation.

12.1.4 Let $x = t$ and $y = -2t + 5$ for $t \in (-\infty, \infty)$.

12.1.5 Let $x = t^2$ and $y = t$ for $t \in (-\infty, \infty)$.

12.1.6 Both pairs of parametric equations describe the same segment of the parabola $x = y^2$, but the positive orientations of the curves are in opposite directions. The first curve starts at $(0, 0)$ and ends at $(1, 1)$; the second curve starts at $(1, 1)$ and ends at $(0, 0)$.

12.1.7 $\left. \frac{d}{dx} \right|_{t=2} = \left(\frac{6t}{-6t^2} \right) \Big|_{t=2} = \left(\frac{-1}{t} \right) \Big|_{t=2} = -\frac{1}{2}.$

12.1.8 With $t = 0$ the corresponding point on the curve is $(-2 \sin 0, 2 \cos 0) = (0, 2)$. As t increases from 0, the x -coordinate becomes negative, while the y coordinate decreases. Thus the curve is generated counterclockwise, running successively through $(0, 2)$, then $(-2, 0)$ for $t = \frac{\pi}{2}$, then $(0, -2)$ for $t = \pi$, then $(2, 0)$ for $t = \frac{3\pi}{2}$, and finally back to $(0, 2)$ for $t = 2\pi$. Then it repeats.

12.1.9 $x = t, y = t, 0 \leq t \leq 6$; $x = 2t, y = 2t, 0 \leq t \leq 3$; $x = 3t, y = 3t, 0 \leq t \leq 2$. (Answers may vary).

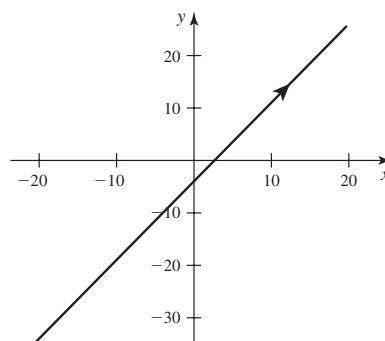
12.1.10 The arc length is given by $\int_0^1 \sqrt{25} dt = 5$. The distance between the endpoints $(1, 0)$ and $(4, 4)$ is $\sqrt{(4-1)^2 + (4-0)^2} = \sqrt{9+16} = 5$.

12.1.11

a.

t	x	y
-10	-20	-34
-5	-10	-19
0	0	-4
5	10	11
10	20	26

b.



c. Solving $x = 2t$ for t yields $t = x/2$, so $y = 3t - 4 = 3x/2 - 4$.

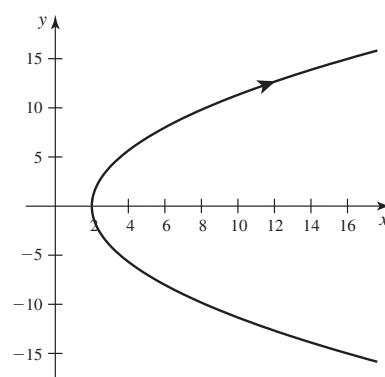
d. The curve is the line segment from $(-20, -34)$ to $(20, 26)$.

12.1.12

a.

t	x	y
-4	18	-16
-2	6	-8
0	2	0
2	6	8
4	18	16

b.



c. Solving $y = 4t$ for t yields $t = y/4$, so $x = t^2 + 2 = y^2/16 + 2$.

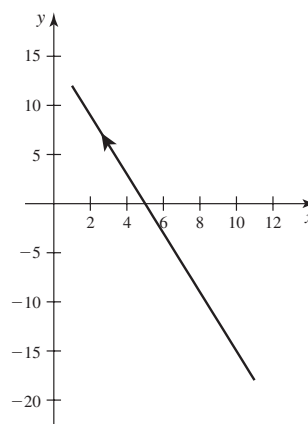
d. The curve is part of the parabola $x = y^2/16 + 2$ from $(18, -16)$ to $(18, 16)$.

12.1.13

a.

t	x	y
-5	11	-18
-3	9	-12
0	6	-3
3	3	6
5	1	12

b.



c. Solving $x = -t + 6$ for t yields $t = 6 - x$, so $y = 3t - 3 = 18 - 3x - 3 = 15 - 3x$.

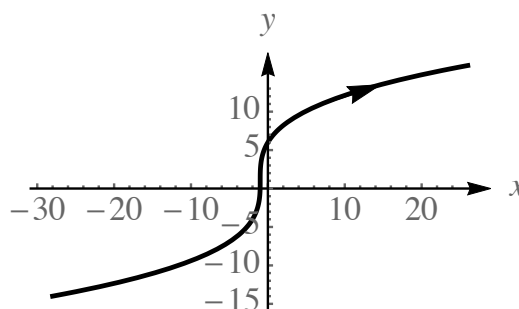
d. The curve is the line segment from $(11, -18)$ to $(1, 12)$.

12.1.14

a.

t	x	y
-3	-28	-14
-2	-9	-9
-1	-2	-4
0	-1	1
1	0	6
2	7	11
3	26	16

b.



c. Because $t = \sqrt[3]{x+1}$, we have $y = 5\sqrt[3]{x+1} + 1$.

d. The curve is a shifted and scaled version of the cube root function.

12.1.15

a. Because $t = x - 3$, we have $y = 1 - (x - 3) = 4 - x$.

b. This is a line segment starting at $(3, 1)$ and ending at $(4, 0)$.

12.1.16

a. Because $t = \frac{4-x}{3}$, we have $y = -2 + 6\left(\frac{4-x}{3}\right) = 6 - 2x$,

b. This is a line segment starting at $(7, -8)$ and ending at $(-5, 16)$.

12.1.17

a. Solving $x = \sqrt{t} + 4$ for t yields $t = (x - 4)^2$. Thus, $y = 3\sqrt{t} = 3(x - 4)$, where x ranges from 4 to 8. Note that all $t \geq 0$, $x > 0$, and $y > 0$.

- b. The curve is the line segment from $(4, 0)$ to $(8, 12)$.

12.1.18

- a. Solving $y = t + 2$ for t yields $t = y - 2$. Thus, $x = (t + 1)^2 = (y - 2 + 1)^2 = (y - 1)^2$, where $-8 \leq y \leq 12$.
- b. The curve is the part of the parabola $x = (y - 1)^2$ from $(81, -8)$ to $(121, 12)$.

12.1.19

- a. Note that $x^2 + y^2 = 9 \cos^2 t + 9 \sin^2 t = 9$.
- b. This represents an arc of the circle of radius 3 centered at the origin from $(-3, 0)$ to $(3, 0)$ traversed counterclockwise.

12.1.20

- a. Note that $x^2 + y^2 = 9 \cos^2 t + 9 \sin^2 t = 9$,
- b. This represents an arc of the circle of radius 3 centered at the origin from $(3, 0)$ to $(0, 3)$ traversed counterclockwise.

12.1.21

- a. Because $\cos^2 t + \sin^2 t = 1$, we have $x^2 + y = 1$, so $y = 1 - x^2$, $-1 \leq x \leq 1$.
- b. This is a parabola opening downward with a vertex at $(0, 1)$, and starting at $(1, 0)$ and ending at $(-1, 0)$.

12.1.22

- a. Note that $(1 - \sin^2 s) - \cos^2 s = 0$, so $x - y^2 = 0$, so $x = y^2$, $-1 \leq y \leq 1$.
- b. This is a parabola opening to the right with a vertex at $(0, 0)$, starting at $(1, -1)$ and ending at $(1, 1)$.

12.1.23

- a. Note that $x^2 + (y - 1)^2 = \cos^2 t + \sin^2 t = 1$,
- b. We have a circle of radius 1 centered at $(0, 1)$, traversed counterclockwise starting at $(1, 1)$.

12.1.24

- a. Replacing $\sin t$ with x in the second equation, we have $y = 2x + 1$.
- b. Because the sine function increases on $[0, \pi/2]$, x and y are both increasing on this interval also. When $t = 0$, $(x, y) = (0, 1)$ and when $t = \frac{\pi}{2}$, $(x, y) = (1, 3)$. So we have a line segment beginning at $(0, 1)$ and ending at $(1, 3)$.

12.1.25

- a. Solving $x = r - 1$ for r yields $r = x + 1$. Thus, $y = r^3 = (x + 1)^3$, where $-5 \leq x \leq 3$.
- b. The curve is the part of the standard cubic curve, shifted one unit to the left, from $(-5, -64)$ to $(3, 64)$.

12.1.26

- a. Solving $x = e^{2t}$ for t yields $t = \ln(\sqrt{x})$. Thus, $y = e^t + 1 = \sqrt{x} + 1$, where $1 \leq x \leq e^{50}$.
- b. The curve is the part of the standard square root function, shifted one unit vertically, from the point $(1, 2)$ to $(e^{50}, e^{25} + 1)$.

12.1.27

- a. Note that $x^2 + y^2 = 49 \cos^2 2t + 49 \sin^2 2t = 49$.
- b. This represents an arc of the circle of radius 7 centered at the origin from $(-7, 0)$ to $(-7, 0)$ traversed counterclockwise. (So the whole circle is represented.)

12.1.28

- a. Note that $(x - 1)^2 + (y - 2)^2 = 9 \sin^2 4\pi t + 9 \cos^2 4\pi t = 9$.
- b. This represents the circle of radius 3 centered at $(1, 2)$ from $(1, 5)$ to $(1, 5)$ traversed counterclockwise.

12.1.29

- a. $y = 1, -\infty < x < \infty$.
- b. Because $y = 1$, this is a horizontal line with slope 0 and y -intercept 1.

12.1.30

- a. $x = 5, -\infty < y < \infty$.
- b. This is a vertical line with x -intercept 5.

12.1.31 Note that $x^2 + y^2 = 4 \sin^2 8t + 4 \cos^2 8t = 4$, so the curve is the circle $x^2 + y^2 = 4$.

12.1.32 Note that $4x^2 + y^2 = 4 \sin^2 8t + 4 \cos^2 8t = 4$, so the curve is the ellipse $4x^2 + y^2 = 4$.

12.1.33 Note that because $t = x$, we have $y = \sqrt{4 - t^2} = \sqrt{4 - x^2}$.

12.1.34 Note that $x^2 = t + 1$, so $y = \frac{1}{t + 1} = \frac{1}{x^2}$.

12.1.35 Because $\sec^2 t - 1 = \tan^2 t$, we have $y = x^2$.

12.1.36 Note that $(\sqrt[n]{x/a})^2 + (\sqrt[n]{y/b})^2 = \sin^2 t + \cos^2 t$, so $(\sqrt[n]{x/a})^2 + (\sqrt[n]{y/b})^2 = 1$.

12.1.37 Let $x = 4 \cos t$ and $y = 4 \sin t$ for $0 \leq t \leq 2\pi$. Then $x^2 + y^2 = 16 \cos^2 t + 16 \sin^2 t = 16$.

12.1.38 Let $x = 12 \sin t$ and $y = 12 \cos t$ for $0 \leq t \leq 2\pi$. Then $x^2 + y^2 = 144 \cos^2 t + 144 \sin^2 t = 144$, and for $t = 0$ the value of (x, y) is $(0, 12)$.

12.1.39 Let $x = \cos t + 2$ and $y = \sin t + 3$ for $0 \leq t \leq 2\pi$. Then $(x - 2)^2 + (y - 3)^2 = 1$, which is a circle with the desired center and radius and orientation.

12.1.40 Let $x = -2 + 8 \sin t$ and $y = -3 + 8 \cos t$ for $0 \leq t \leq 2\pi$. Then $(x + 2)^2 + (y + 3)^2 = 64 \sin^2 t + 64 \cos^2 t = 64$.

12.1.41 Let $x = x_0 + at$ and $y = y_0 + bt$. Letting $(x_0, y_0) = (0, 0)$, and then finding a and b so that the curve is at the point Q when $t = 1$ yields $x = 2t, y = 8t$ for $0 \leq t \leq 1$.

12.1.42 Let $x = x_0 + at$ and $y = y_0 + bt$. Letting $(x_0, y_0) = (-1, -3)$, and then finding a and b so that the curve is at the point Q when $t = 1$ yields $x = -1 + 7t, y = -3 - 13t$ for $0 \leq t \leq 1$.

12.1.43 Let $x = t$ and $y = 2t^2 - 4, -1 \leq t \leq 5$.

12.1.44 Let $x = t^3 - 3t$ and $y = t, -\infty < t < \infty$.

12.1.45 Let $x = 2$

12.1.46 Let $x = x_0 + at$ and $y = y_0 + bt$, and parametrize from $t = 0$ to $t = 1$. Because the point is at $(8, 2)$ when $t = 0$, we have $x_0 = 8$ and $y_0 = 2$. At $t = 1$, the point is at $(-2, 2)$, so that $-2 = 8 + a$ and $2 = 2 + b$. Thus $a = -10$ and $b = 0$, and our equations are $x = 8 - 10t$ and $y = 2$ for $0 \leq t \leq 1$.

12.1.47 Let $x = -2 + 4t$ and $y = 3 - 6t$, $0 \leq t \leq 1$, and $x = t + 1$, $y = 8t - 11$ for $1 \leq t \leq 2$.

12.1.48 Let $x = -4 + 4t$ and $y = 4 + 4t$, $0 \leq t \leq 1$, and $x = t - 1$, $y = 8 - 2(t - 1)^2$ for $1 \leq t \leq 3$.

12.1.49 Let $x = 1 + 2t$ and $y = 1 + 4t$, for $-\infty < t < \infty$. Note that $y = 2(1 + 2t) - 1$, so $y = 2x - 1$.

12.1.50 Let $x = -t$ and $y = t^2 + 1$, for $0 \leq t < \infty$.

12.1.51 Let $x = t^2$ and $y = t$, for $0 \leq t < \infty$. Note that $x = t^2 = y^2$, and that the starting point is $(0, 0)$.

12.1.52 Let $x = -2 - 6 \cos t$ and $y = 2 - 6 \sin t$, for $0 \leq t \leq \pi$. Then $(x + 2)^2 + (y - 2)^2 = 36$, so the curve represented is part of the circle of radius 6 centered at $(-2, 2)$. Note also that as t runs from 0 to π , the portion of the circle traversed is the lower portion, from $(-8, 2)$ to $(4, 2)$.

12.1.53 Let t be time in minutes, so $0 \leq t \leq 1.5$. Let $x = 400 \cos(4\pi/3)t$ and $y = 400 \sin(4\pi/3)t$. Then because $x^2 + y^2 = 400^2$, the path is a circle of radius 400. Note that the values of x and y are the same at $t = 0$ and $t = 1.5$, and that the circle is traversed counterclockwise.

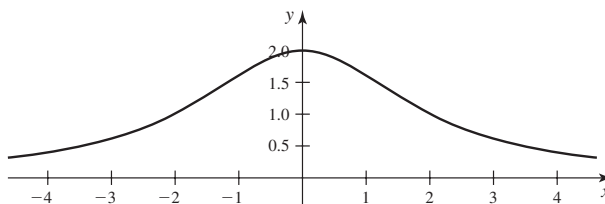
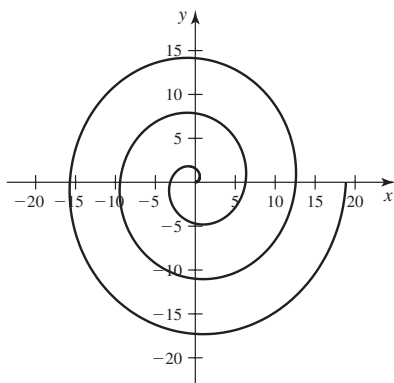
12.1.54 Let t be time in seconds, so $0 \leq t \leq 60$. Let $x = 15 \sin(\pi/30)t$ and $y = 15 \cos(\pi/30)t$. Then because $x^2 + y^2 = 15^2$, the path is a circle of radius 15. Note that the values of x and y are the same at $t = 0$ and $t = 60$, and that the circle is traversed clockwise.

12.1.55 Let t be time in seconds, so $0 \leq t \leq 24$. Let $x = 50 \cos(\pi/12)t$ and $y = 50 \sin(\pi/12)t$. Then because $x^2 + y^2 = 50^2$, the path is a circle of radius 50. Note that the values of x and y are the same at $t = 0$ and $t = 24$, and that the circle is traversed counterclockwise.

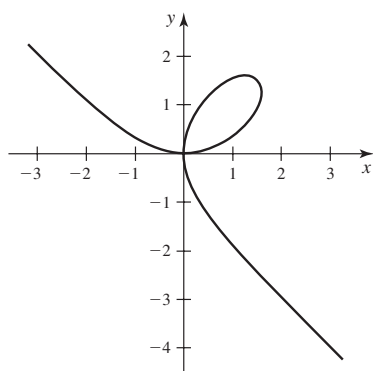
12.1.56 Let t be time in minutes, so $0 \leq t \leq 3$. Because the low point is the origin, the circle we seek has its center at $(0, 20)$ and a radius of 20. Let $x = -20 \sin(2\pi/3)t$ and $y = 20 - 20 \cos(2\pi/3)t$. Then because $x^2 + (y - 20)^2 = 20^2$, the path is a circle of radius 20. Note that the values of x and y are the same for $t = 0$ and $t = 3$.

12.1.57

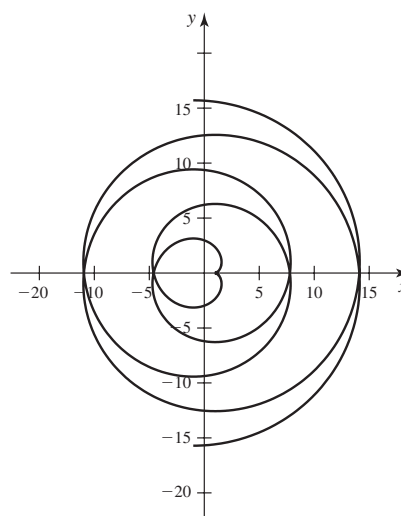
12.1.58



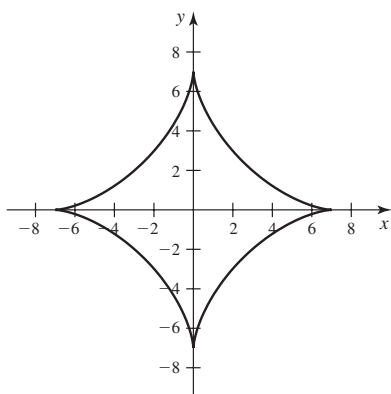
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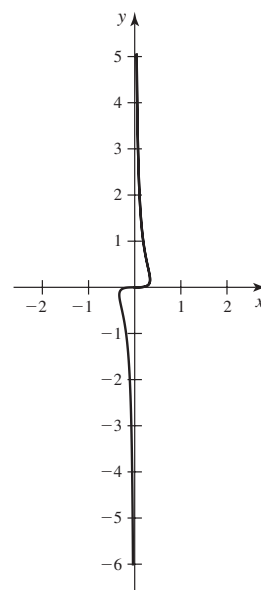
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12.1.61



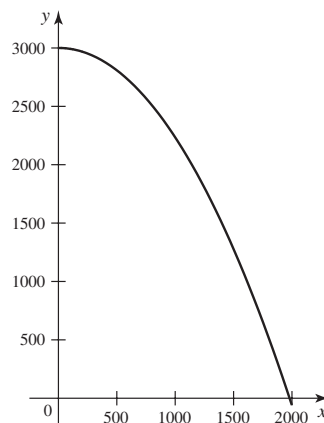
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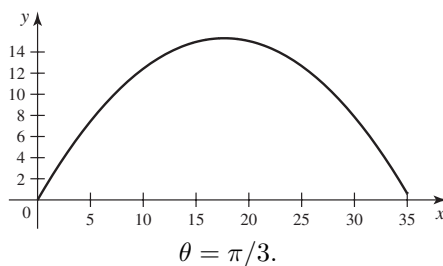
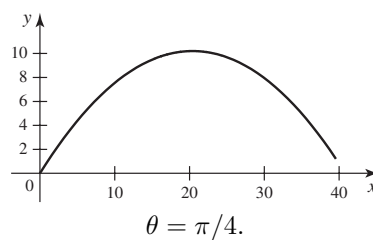
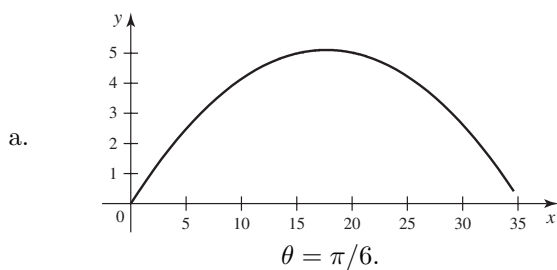
12.1.63 Let $x = 1 + \cos^2 t - \sin^2 t$ and $y = t$, for $-\infty < t < \infty$. Note that because $1 - \sin^2 t = \cos^2 t$, we have $x = 2 \cos^2 t$, $y = t$.

12.1.64

The packet lands when $y = 0$, so we seek a solution to $0 = -4.9t^2 + 3000$. So $t = \sqrt{\frac{3000}{4.9}} \approx 24.744$ seconds, at which point $x \approx 80 \cdot 24.744 \approx 1979.487$ meters.



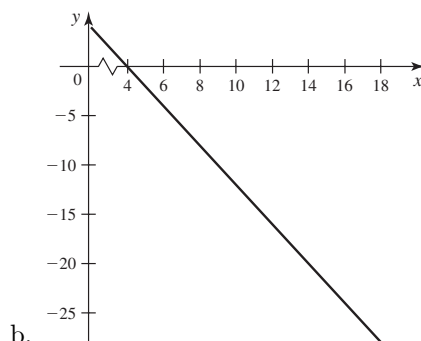
12.1.65 The packet lands when $y = 0$, so when $-4.9t^2 + 4000 = 0$ for $t > 0$. This occurs when $t = \sqrt{\frac{4000}{4.9}} \approx 28.571$ seconds. At that time, $x \approx 100 \cdot 28.57 = 2857$ meters.

12.1.66

b. The maximum appears to be reached when $\theta = \pi/4$.

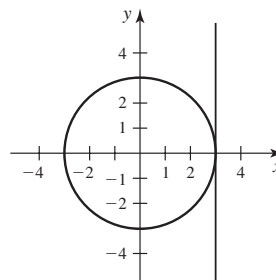
12.1.67

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{8}{4} = -2$ for all t . Because the curve is a line, the tangent line to the curve at the given point is the line itself.



12.1.68

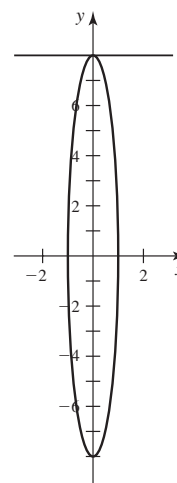
- a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{3 \sin t}{3 \cos t} = -\tan t$. At the given value of t , the value of $\frac{dy}{dx}$ doesn't exist, and the tangent line is the vertical line $x = 3$.



b.

12.1.69

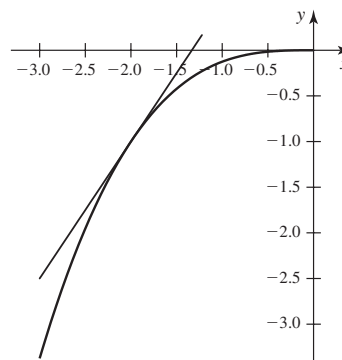
- a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8 \cos t}{-\sin t} = -8 \cot t$. At the given value of t , the value of $\frac{dy}{dx}$ is $-8 \cot \pi/2 = 0$. The tangent line at the point $(0, 8)$ is thus the horizontal line $y = 8$.



b.

12.1.70

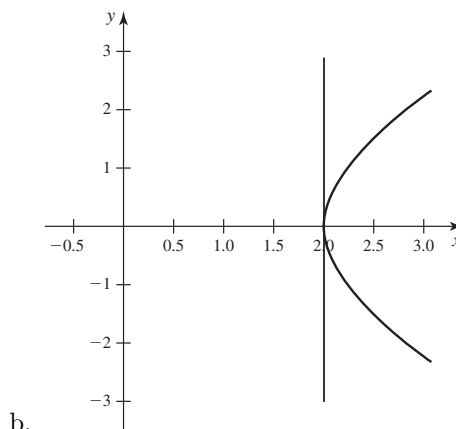
- a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2}$. At the given value of t , the value of $\frac{dy}{dx}$ is $\frac{3}{2}$, and the tangent line is $y = \frac{3}{2}x + 2$, tangent at the point $(-2, -1)$.



b.

12.1.71

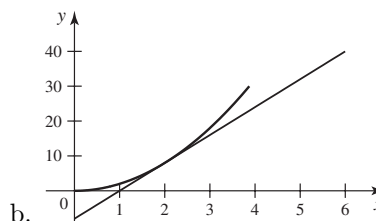
- a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$. At the given value of t , the derivative doesn't exist, and the tangent line is the vertical line $x = 2$, tangent at the point $(2, 0)$.



b.

12.1.72

- a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{1/(2\sqrt{t})} = 4\sqrt{t}$. At the given value of t , the value of $\frac{dy}{dx}$ is 8. The equation of the tangent line is $y = 8x - 8$, tangent at the point $(2, 8)$.



b.

- 12.1.73 The point corresponding to $t = 2$ is $(3, 10)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 1}{2t}$, so the slope at $t = 2$ is $\frac{13}{4}$. The equation of the tangent line is therefore $y - 10 = \frac{13}{4}(x - 3)$, or $y = \frac{13}{4}x + \frac{1}{4}$.

- 12.1.74 The point corresponding to $t = \pi/4$ is $(\sqrt{2}/2, \sqrt{2}/2)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t}$. At $t = \pi/4$, we have a slope of -1 . The equation of the tangent line is thus $y - \sqrt{2}/2 = -1(x - \sqrt{2}/2)$, or $y = -x + \sqrt{2}$.

- 12.1.75 The point corresponding to $t = \pi/4$ is

$$\left(\frac{4\sqrt{2} + \pi\sqrt{2}}{8}, \frac{4\sqrt{2} - \pi\sqrt{2}}{8} \right).$$

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t - (\cos t - t \sin t)}{-\sin t + (\sin t + t \cos t)} = \tan t$. At $t = \pi/4$, we have a slope of 1. The equation of the tangent line is thus $y - \frac{4\sqrt{2} - \pi\sqrt{2}}{8} = 1 \left(x - \frac{4\sqrt{2} + \pi\sqrt{2}}{8} \right)$, or $y = x - \frac{\pi\sqrt{2}}{4}$.

- 12.1.76 The point corresponding to $t = 0$ is $(1, 0)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{(t+1)e^t}$, so the slope at $t = 0$ is 1. The equation of the tangent line is thus $y = x - 1$.

- 12.1.77 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4 \cos t}{-4 \sin t} = -\cot t$. We seek t so that $\cot t = -1/2$, so $t = \cot^{-1}(-1/2)$. The corresponding points on the curve are $\left(-\frac{4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5} \right)$ and $\left(\frac{4\sqrt{5}}{5}, -\frac{8\sqrt{5}}{5} \right)$.

12.1.78 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8 \cos t}{-2 \sin t} = -4 \cot t$. We seek t so that $\cot t = 1/4$, so $t = \cot^{-1}(1/4)$. The corresponding points on the curve are $\left(\frac{2\sqrt{17}}{17}, \frac{32\sqrt{17}}{17}\right)$ and $\left(-\frac{2\sqrt{17}}{17}, -\frac{32\sqrt{17}}{17}\right)$.

12.1.79 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + (1/t^2)}{1 - (1/t^2)} = \frac{t^2 + 1}{t^2 - 1}$. We seek t so that $\frac{t^2 + 1}{t^2 - 1} = 1$, which never occurs. Thus, there are no points on this curve with slope 1.

12.1.80 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{4}{(1/2\sqrt{t})} = -8\sqrt{t}$ for $t \neq 0$. Note that this isn't 0 for t on the interval $(0, \infty)$, but it is the case that $\lim_{t \rightarrow 0^+} \frac{dy}{dx} = 0$, so there is a flat tangent line at the point $(2, 2)$, as long as the point is approached from the right.

12.1.81 The arc length is $\int_0^2 \sqrt{9 + 16t} dt = 5t \Big|_0^2 = 10$.

12.1.82 The arc length is $\int_0^{2\pi} \sqrt{9 \sin^2 t + 9 \cos^2 t} dt = 3t \Big|_0^{2\pi} = 6\pi$.

12.1.83 The arc length is

$$\int_0^\pi \sqrt{(-\sin t - \cos t)^2 + (-\sin t + \cos t)^2} dt = \int_0^\pi \sqrt{2 \sin^2 t + 2 \cos^2 t} dt = \sqrt{2}t \Big|_0^\pi = \pi\sqrt{2}.$$

12.1.84 The arc length is

$$\begin{aligned} \int_0^{2\pi} \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2} dt &= \int_0^{2\pi} \sqrt{2e^{2t} \sin^2 t + 2e^{2t} \cos^2 t} dt \\ &= \int_0^{2\pi} \sqrt{2}e^t dt = \sqrt{2}e^t \Big|_0^{2\pi} = \sqrt{2}(e^{2\pi} - 1). \end{aligned}$$

12.1.85 The arc length is

$$\int_0^1 \sqrt{(4t^3)^2 + (2t^5)^2} dt = \int_0^1 \sqrt{16t^6 + 4t^{10}} dt = \int_0^1 t^3 \sqrt{16 + 4t^4} dt = \frac{1}{16} \int_0^1 16t^3 \sqrt{16 + 4t^4} dt.$$

Let $u = 16 + t^4$ so that $du = 4t^3 dt$. Then we have

$$\frac{1}{16} \int_{16}^{20} u^{1/2} du = \frac{2}{48} u^{3/2} \Big|_{16}^{20} = \frac{1}{3} (5\sqrt{5} - 8).$$

12.1.86 The arc length is

$$\begin{aligned} \int_0^\pi \sqrt{(2 \sin t + t^2 \sin t)^2 + (2 \cos t + t^2 \cos t)^2} dt \\ &= \int_0^\pi \sqrt{4 + 4t^2 \sin^2 t + t^4 \sin^2 t + 4t^2 \cos^2 t + t^4 \cos^2 t} dt \\ &= \int_0^\pi \sqrt{4 + 4t^2 + t^4} dt = \int_0^\pi \sqrt{(t^2 + 2)^2} dt \\ &= \int_0^\pi (t^2 + 2) dt = \left(\frac{t^3}{3} + 2t\right) \Big|_0^\pi = \frac{\pi^3}{3} + 2\pi. \end{aligned}$$

12.1.87 The arc length is

$$\begin{aligned} \int_0^{\pi/4} \sqrt{((3 \cos^2 2t)(-2 \sin 2t))^2 + ((3 \sin^2 2t)(2 \cos 2t))^2} dt &= \int_0^{\pi/4} \sqrt{36 \cos^4 2t \sin^2 2t + 36 \sin^4 2t \cos^2 2t} dt \\ &= 6 \int_0^{\pi/4} \cos 2t \sin 2t \sqrt{\cos^2 2t + \sin^2 2t} dt = 6 \int_0^{\pi/4} \cos 2t \sin 2t dt = 3 \int_0^{\pi/4} \sin 4t dt \\ &= -\frac{3}{4} \cos 4t \Big|_0^{\pi/4} = -\frac{3}{4} (-1 - 1) = \frac{3}{2}. \end{aligned}$$

12.1.88 The arc length is

$$\begin{aligned} \int_0^{\pi/2} \sqrt{\cos^2 t + (1 + \sin t)^2} dt &= \sqrt{2} \int_0^{\pi/2} \sqrt{1 + \sin t} dt \\ &= \sqrt{2} \int_0^{\pi/2} \sqrt{1 + \sin t} \cdot \frac{\sqrt{1 - \sin t}}{\sqrt{1 - \sin t}} dt = \sqrt{2} \int_0^{\pi/2} \frac{\cos t}{\sqrt{1 - \sin t}} dt \\ &= -\sqrt{2} \int_1^0 u^{-1/2} du \end{aligned}$$

Then let $u = 1 - \sin t$ so that $du = -\cos t dt$. We have

$$\sqrt{2} \int_0^1 u^{-1/2} du = \sqrt{2} \lim_{b \rightarrow 0^+} \int_b^1 u^{-1/2} du = \sqrt{2} \lim_{b \rightarrow 0^+} (2 - 2\sqrt{b}) = 2\sqrt{2}.$$

12.1.89

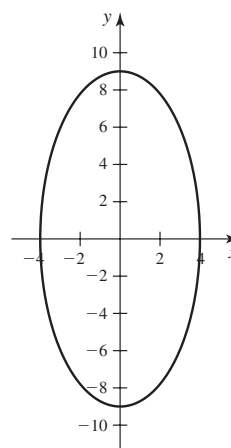
- False. This generates a circle in the counterclockwise direction.
- True. Note that when t is increased by one, the value of $2\pi t$ is increased by 2π , which is the period of both the sine and the cosine functions.
- False. This generates only the portion of the parabola in the first quadrant, omitting the portion in the second quadrant.
- True. They describe the portion of the unit circle in the 4th and 1st quadrants.
- True. This ellipse has vertical tangents at $t = 0$ and $t = \pi$.

12.1.90

- This corresponds to graph (D). Note that $t = 0$ corresponds to the point $(-2, 0)$ and as $t \rightarrow \infty$, both $x \rightarrow \infty$ and $y \rightarrow \infty$.
- This corresponds to graph (B). Note that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ for all values of t .
- This corresponds to graph (A). Note that as $t \rightarrow -\infty$, we have $x \rightarrow -\infty$ and $y \rightarrow -\infty$.
- This corresponds to graph (C). Note that $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$ for all values of t .

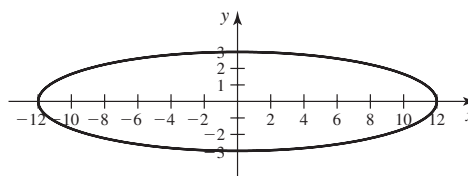
12.1.91

The entire curve is traversed for $0 \leq t \leq 2\pi$.



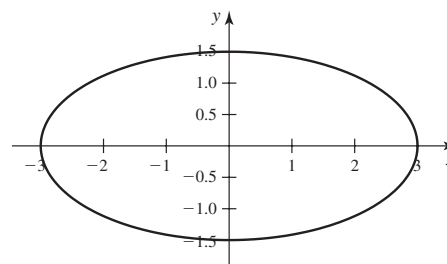
12.1.92

The entire curve is traversed for $0 \leq t \leq \pi$.



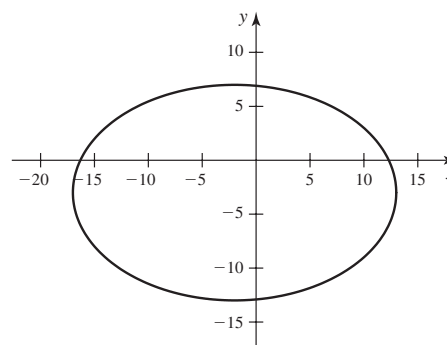
12.1.93

Let $x = 3 \cos t$ and $y = \frac{3}{2} \sin t$ for $0 \leq t \leq 2\pi$. Then the major axis on the x -axis has length $2 \cdot 3 = 6$ and the minor axis on the y -axis has length $2 \cdot \frac{3}{2} = 3$. Note that $\left(\frac{x}{3}\right)^2 + \left(\frac{2y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$.



12.1.94

Let $x = 15 \cos t - 2$ and $y = 10 \sin t - 3$ for $0 \leq t \leq 2\pi$. Note that $\left(\frac{x+2}{15}\right)^2 + \left(\frac{y+3}{10}\right)^2 = \cos^2 t + \sin^2 t = 1$. Then the major axis has length 30 and the minor axis has length 20.

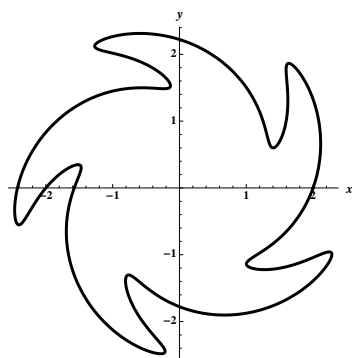


12.1.95

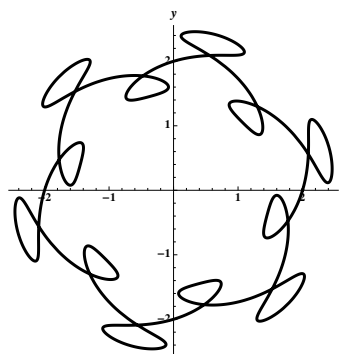
- a. For $(1 + s, 2s) = (1 + 2t, 3t)$, we must have $1 + s = 1 + 2t$ and $2s = 3t$, so that $s = 2t$ and $2s = 3t$. The only solution to this pair of equations is $s = t = 0$, so these two lines intersect when $s = t = 0$, at the point $(1, 0)$.
- b. For $(2 + 5s, 1 + s) = (4 + 10t, 3 + 2t)$, we must have $2 + 5s = 4 + 10t$ and $1 + s = 3 + 2t$, so that $s = 2 + 2t$ and $s = 2 + 2t$. This pair of equations has no solutions, so the lines are parallel.
- c. For $(1 + 3s, 4 + 2s) = (4 - 3t, 6 + 4t)$, we must have $1 + 3s = 4 - 3t$ and $4 + 2s = 6 + 4t$, so that $s = 1 - t$ and $s = 1 + 2t$. The only solution to this pair of equations is $s = 1$ and $t = 0$, so these two lines intersect for these values of s and t , at the point $(4, 6)$.

12.1.96 All three represent portions of the parabola $x = 2 \cdot (y - 4)^2$ where x is between 0 and 32. However, the curve in part **b** only represents the portion of the parabola where $y \geq 4$, because for that curve, $y = 4 + t^2 \geq 4$.

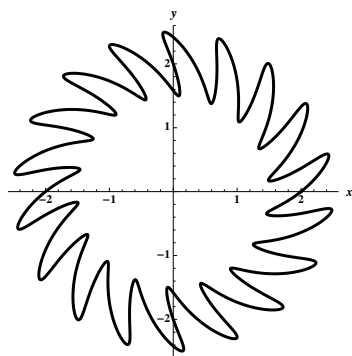
12.1.97



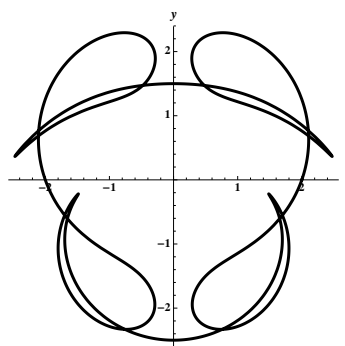
12.1.98



12.1.99

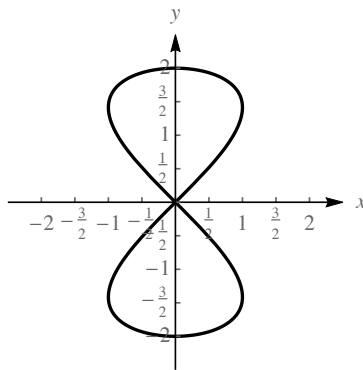


12.1.100

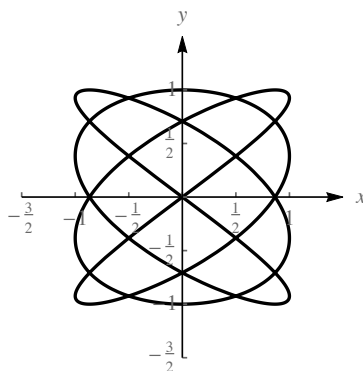


12.1.101

- a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{2 \cos 2t}$. This is zero when $\cos t = 0$ but $\cos 2t \neq 0$, which occurs for $t = \pi/2$ and $t = 3\pi/2$. The corresponding points on the graph are $(0, 2)$ and $(0, -2)$.
- b. Using the derivative obtained above, we seek points where $\cos 2t = 0$ but $\cos t \neq 0$. This occurs for $t = \pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$. The corresponding points on the curve are $(1, \sqrt{2})$, $(-1, \sqrt{2})$, $(-1, -\sqrt{2})$, and $(1, -\sqrt{2})$.

**12.1.102**

- a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3 \cos 3t}{4 \cos 4t}$. This is zero when $\cos 3t = 0$ but $\cos 4t \neq 0$, which occurs for $t = \pi/6, \pi/2, 5\pi/6, 7\pi/6, 3\pi/2$, and $t = 11\pi/6$. The corresponding points on the graph are the four points $(\pm \frac{\sqrt{3}}{2}, \pm 1)$ and the two points $(0, \pm 1)$.
- b. Using the derivative obtained above, we seek points where $\cos 4t = 0$ but $\cos 3t \neq 0$. This occurs for $t = \frac{(2n+1)\pi}{8}$, $n = 0, 1, \dots, 7$. The corresponding points on the curve are the four points $(\pm 1, \pm \sin(\pi/8))$ and $(\pm 1, \pm \sin(3\pi/8))$.



12.1.103 One arch is traced out by the parametric equations for $0 \leq t \leq 2\pi$. The base of the arch is in the interval $0 \leq x \leq 6\pi$. So the area is given by

$$\begin{aligned} \int_0^{6\pi} y \, dx &= 9 \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt = 9 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) \, dt \\ &= 9 \int_0^{2\pi} \left(1 - 2 \cos t + \frac{1}{2} + \frac{1}{2} \cos 2t \right) \, dt = 9 \left(\frac{3t}{2} - 2 \sin t + \frac{1}{4} \sin 2t \right) \Big|_0^{2\pi} = 27\pi. \end{aligned}$$

12.1.104 We find the area bounded by the ellipse in the first quadrant and multiply by 4. Let $f(t) = 3 \cos t$ and $g(t) = 4 \sin t$. Then we have $f'(t) = -3 \sin t$, $\alpha = 0 = f(\pi/2)$ and $\beta = 3 = f(0)$. It follows that the area is $4 \int_0^3 y \, dx = 4 \int_{\pi/2}^0 4 \sin t (-3 \sin t) \, dt$. Evaluating the integral, we have

$$4 \int_{\pi/2}^0 4 \sin t (-3 \sin t) \, dt = 48 \int_0^{\pi/2} \sin^2 t \, dt = 24 \int_0^{\pi/2} (1 - \cos 2t) \, dt = 24 \left(t - \frac{1}{2} \sin 2t \right) \Big|_0^{\pi/2} = 12\pi.$$

12.1.105 The area is given by

$$\begin{aligned}
 4 \int_{\pi/2}^0 \sin^3 t (3 \cos^2 t) (-\sin t) dt &= 12 \int_0^{\pi/2} \sin^4 t \cos^2 t dt = \frac{3}{2} \int_0^{\pi/2} (1 - \cos 2t)^2 (1 + \cos 2t) dt \\
 &= \frac{3}{2} \int_0^{\pi/2} (1 - \cos^2 2t)(1 - \cos 2t) dt \\
 &= \frac{3}{2} \int_0^{\pi/2} (1 - \cos 2t - \cos^2 2t + \cos^3 2t) dt \\
 &= \frac{3}{2} \int_0^{\pi/2} \left(1 - \cos 2t - \frac{1 + \cos 4t}{2} + (1 - \sin^2 2t)(\cos 2t) \right) dt \\
 &= \frac{3}{2} \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos 4t}{2} - \sin^2 2t \cos 2t \right) dt \\
 &= \frac{3}{2} \left(\frac{t}{2} - \frac{\sin 4t}{8} - \frac{\sin^3 2t}{6} \right) \Big|_0^{\pi/2} = \frac{3\pi}{8}.
 \end{aligned}$$

12.1.106 Revolving the parametric equations about the x -axis produces a sphere of radius 1. Therefore the area of this sphere is

$$\int_0^\pi 2\pi \sin t \sqrt{(-\sin t)^2 + (\cos t)^2} dt = 2\pi \int_0^\pi \sin t dt = 2\pi(-\cos t) \Big|_0^\pi = 2\pi(2) = 4\pi.$$

12.1.107

- Observe that $x^2 + (y - 4)^2 = 9 \cos^2 t + 9 \sin^2 t = 9$. So the curve is a circle of radius 3 centered at $(0, 4)$.
- Because the circle lies above the x -axis, the corresponding solid of revolution is a torus. The surface area is

$$\begin{aligned}
 S &= \int_0^{2\pi} 2\pi g(t) \sqrt{(f'(t))^2 + (g'(t))^2} dt = \int_0^{2\pi} 2\pi(3 \sin t + 4) \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} dt \\
 &= 6\pi \int_0^{2\pi} (3 \sin t + 4) dt = 6\pi(-3 \cos t + 4t) \Big|_0^{2\pi} = 6\pi(8\pi) = 48\pi^2.
 \end{aligned}$$

12.1.108 The area is given by

$$\begin{aligned}
 \int_0^{\pi/2} 2\pi \sin^3 t \sqrt{((3 \cos^2 t)(-\sin t))^2 + ((3 \sin^2 t)(\cos t))^2} dt &= \int_0^{\pi/2} 2\pi \sin^3 t \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} dt \\
 &= \int_0^{\pi/2} 6\pi \sin^3 t \sqrt{(\sin^2 t \cos^2 t)(\cos^2 t + \sin^2 t)} dt = 6\pi \int_0^{\pi/2} \sin^4 t \cos t dt = \frac{6}{5} \pi \sin^5 t \Big|_0^{\pi/2} = \frac{6\pi}{5}.
 \end{aligned}$$

12.1.109 The area is given by

$$\begin{aligned}
 \int_0^{2\pi} 2\pi(1 - \cos t) \sqrt{(1 - \cos t)^2 + \sin^2 t} dt &= \int_0^{2\pi} 2\pi(1 - \cos t) \sqrt{2 - 2 \cos t} dt \\
 &= 2\pi\sqrt{2} \int_0^{2\pi} (1 - \cos t)^{3/2} dt = 2\pi\sqrt{2} \int_0^{2\pi} \left(2 \sin^2 \frac{t}{2} \right)^{3/2} dt \\
 &= 8\pi \int_0^{2\pi} \sin^3 \frac{t}{2} dt = 8\pi \int_0^{2\pi} \left(1 - \cos^2 \frac{t}{2} \right) \left(\sin \frac{t}{2} \right) dt.
 \end{aligned}$$

Now let $u = \cos \frac{t}{2}$ so that $du = -\frac{1}{2} \sin \frac{t}{2} dt$. Substituting gives

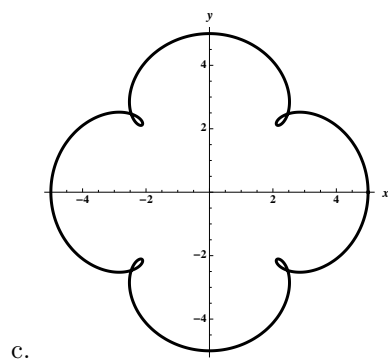
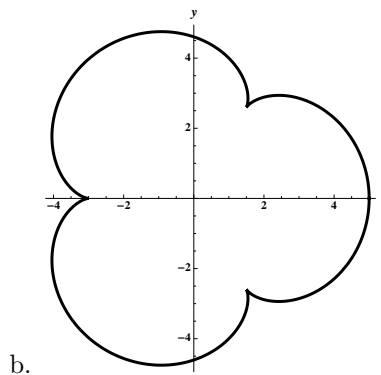
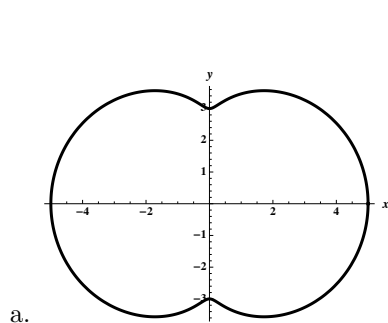
$$\begin{aligned}
 -16\pi \int_1^{-1} (1 - u^2) du &= 16\pi \left(u - \frac{u^3}{3} \right) \Big|_{-1}^1 \\
 &= 16\pi \cdot \frac{4}{3} = \frac{64\pi}{3}.
 \end{aligned}$$

12.1.110 The area is given by

$$\int_1^5 2\pi\sqrt{t+1}\sqrt{\frac{1}{4(t+1)}+1} dt = \int_1^5 \pi\sqrt{4t+5} dt = \frac{\pi}{6}(4t+5)^{3/2}\bigg|_1^5 = \frac{\pi}{6}(125-27) = \frac{49\pi}{3}.$$

12.1.111 The area is given by $\int_0^1 2\pi(e^{3t}+1)\sqrt{4e^{4t}+9e^{6t}} dt$ which can be evaluated with a CAS to give about 1445.9.

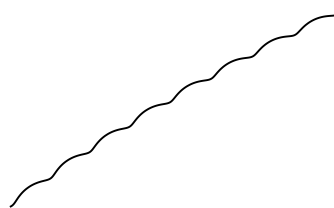
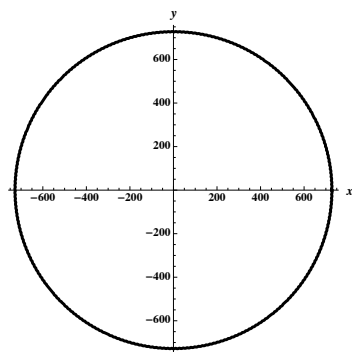
12.1.112



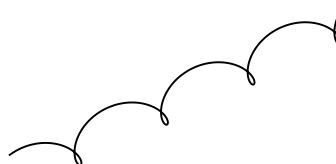
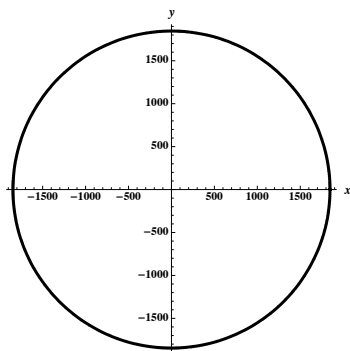
For a fixed a , there appear to be loops when $n > a$.

12.1.113

a.



b.



12.1.114 Note that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so $\frac{d^2y}{dt^2} = \frac{d}{dt} \left(\frac{dy}{dx} \cdot \frac{dx}{dt} \right) = \frac{dy}{dx} \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{d}{dt} \left(\frac{dy}{dx} \right)$.

Also $\frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \cdot \frac{dx}{dt}$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Thus,

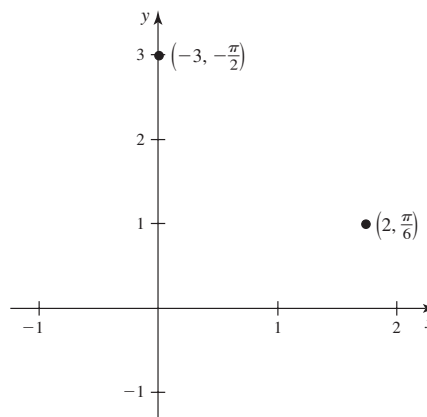
$$\frac{d^2y}{dt^2} = \frac{dy}{dx} \cdot \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dt} = \frac{dy/dt}{dx/dt} \cdot \frac{d^2x}{dt^2} + \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{dt} \right)^2.$$

Solving for $\frac{d^2y}{dx^2}$ yields $y'' = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t))^3} = \frac{f'(t)g''(t) - f''(t)g'(t)}{(f'(t))^3}$.

12.2 Polar Coordinates

12.2.1

The coordinates $(2, \pi/6)$, $(2, -11\pi/6)$, and $(-2, 7\pi/6)$ all give rise to the same point. Also, the coordinates $(-3, -\pi/2)$, $(3, \pi/2)$ and $(-3, 3\pi/2)$ give rise to the same point.



12.2.2 For a point with polar coordinates (r, θ) , we have the Cartesian coordinates $x = r \cos \theta$ and $y = r \sin \theta$.

12.2.3 If a point has Cartesian coordinates (x, y) then $r^2 = x^2 + y^2$ and $\tan \theta = y/x$ for $x \neq 0$. If $x = 0$, then $\theta = \pi/2$ and $r = y$.

12.2.4 $r = 2a \cos \theta + 2b \sin \theta$.

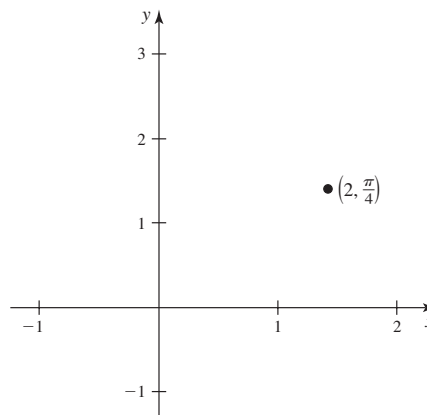
12.2.5 Because $x = r \cos \theta$, we have that the vertical line $x = 5$ has polar equation $r = 5 \sec \theta$.

12.2.6 Because $y = r \sin \theta$, the horizontal line $y = 5$ has polar equation $r = 5 \csc \theta$.

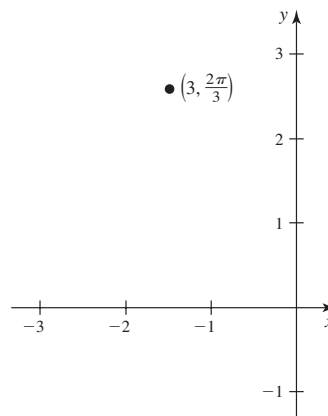
12.2.7 x -axis symmetry occurs if (r, θ) on the graph implies $(r, -\theta)$ is on the graph. y -axis symmetry occurs if (r, θ) on the graph implies $(r, \pi - \theta) = (-r, -\theta)$ is on the graph. Symmetry about the origin occurs if (r, θ) on the graph implies $(-r, \theta) = (r, \theta + \pi)$ is on the graph.

12.2.8 $(0, 0)$, $(0, \pi)$, $(0, 2\pi)$.

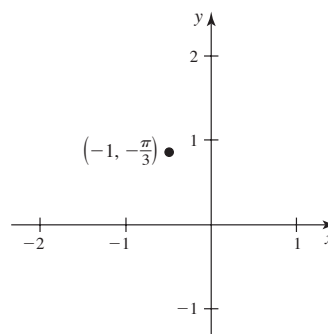
12.2.9 The coordinates $(2, \pi/4)$, $(-2, 5\pi/4)$, and $(2, 9\pi/4)$ represent the same point.



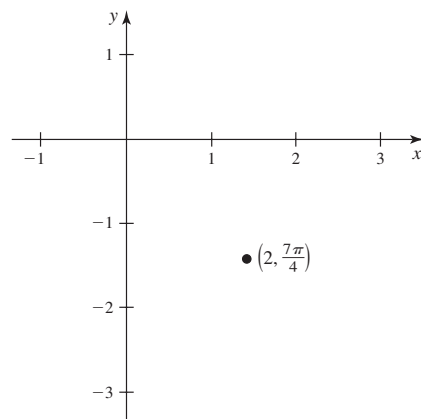
12.2.10 The coordinates $(3, 2\pi/3)$, $(-3, 5\pi/3)$ and $(3, 8\pi/3)$ represent the same point.



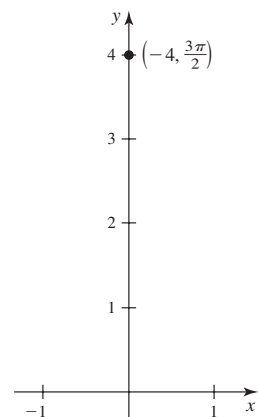
12.2.11 The coordinates $(-1, -\pi/3)$, $(1, 2\pi/3)$ and $(1, -4\pi/3)$ represent the same point.



- 12.2.12** The coordinates $(2, 7\pi/4)$, $(-2, 3\pi/4)$ and $(2, -\pi/4)$ represent the same point.

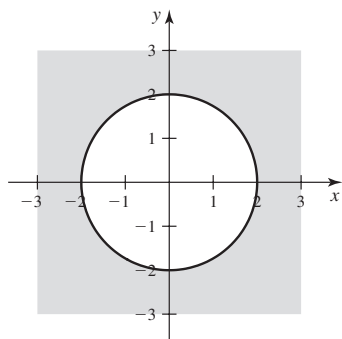


- 12.2.13** The coordinates $(-4, 3\pi/2)$, $(4, \pi/2)$ and $(-4, -\pi/2)$ represent the same point.

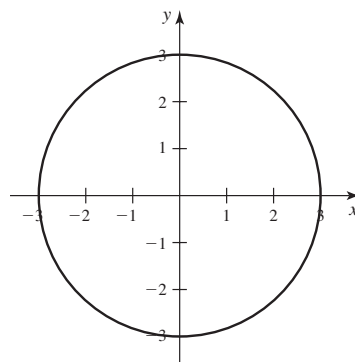


- 12.2.14** $A = (4, \pi/6) = (-4, 7\pi/6)$. $B = (3, \pi/4) = (-3, 5\pi/4)$. $C = (2, \pi/3) = (-2, 4\pi/3)$. $D = (4, \pi/2) = (-4, 3\pi/2)$. $E = (2, 4\pi/3) = (-2, \pi/3)$. $F = (4, -\pi/3) = (-4, 2\pi/3)$.

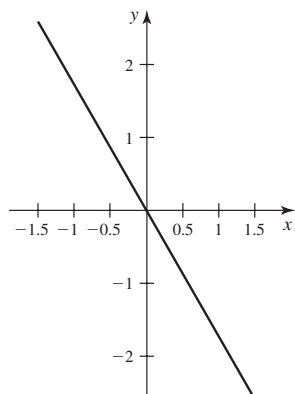
12.2.15



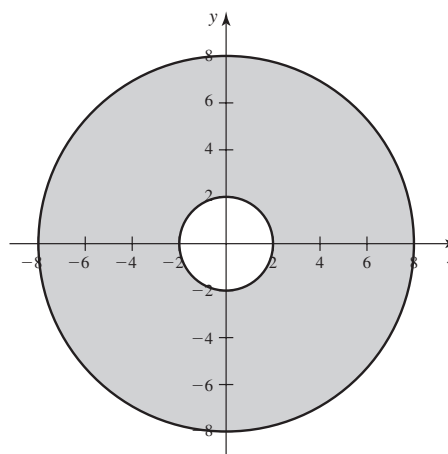
12.2.16



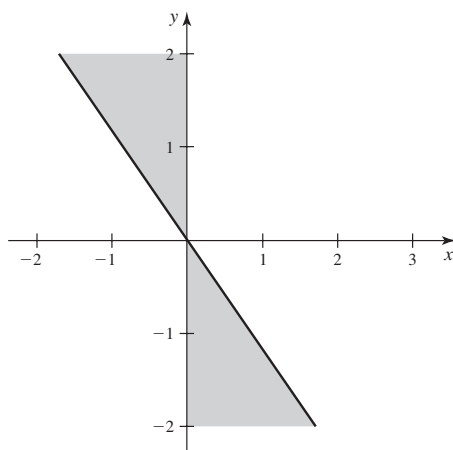
12.2.17



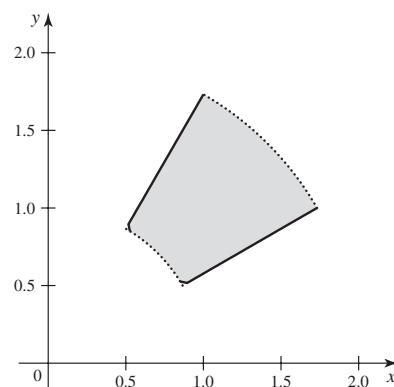
12.2.18



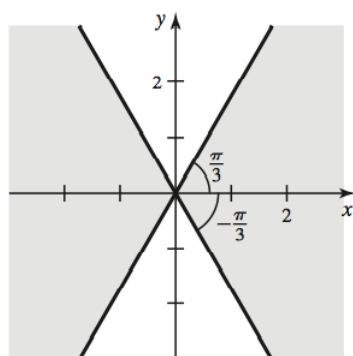
12.2.19



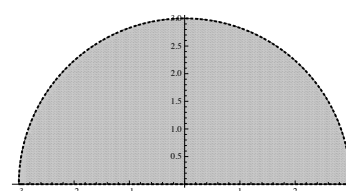
12.2.20



12.2.21



12.2.22



12.2.23 The angle (in radians) is $\frac{\pi}{2} - \frac{3\pi}{4}$, so the polar coordinates are $\left(100, \frac{\pi}{2} - \frac{3\pi}{4}\right) = \left(100, -\frac{\pi}{4}\right)$.

12.2.24 $\left(50, -\frac{2\pi}{3}\right)$.

12.2.25 $x = 3 \cos(\pi/4) = \frac{3\sqrt{2}}{2}$. $y = 3 \sin(\pi/4) = \frac{3\sqrt{2}}{2}$.

$$12.2.26 \quad x = \cos(2\pi/3) = -1/2. \quad y = \sin(2\pi/3) = \frac{\sqrt{3}}{2}.$$

$$12.2.27 \quad x = \cos(-\pi/3) = \frac{1}{2}. \quad y = \sin(-\pi/3) = -\frac{\sqrt{3}}{2}.$$

$$12.2.28 \quad x = 2\cos(7\pi/4) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}. \quad y = 2\sin(7\pi/4) = -\sqrt{2}.$$

$$12.2.29 \quad x = -4\cos(3\pi/4) = 2\sqrt{2}. \quad y = -4\sin(3\pi/4) = -2\sqrt{2}.$$

$$12.2.30 \quad x = 4\cos(5\pi) = -4. \quad y = 4\sin(5\pi) = 0.$$

12.2.31 $r^2 = x^2 + y^2 = 4 + 4 = 8$, so $r = \sqrt{8}$. $\tan \theta = 1$, so $\theta = \pi/4$, so $(2\sqrt{2}, \pi/4)$ is one representation of this point, and $(-2\sqrt{2}, -3\pi/4)$ is another.

12.2.32 $r^2 = x^2 + y^2 = 1 + 0$, so $r = \pm 1$. $\tan \theta = 0$, so $\theta = 0, \pi$. $(-1, 0)$ is one representation of this point, and $(1, \pi)$ is another.

12.2.33 $r^2 = x^2 + y^2 = 1 + 3 = 4$, so $r = \pm 2$. $\tan \theta = \sqrt{3}$, so $\theta = \pi/3, 4\pi/3$. $(2, \pi/3)$ is one representation of this point, and $(-2, -2\pi/3)$ is another.

12.2.34 $r^2 = 81$, so $r = \pm 9$. $\cos \theta = 0$, so $\theta = \pm \frac{\pi}{2}$. One representation of the given point is $(9, -\pi)/2$, and $(-9, \pi/2)$ is another.

12.2.35 $r^2 = 64$, so $r = \pm 8$. $\tan \theta = -\sqrt{3}$, so $\theta = -\pi/3, 2\pi/3$. One representation of the given point is $(8, 2\pi/3)$, and $(-8, -\pi/3)$ is another.

12.2.36 $r^2 = 64 \cdot 3 + 64 = 256$, so $r = 16$. $\tan \theta = \frac{1}{\sqrt{3}}$, so $\theta = \pi/6, 7\pi/6$. One representation of the given point is $(16, 7\pi/6)$ and another is $(16, -5\pi/6)$.

12.2.37 $x = -4$, so this is the vertical line through $(-4, 0)$.

12.2.38 Rewriting the equation in terms of sine and cosine, we have $r = \frac{\cos \theta}{\sin \theta} \frac{1}{\sin \theta}$, or $r \sin^2 \theta = \cos \theta$. Multiplying both sides by r , we have $r^2 \sin^2 \theta = r \cos \theta$, or $y^2 = x$. This curve is a parabola with vertex at $(0, 0)$ which opens to the right.

12.2.39 Because $x^2 + y^2 = r^2 = 4$, this is a circle of radius 2 centered at the origin.

12.2.40 Because $y = r \sin \theta = 3 \csc \theta \sin \theta = 3$, this is the horizontal line $y = 3$.

12.2.41 Note that $x^2 + y^2 = r^2 = 4 \sin^2 \theta + 8 \sin \theta \cos \theta + 4 \cos^2 \theta = 4 + 8 \sin \theta \cos \theta$. Also note that $x = r \cos \theta = 2 \sin \theta \cos \theta + 2 \cos^2 \theta$ and $y = r \sin \theta = 2 \sin^2 \theta + 2 \sin \theta \cos \theta$. Thus, $2x + 2y = 4 + 8 \sin \theta \cos \theta$. If we combine these, we see that $x^2 + y^2 - (2x + 2y) = 0$. Thus $(x^2 - 2x + 1) + (y^2 - 2y + 1) = 2$, so we have the circle $(x - 1)^2 + (y - 1)^2 = 2$. This is a circle of radius $\sqrt{2}$ centered at $(1, 1)$.

12.2.42 Multiplying each side of the equation by r , we have $r^2 = -2r \cos \theta$, or $x^2 + 2x + y^2 = 0$. Completing the square, we have $(x + 1)^2 + y^2 = 1$, which is a circle of radius 1 centered at $(-1, 0)$. This result can also be obtained by using the summary box on circles in polar coordinates (we have $a = -1$, $b = 0$, and $r = \sqrt{(-1)^2 + 0} = 1$).

12.2.43 Multiplying each side of the equation by r , we have $r^2 = 6r \cos \theta + 8r \sin \theta$. Converting to Cartesian coordinates, we have $x^2 + y^2 = 6x + 8y$. Completing the square leads to $x^2 - 6x + 9 + y^2 - 8y + 16 = 25$ and then factoring we arrive at $(x - 3)^2 + (y - 4)^2 = 5^2$. So the curve is a circle of radius 5 centered at $(3, 4)$. This result can also be obtained by inspection using the summary box on circles in polar coordinates.

12.2.44 We have $r \sin \theta = \pm r \cos \theta$, so $y = \pm x$. These are lines through the origin with slopes ± 1 .

12.2.45 $r \cos \theta = \sin 2\theta = 2 \sin \theta \cos \theta$. Note that if $\cos \theta = 0$, then r can be any real number, and the equation is satisfied. For $\cos \theta \neq 0$, we have $x = r \cos \theta = 2 \sin \theta \cos \theta$, so $r = 2 \sin \theta$, and thus $y = r \sin \theta = 2 \sin^2 \theta$. Thus $x^2 + y^2 - 2y = 4 \sin^2 \theta \cos^2 \theta + 4 \sin^2 \theta \sin^2 \theta - 4 \sin^2 \theta = 4 \sin^2 \theta (\sin^2 \theta + \cos^2 \theta) - 4 \sin^2 \theta = 4 \sin^2 \theta - 4 \sin^2 \theta = 0$. Note also that $x^2 + y^2 - 2y = 0$ is equivalent to $x^2 + (y - 1)^2 = 1$, so we have a circle of radius one centered at $(0, 1)$, as well as the line $x = 0$ which is the y -axis.

12.2.46 $r = \sin \theta \sec^2 \theta$, so $x = r \cos \theta = \tan \theta = \frac{y}{x}$, so $y = x^2$, the standard parabola.

12.2.47 $r = 8 \sin \theta$, so $r^2 = 8r \sin \theta$, so $x^2 + y^2 = 8y$. This can be written $x^2 + (y - 4)^2 = 16$, which represents a circle of radius 4 centered at $(0, 4)$.

12.2.48 The given equation implies that $2r \cos \theta + 3r \sin \theta = 1$, so $2x + 3y = 1$. This is a line with slope $-\frac{2}{3}$ and y -intercept $\frac{1}{3}$.

12.2.49 We have $r \sin \theta = r^2 \cos^2 \theta$, so $r = \frac{\sin \theta}{\cos^2 \theta} = \tan \theta \sec \theta$.

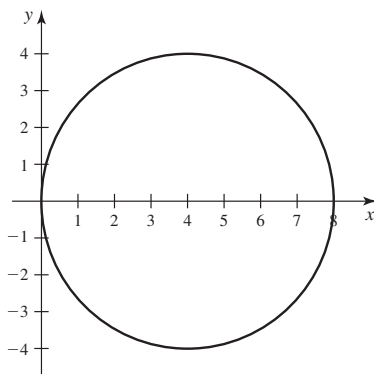
12.2.50 We have $y = r \sin \theta = 3$, so $r = \frac{3}{\sin \theta} = 3 \csc \theta$.

12.2.51 We have $r \sin \theta = \frac{1}{r \cos \theta}$, so $r^2 = \sec \theta \csc \theta$.

12.2.52 We have $(r \cos \theta - 1)^2 + r^2 \sin^2 \theta = 1$, so $r^2 \cos^2 \theta - 2r \cos \theta + 1 + r^2 \sin^2 \theta = 1$, and thus $2r \cos \theta = r^2$. Thus $r = 2 \cos \theta$.

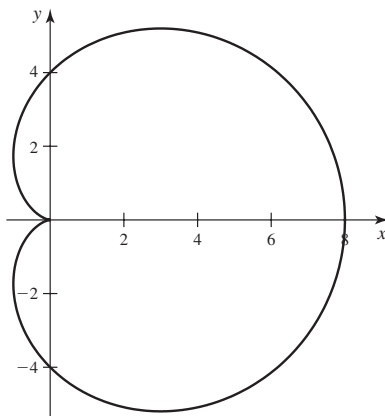
12.2.53

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	8	$4\sqrt{3}$	$4\sqrt{2}$	4	0	-4	$-4\sqrt{2}$	$-4\sqrt{3}$	-8

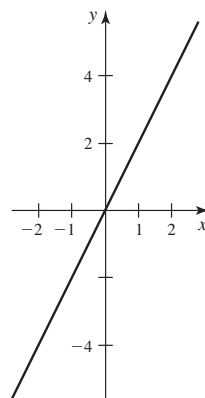


12.2.54

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	8	$4 + 2\sqrt{2}$	4	$4 - 2\sqrt{2}$	0	$4 - 2\sqrt{2}$	4	$4 + 2\sqrt{2}$	8

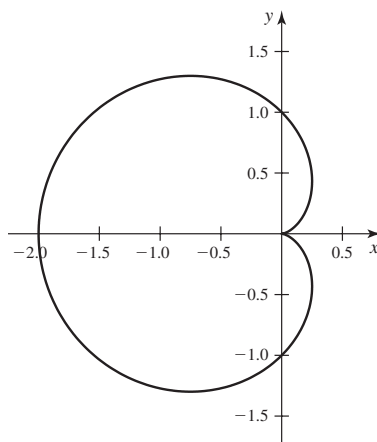


- 12.2.55** $r(\sin \theta - 2 \cos \theta) = 0$ when $r = 0$ or when $\tan \theta = 2$, so the curve is a straight line through the origin of slope 2.

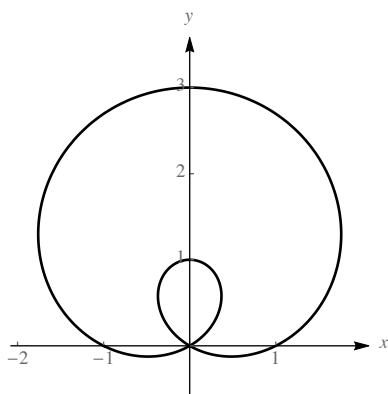


12.2.56

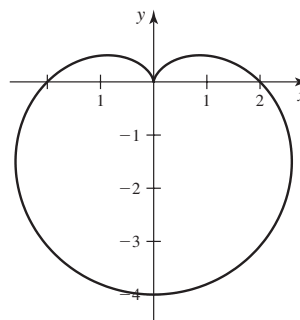
θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	0	$\frac{\sqrt{2}-1}{\sqrt{2}}$	1	$\frac{\sqrt{2}+1}{\sqrt{2}}$	2	$\frac{\sqrt{2}+1}{\sqrt{2}}$	1	$\frac{\sqrt{2}-1}{\sqrt{2}}$	0



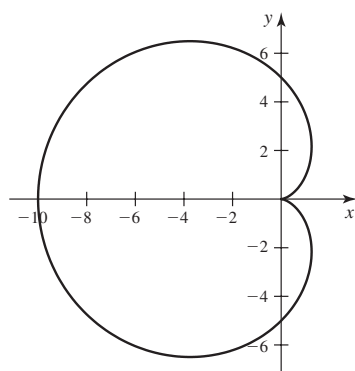
12.2.57



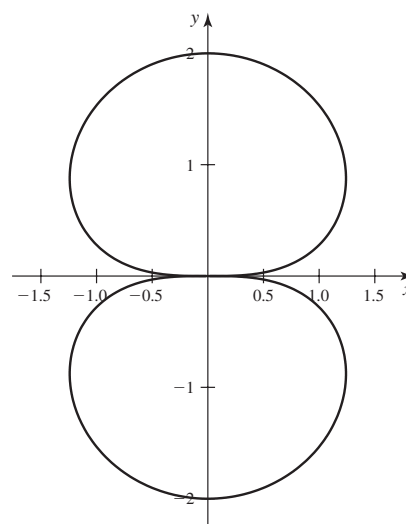
12.2.58



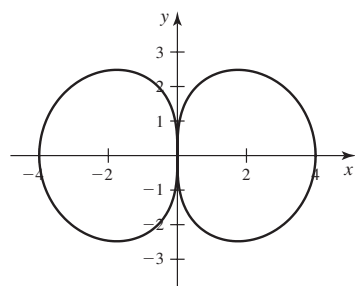
12.2.59



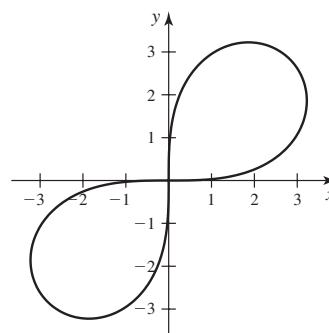
12.2.60



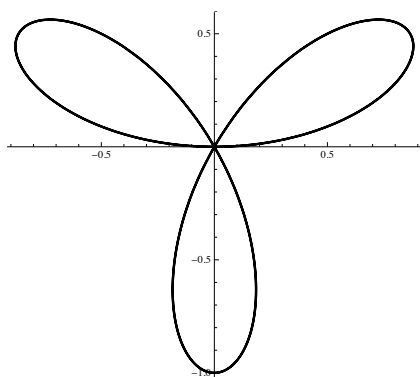
12.2.61



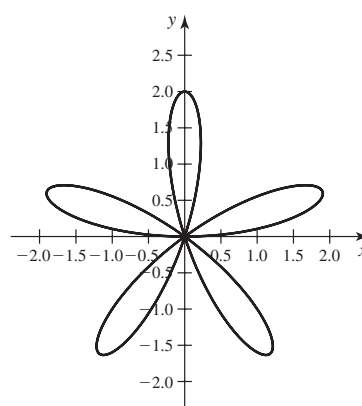
12.2.62



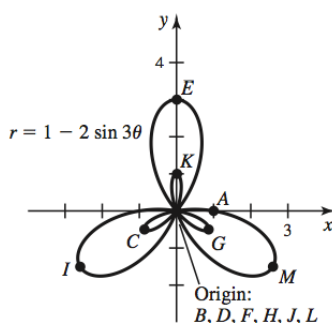
12.2.63



12.2.64

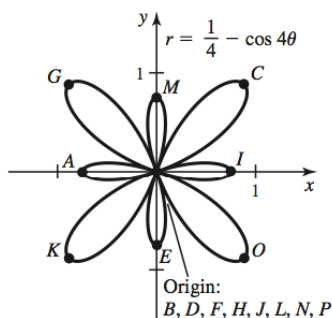


12.2.65 Points B, D, F, H, J and L have y -coordinate 0, so the graph is at the pole for each of these points. Points E, I , and M have maximal radius, so these correspond to the points at the tips of the outer loops. The points C, G and K correspond to the tips of the smaller loops. Point A corresponds to the polar point $(1, 0)$.



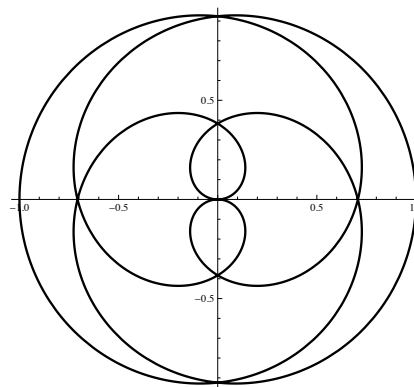
12.2.66 Points B, D, H and J have y -coordinate 0, so the graph is at the pole for each of these points. Points A and K lie where the graph intersects the negative x -axis. C and I are at the top of the two large loops, while F is where the graph intersects the positive x -axis. E and G are the extreme points of the large wide loop.

12.2.67 Points B, D, F, H, J, L, N and P are at the origin. C, G, K and O are on the ends of the long loops, while A, E, I and M are at the ends of the smaller loops.

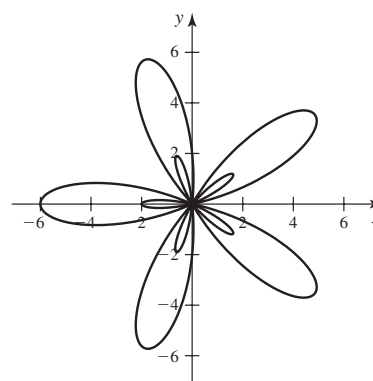


12.2.68 Points C, E, G and I are at the origin. B and D are at the ends of the two bigger loops, F and H are at ends of the two smaller loops. A and J are the points where the graph intersects the positive x -axis.

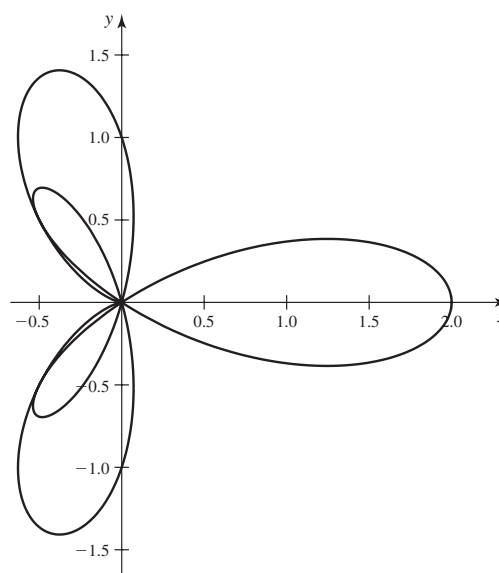
12.2.69 The interval $[0, 8\pi]$ generates the entire graph.



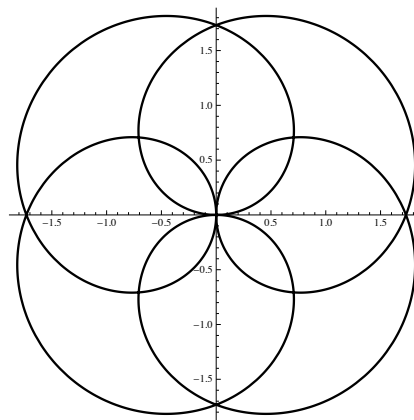
12.2.70 The interval $[0, 2\pi]$ generates the entire graph.



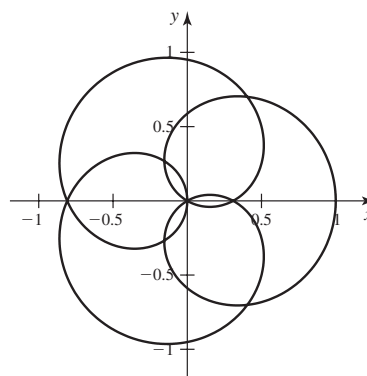
12.2.71 The interval $[0, 2\pi]$ generates the entire graph.



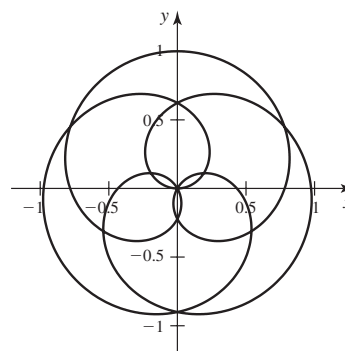
12.2.72 The interval $[0, 6\pi]$ generates the entire graph.



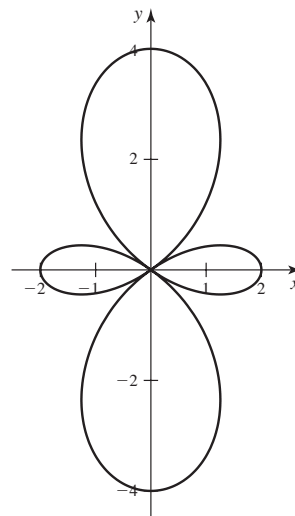
12.2.73 The interval $[0, 5\pi]$ generates the entire graph.



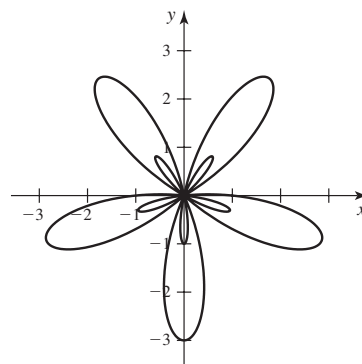
12.2.74 The interval $[0, 7\pi]$ generates the entire graph.



- 12.2.75** The interval $[0, 2\pi]$ generates the entire graph.



- 12.2.76** The interval $[0, 2\pi]$ generates the entire graph.



12.2.77

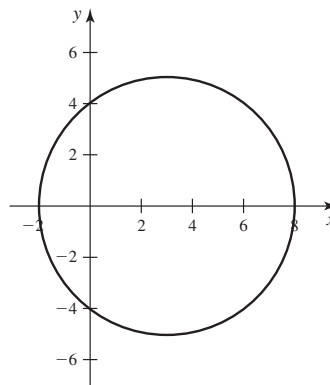
- True. Note that $r^2 = 8$ and $\tan \theta = -1$.
- True. Their intersection point (in Cartesian coordinates) is $(4, -2)$.
- False. They intersect at the polar coordinates $(2, \pi/4)$ and $(2, 5\pi/4)$.
- True. Note that for $\theta = \frac{3\pi}{2}$ we have $r = -3$. But the polar point $\left(-3, \frac{3\pi}{2}\right)$ is the same as the polar point $\left(3, \frac{\pi}{2}\right)$.
- True. The first is the line $x = 2$ because $x = r \cos \theta = 2 \sec \theta \cos \theta = 2$, and the second is $y = 3$ because $y = r \sin \theta = 3 \csc \theta \sin \theta = 3$.

12.2.78 Let $x = r \cos \theta$ and $y = r \sin \theta$, so that $r^2 = x^2 + y^2$. Then the given equation can be written $(x^2 + y^2) - 2ax - 2by + (a^2 + b^2) = R^2$, which in turn can be written as $(x - a)^2 + (y - b)^2 = R^2$, which is the equation of a circle of radius R centered at (a, b) .

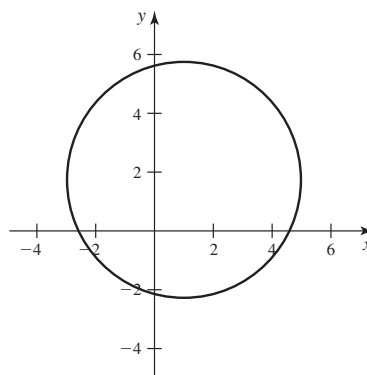
12.2.79 Consider the circle with center $C(r_0, \theta_0)$, and let A be the origin and $B(r, \theta)$ be a point on the circle not collinear with A and C . Note that the length of side BC is R , and that the angle CAB has measure $\theta - \theta_0$. Applying the law of cosines to triangle CAB yields the equation $R^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)$, which is equivalent to the given equation.

12.2.80

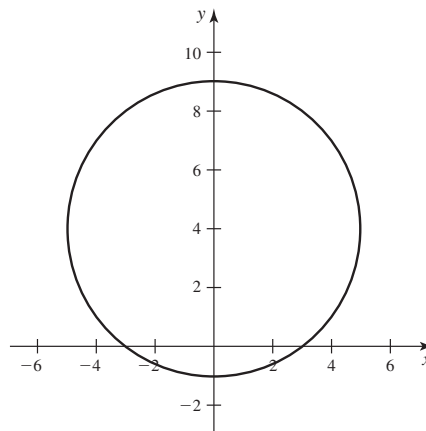
In relation to number 78, we have $2a = 6$, so $a = 3$ and $b = 0$. So $R^2 - a^2 - b^2 = R^2 - 9 = 16$, and thus $R^2 = 25$. Thus we have a circle centered at $(3, 0)$ with radius 5.

**12.2.81**

In relation to number 79, we have $r_0 = 2$ and $\theta_0 = \pi/3$, and $R^2 - 4 = 12$, so $R^2 = 16$. Thus this is a circle with polar center $(2, \pi/3)$ and radius 4.

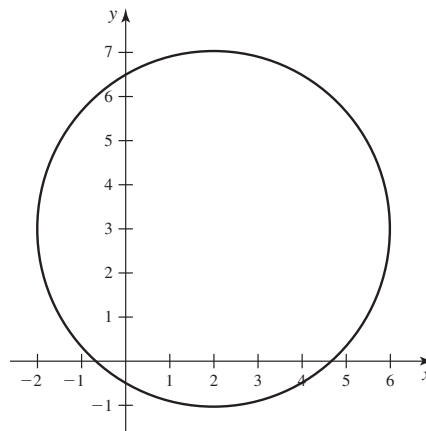
**12.2.82**

In relation to number 79, we have $r_0 = 4$ and $\theta_0 = \pi/2$, and $R^2 - 16 = 9$, so $R^2 = 25$. Thus this is a circle with polar center $(4, \pi/2)$ and radius 5.



12.2.83

In relation to number 78, we have $a = 2$ and $b = 3$. So $R^2 - a^2 - b^2 = R^2 - 13 = 3$, and thus $R^2 = 16$. Thus we have a circle centered at $(2, 3)$ with radius 4.



12.2.84 The radius of a circle inscribed in a triangle with side lengths a , b , and c is $\frac{2A}{a+b+c}$ where A is the area of the triangle. So for the bigger circle, $R = r_0 = \frac{2}{2+2\sqrt{2}} = \frac{1}{1+\sqrt{2}}$. For each of the smaller circles, we have $R = \frac{1}{2+\sqrt{2}}$. The area inside the three circles is thus $2\pi \cdot \frac{1}{(2+\sqrt{2})^2} + \pi \cdot \frac{1}{(1+\sqrt{2})^2} \approx 1.078$. Because the area of the square is 2, there is more area inside the circles than outside the circles but inside the square. Using problem 75, the equation of the largest circle is $r^2 - 2r \left(\frac{1}{1+\sqrt{2}} \right) \cos(\theta - \pi/2) = 0$. The smaller circle in the 3rd quadrant has center with polar radius $r_0 = \frac{\sqrt{2}}{2} - \frac{1}{2+\sqrt{2}} = \sqrt{2} - 1$, so its equation is $r^2 - 2r(\sqrt{2} - 1) \cos(\theta - 5\pi/4) = R^2 - r_0^2 = \sqrt{2} - 3/2$, and the other circle has equation $r^2 - 2r(\sqrt{2} - 1) \cos(\theta + \pi/4) = \sqrt{2} - 3/2$.

12.2.85

- a. During this 30-minute period, the plane moves on a straight line from the point (using Cartesian coordinates) $(0, 150)$, to $(100 \cos \frac{\pi}{6}, 100 \sin \frac{\pi}{6}) = (50\sqrt{3}, 50)$. So the distance travelled is

$$\sqrt{(50\sqrt{3})^2 + 100^2} \approx 132.3 \text{ miles.}$$

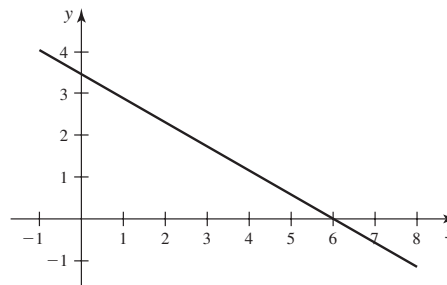
- b. The average velocity is $\frac{132.3}{1/2} = 264.6$ miles per hour.

12.2.86

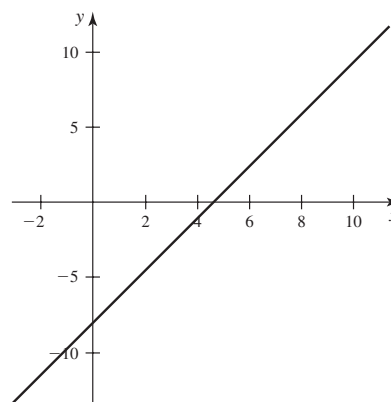
- a. Given $y = mx + b$, let $x = r \cos \theta$ and $y = r \sin \theta$. Then $r \sin \theta = m(r \cos \theta) + b$, so $r \sin \theta - mr \cos \theta = b$, and thus $r(\sin \theta - m \cos \theta) = b$, and $r = \frac{b}{\sin \theta - m \cos \theta}$, provided $\sin \theta - m \cos \theta \neq 0$.
- b. Using the right triangle shown, we see that $\frac{r_0}{r} = \cos(\theta_0 - \theta)$, so $r_0 = r \cos(\theta_0 - \theta)$.

12.2.87

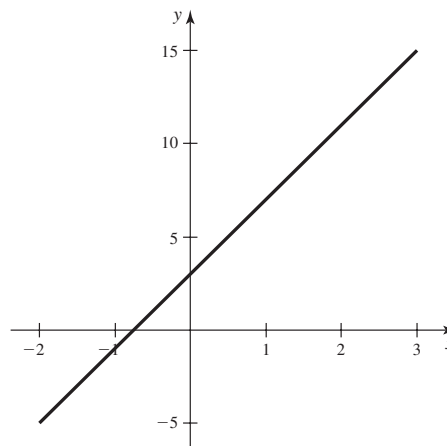
Using problem 86b, this is the line with $r_0 = 3$ and $\theta_0 = \frac{\pi}{3}$. So it is the line through the polar point $(3, \pi/3)$ in the direction of angle $\pi/3 + \pi/2 = 5\pi/6$. The Cartesian equation is $y = -\frac{x}{\sqrt{3}} + 2\sqrt{3}$.

**12.2.88**

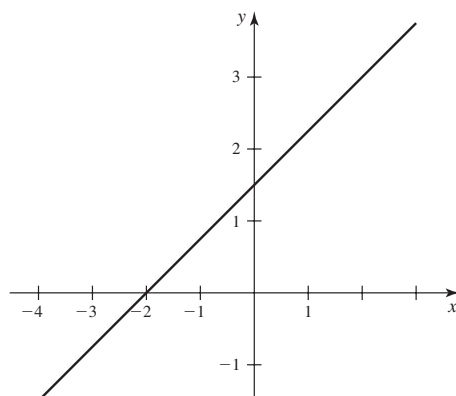
Using problem 86b, this is the line with $r_0 = 4$ and $\theta_0 = -\frac{\pi}{6}$. So it is the line through the polar point $(4, -\pi/6)$ in the direction of angle $-\pi/6 + \pi/2 = \pi/3$. The Cartesian equation is $y = \sqrt{3}x - 8$.

**12.2.89**

Using problem 86a, this is the line with $b = 3$ and $m = 4$, so $y = 4x + 3$.



12.2.90 Using problem 86a, this is the line with $b = 3/2$ and $m = 3/4$, so $y = \frac{3}{4}x + \frac{3}{2}$.

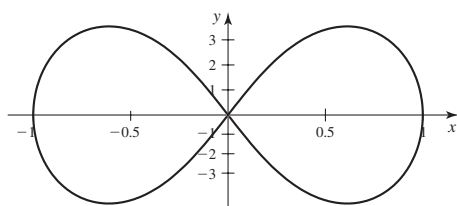


12.2.91

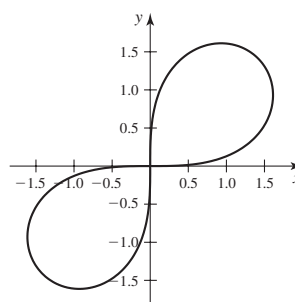
- This matches (A), because we have $|a| = 1 = |b|$, and the graph is a cardioid.
- This matches (C). This has an inner loop because $|a| = 1 < 2 = |b|$. Note that $r = 1$ when $\theta = 0$, so it can't be (D).
- This matches (B). This has $|a| = 2 > 1 = |b|$, so it has an oval-like shape.
- This matches (D). This has an inner loop because $|a| = 1 < 2 = |b|$. Note that $r = -1$ when $\theta = 0$, so this can't be (C).
- This matches (E). Note that there is an inner loop because $|a| = 1 < 2 = |b|$, and that $r = 3$ when $\theta = \pi/2$.
- This matches (F).

12.2.92 As $b \rightarrow \infty$, the inner loop approaches the outer loop, so that for large b the graph appears to be a single circle with diameter b . Thus, there is no limiting curve as $b \rightarrow \infty$.

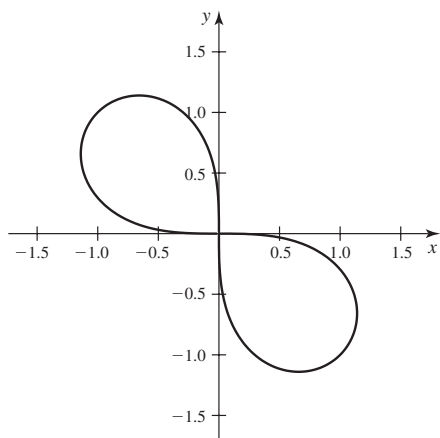
12.2.93



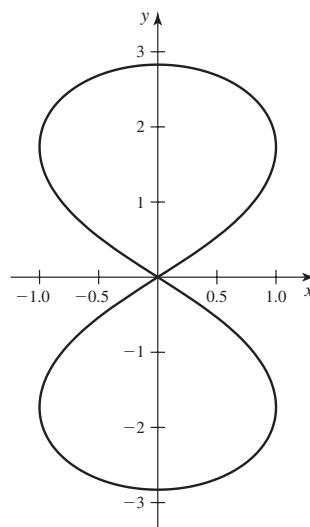
12.2.94



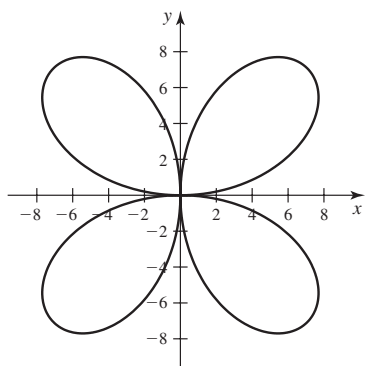
12.2.95



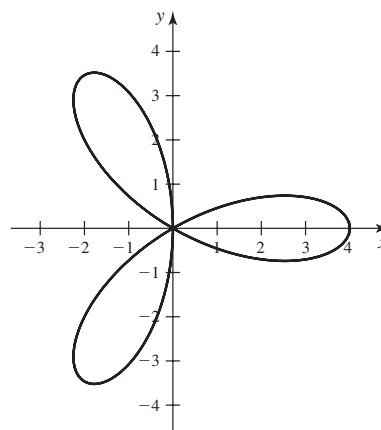
12.2.96



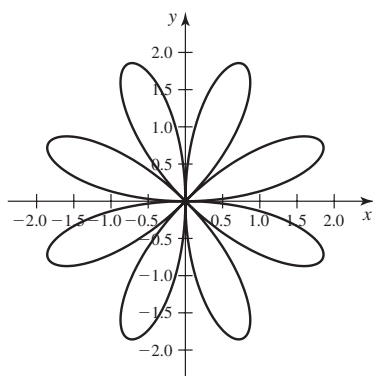
12.2.97



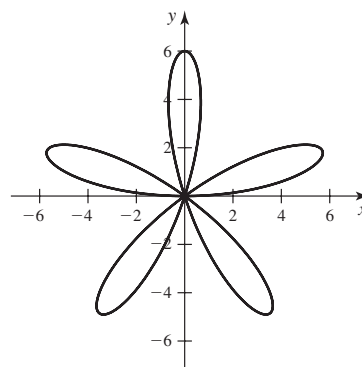
12.2.98



12.2.99



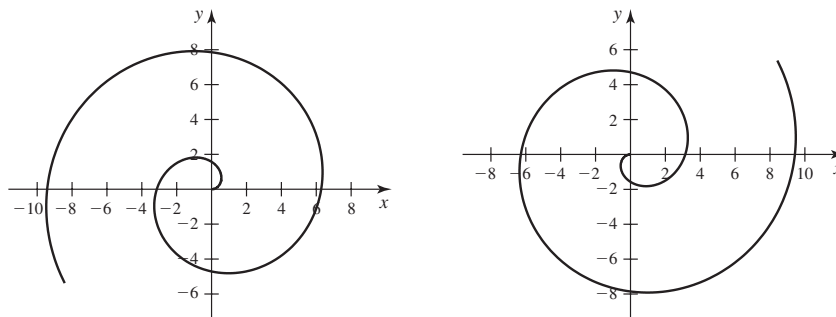
12.2.100



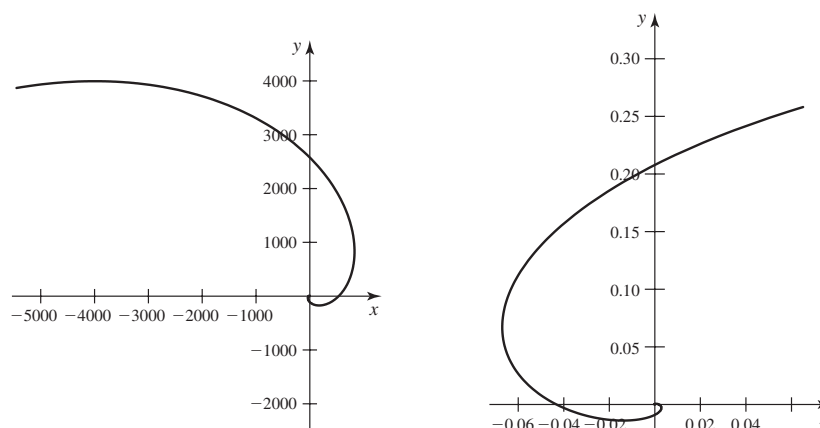
12.2.101 Note that $a \sin m\theta = 0$ for $\theta = \frac{k\pi}{m}$, $k = 1, 2, \dots, 2m$. Thus the graph is back at the pole $r = 0$ for each of these values, and each of these gives rise to a distinct petal of the rose if m is odd. If m is even, then by symmetry, each petal for $k = 1, 2, \dots, \frac{m}{2}$ is equivalent to one for $k = \frac{m}{2} + 1, \frac{m}{2} + 2, \dots, m$. (Note

that this follows because the sine function is odd.) A similar result holds for the rose $r = a \cos \theta$.

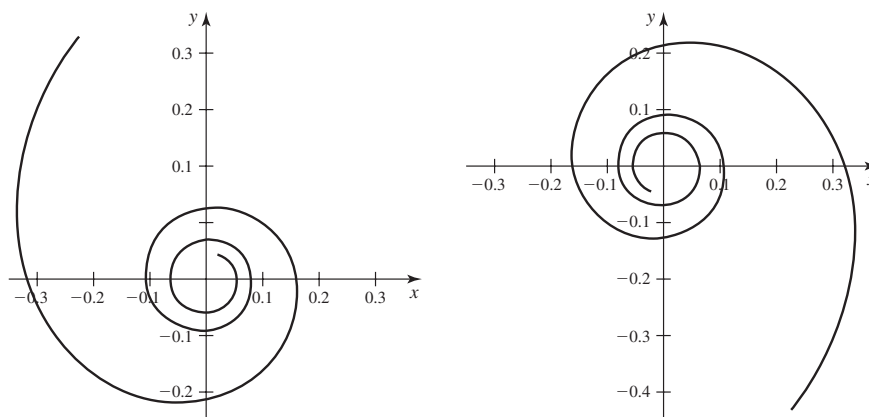
12.2.102 The spirals wind outward counterclockwise.



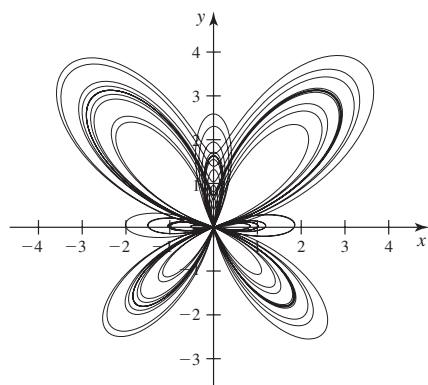
12.2.103 For $a = 1$, the spiral winds outward counterclockwise. For $a = -1$, the spiral winds inward counterclockwise.



12.2.104 The spirals wind inward counterclockwise for $a = 1$ and outward clockwise for $a = -1$.



12.2.105

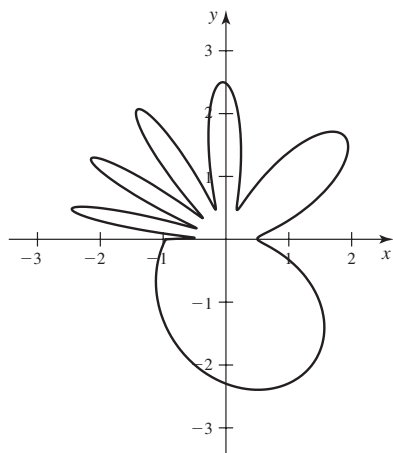


a.

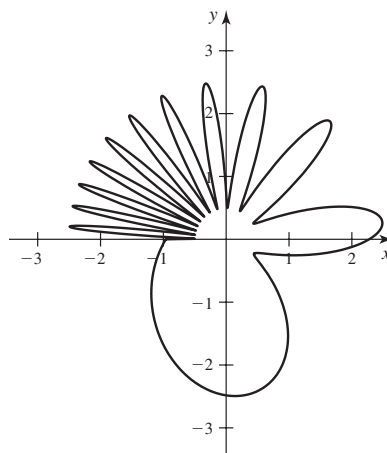
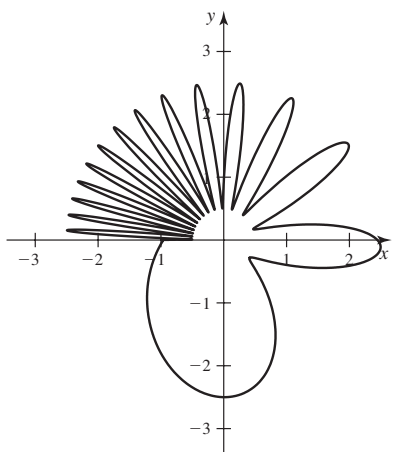
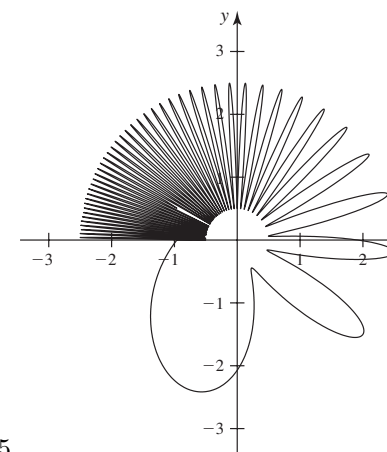
- b. It adds multiple layers of the same type of curve as $\sin^5 \theta / 12$ oscillates between -1 and 1 for $0 \leq \theta \leq 24\pi$.

12.2.106

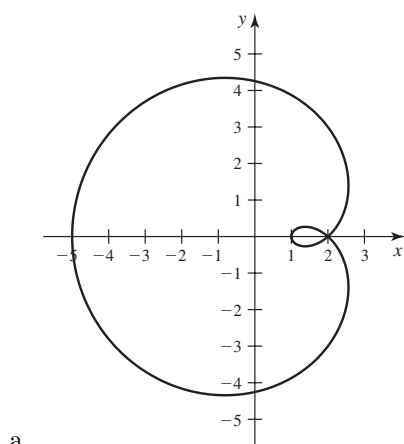
- a. $f(0) = \cos(1) - 1.5$, and $f(2\pi) = \cos(((1+12\pi)^{1/2\pi})^{2\pi}) - 1.5 = \cos(1+12\pi) - 1.5 = \cos(1) - 1.5 = f(0)$. The points correspond to the polar points $(-0.960, 0)$.
- b. No. The curve for $-\pi \leq \theta \leq 0$ has nowhere where the absolute value of the radius is equal to 1, whereas the curve for $\pi \leq \theta \leq 2\pi$ has numerous places where this is true, because a^x has a much bigger range on $[0, \pi]$ than on $[-\pi, 0]$.
- c. Because $((1+2k\pi)^{1/2\pi})^0 = 1$ and $((1+2k\pi)^{1/2\pi})^{2\pi} = 1+2k\pi$, we have that $f(0) = \cos(1) - b = \cos(1+2k\pi) - b = \cos(1) - b = f(2\pi)$.



d.

 $k = 6$  $k = 12$  $k = 15$  $k = 45$

12.2.107



a.

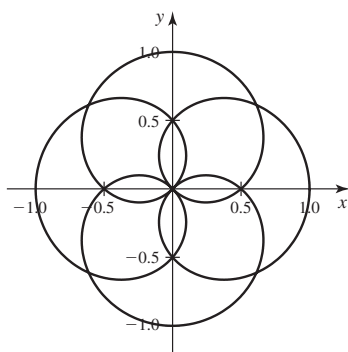
- b. $r = 3 - 4 \cos \pi t$ is a limaçon, and $x - 2 = r \cos \pi t$ and $y = r \sin \pi t$ is a circle, and the composition of a limaçon and a circle is a limaçon.

12.2.108

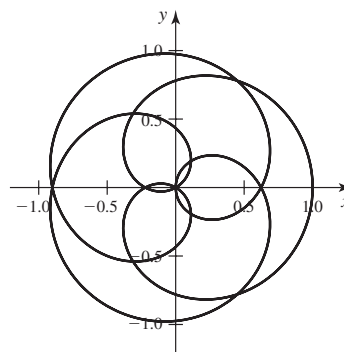
- a. The region is given by $\{(r, \theta) : 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$.
- b. The inflow is given by $\{(r, \theta) : 1 \leq r \leq 2, \theta = 0\}$. The outflow is given by $\{(r, \theta) : 1 \leq r \leq 2, \theta = \pi\}$.
- c. The tangential velocity at $(1.5, \pi/4)$ is $v(1.5) = 10 \cdot 1.5 = 15$ meters per second. At $(1.2, 3\pi/4)$ it is $v(1.2) = 10 \cdot 1.2 = 12$ meters per second, so it is greater at 1.5.
- d. The velocity is greater at $r = 1.3$, because $\frac{20}{1.3} > \frac{20}{1.8}$.
- e. $\int_1^2 10r \, dr = 5r^2 \Big|_1^2 = 15$, while $\int_1^2 \frac{20}{r} \, dr = 20 \ln r \Big|_1^2 \approx 13.86$, so the flow is greater in part c).

12.2.109 Because $\sin(\theta/2) = \sin(\pi - \theta/2) = \sin((2\pi - \theta)/2)$, we have that the graph is symmetric with respect to the x -axis.

12.2.110



a.



b.

- c. If n is even, then the whole curve is generated for $0 \leq \theta \leq 2m\pi$. If n is odd, then the whole curve is generated for $0 \leq \theta \leq m\pi$.

12.2.111 Note that $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$, so $r^2 = a^2(\cos^2 \theta - \sin^2 \theta)$, so $r^4 = a^2(r^2 \cos^2 \theta - r^2 \sin^2 \theta)$, so $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

12.3 Calculus in Polar Coordinates

12.3.1 Because $x = r \cos \theta$ and $y = r \sin \theta$, we have $x = f(\theta) \cos \theta$ and $y = f(\theta) \sin(\theta)$.

12.3.2 This line is a vertical line passing through the origin and vertical lines have undefined slope.

12.3.3 Because slope is given relative to the horizontal and vertical coordinates, it is given by $\frac{dy}{dx}$, not by $\frac{dr}{d\theta}$.

12.3.4 This would be given by $\frac{1}{2} \int_{\alpha}^{\beta} (f(\theta)^2 - g(\theta)^2) d\theta$.

12.3.5 The slope is $\tan \frac{\pi}{3} = \sqrt{3}$.

12.3.6 $\theta = \frac{3\pi}{4}$.

12.3.7 The area is given by $\int_0^{\pi} \frac{1}{2} \theta d\theta = \frac{\theta^2}{4} \Big|_0^{\pi} = \frac{\pi^2}{4}$.

12.3.8 The curve passes through the origin for $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$. The corresponding slopes are $\tan \frac{\pi}{3} = \sqrt{3}$ and $\tan \frac{5\pi}{3} = -\sqrt{3}$.

12.3.9 Both curves pass through the origin, but for different values of θ .

12.3.10 The polar point $(-1, 3\pi/2)$ is equivalent to the polar point $(1, \pi/2)$ which does satisfy the equation.

12.3.11 $\frac{dy}{dx} = \frac{-\cos \theta \sin \theta + (1 - \sin \theta) \cos \theta}{-\cos^2 \theta - (1 - \sin \theta) \sin \theta}$. At $(1/2, \pi/6)$, we have $\frac{dy}{dx} = \frac{0}{-1} = 0$.

12.3.12 $\frac{dy}{dx} = \frac{-4 \sin^2 \theta + 4 \cos^2 \theta}{-8 \cos \theta \sin \theta}$. At $(2, \pi/3)$ we have $\frac{dy}{dx} = \frac{-2}{-2\sqrt{3}} = \frac{\sqrt{3}}{3}$.

12.3.13 $\frac{dy}{dx} = \frac{16 \cos \theta \sin \theta}{-8 \sin^2 \theta + 8 \cos^2 \theta}$. At $(4, 5\pi/6)$ we have $\frac{dy}{dx} = \frac{-4\sqrt{3}}{4} = -\sqrt{3}$.

12.3.14 $\frac{dy}{dx} = \frac{\cos \theta \sin \theta + (4 + \sin \theta) \cos \theta}{\cos^2 \theta - (4 + \sin \theta) \sin \theta}$. At $(4, 0)$ we have $\frac{dy}{dx} = \frac{4}{1} = 4$. At $(3, 3\pi/2)$ we have $\frac{dy}{dx} = \frac{0}{3} = 0$.

12.3.15 $\frac{dy}{dx} = \frac{-3 \sin^2 \theta + (6 + 3 \cos \theta) \cos \theta}{-3 \cos \theta \sin \theta - (6 + 3 \cos \theta) \sin \theta}$. At both $(3, \pi)$ and $(9, 0)$, this doesn't exist.

12.3.16 $\frac{dy}{dx} = \frac{6 \cos(3\theta) \sin \theta + 2 \sin(3\theta) \cos \theta}{6 \cos(3\theta) \cos \theta - 2 \sin(3\theta) \sin \theta}$. The tips of the leaves occur at $\theta = \pi/6, \pi/2$ and $5\pi/6$. At $\pi/6$, we have $\frac{dy}{dx} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. At $\pi/2$ we have $\frac{dy}{dx} = \frac{0}{2} = 0$. At $5\pi/6$ we have $\frac{dy}{dx} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$.

12.3.17 $\frac{dy}{dx} = \frac{-8 \sin(2\theta) \sin \theta + 4 \cos(2\theta) \cos \theta}{-8 \sin(2\theta) \cos \theta - 4 \cos(2\theta) \sin \theta}$. The tips of the leaves occur at $\theta = 0, \pi/2, \pi$ and $3\pi/2$. At 0 and at π , we have that $\frac{dy}{dx}$ doesn't exist. At $\pi/2$ and $3\pi/2$ we have $\frac{dy}{dx} = 0$.

12.3.18 $\frac{dy}{dx} = \frac{0 + 3(\sqrt{2}/2)}{0 - 3(\sqrt{2}/2)} = -1$.

12.3.19 The slopes of the tangent lines are given by $\tan(\pi/4) = 1$ and $\tan(-\pi/4) = -1$.

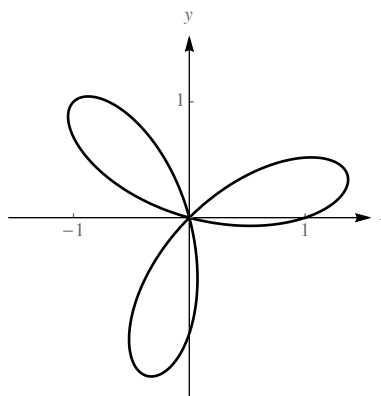
12.3.20 $\frac{dy}{dx} = \frac{2 \sin \theta + 2\theta \cos \theta}{2 \cos \theta - 2\theta \sin \theta}$. At $(\pi/2, \pi/4)$ this is $\frac{\sqrt{2} + \pi\sqrt{2}/4}{\sqrt{2} - \pi\sqrt{2}/4} \approx 8.32$.

12.3.21 $\cos \theta + \sin \theta = 0$ implies that $\cos \theta = -\sin \theta$, or $\tan \theta = -1$. So $\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$. Both of these represent the same line. So the line tangent to the curve at the origin is $\theta = \frac{3\pi}{4}$, which has a slope of -1 .

12.3.22 The complete graph of this curve is generated for $0 \leq \theta \leq \pi$, and $4 \cos \theta = 0$ implies that $\theta = \frac{\pi}{2}$. So the equation of the line tangent to the curve at the origin is $\theta = \frac{\pi}{2}$, and because this is a vertical line, the slope of the tangent line is undefined.

12.3.23

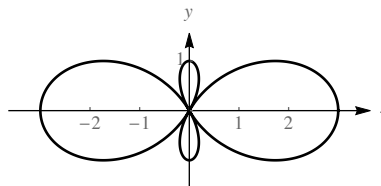
a. The curve is generated for $[0, \pi]$.



b. $\cos 3\theta + \sin 3\theta = 0$ implies that $\tan 3\theta = -1$, which means that $3\theta = \frac{3\pi}{4}, \frac{7\pi}{4},$ and $\frac{11\pi}{4}$. So $\theta = \frac{\pi}{4}, \frac{7\pi}{12},$ and $\frac{11\pi}{12}$. The slopes are $\tan \frac{\pi}{4} = 1$, $\tan \frac{7\pi}{12} \approx -3.73$, and $\tan \frac{11\pi}{12} \approx -0.27$.

12.3.24

a. The curve is generated for $[0, 2\pi]$.



b. $1 + 2 \cos 2\theta = 0$ for $\cos 2\theta = -\frac{1}{2}$, which implies that $2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3},$ and $\frac{10\pi}{3}$.

Thus $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3},$ and $\frac{5\pi}{3}$. The equations $\theta = \frac{\pi}{3}$ and $\theta = \frac{4\pi}{3}$ describe the same line, as do the lines $\theta = \frac{2\pi}{3}$ and $\theta = \frac{5\pi}{3}$. There are therefore two distinct tangent lines, with slopes $m = \tan \frac{\pi}{3} = \sqrt{3}$ and with slope $m = \tan \frac{2\pi}{3} = -\sqrt{3}$.

12.3.25 Note that the curve is at the origin at $\pi/2$, so there is vertical tangent at $(0, \pi/2)$. Also, $\frac{dy}{dx} = \frac{-4\sin^2\theta + 4\cos^2\theta}{-8\sin\theta\cos\theta} = \frac{1 - 2\sin^2\theta}{\sin(2\theta)}$. Thus, there are horizontal tangents at $\pi/4$ and $3\pi/4$ (at the polar points $(2\sqrt{2}, \pi/4)$ and $(-2\sqrt{2}, 3\pi/4)$). There is also a vertical tangent where $\theta = 0$, at the point $(4, 0)$.

12.3.26 Note that the curve is at the origin at $3\pi/2$, so there is vertical tangent at $(0, 3\pi/2)$. Also, $\frac{dy}{dx} = \frac{2\cos\theta\sin\theta + (2 + 2\sin\theta)\cos\theta}{2\cos^2\theta - (2 + 2\sin\theta)\sin\theta} = \frac{\cos\theta(2 + 4\sin\theta)}{(2 - 4\sin^2\theta) - 2\sin\theta}$. Thus, there are horizontal tangents where this expression is 0 at $\pi/2$ and $7\pi/6$ and $(11\pi/6)$ (at the polar points $(4, \pi/2)$ and $(1, 7\pi/6)$ and $(1, 11\pi/6)$). There are also vertical tangents where the denominator is 0 and the numerator isn't, which occurs at the point $(3, \pi/6)$ and at $(3, 5\pi/6)$.

12.3.27 Using the double angle identities somewhat liberally:

$$\frac{dy}{dx} = \frac{2\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta} = \frac{\sin\theta(\cos 2\theta + \cos^2\theta)}{\cos\theta(\cos(2\theta) - \sin^2\theta)} = \frac{\sin\theta(3\cos^2\theta - 1)}{\cos\theta(1 - 3\sin^2\theta)} = \frac{\sin\theta(3\cos^2\theta - 1)}{\cos\theta(3\cos^2\theta - 2)}.$$

The numerator is 0 for $\theta = 0$ and for $\theta = \pm\cos^{-1}(\pm\sqrt{3}/3)$, so there are horizontal tangents at the corresponding points $(0, 0)$, $(0.943, 0.955)$, $(-0.943, 2.186)$, $(0.943, 4.097)$, and $(-0.943, 5.328)$. The denominator is 0 for $\theta = \pi/2$ and $3\pi/2$, and for $\theta = \pm\cos^{-1}(\pm\sqrt{6}/3)$, so there are vertical tangents at $(0, 0)$, $(0.943, 0.615)$, $(-0.943, 2.526)$, $(0.943, 3.757)$, and $(-0.943, 5.668)$.

12.3.28 The curve intersects the origin at $\theta = 7\pi/6$ and $\theta = 11\pi/6$, so those don't give rise to vertical or horizontal tangents. We have $\frac{dy}{dx} = \frac{6\cos\theta\sin\theta + (3 + 6\sin\theta)\cos\theta}{6\cos\theta\cos\theta - (3 + 6\sin\theta)\sin\theta} = \frac{\cos\theta(1 + 4\sin\theta)}{(2 - \sin\theta - 4\sin^2\theta)}$. Thus there are horizontal tangents for $\theta = \pi/2$ and $3\pi/2$, at the corresponding points $(9, \pi/2)$ and $(-3, 3\pi/2)$, and at the points where $\sin(\theta) = -1/4$, which are $(3/2, 3.394)$ and $(3/2, 6.031)$. There are vertical tangents where the denominator is 0, which occurs for $\theta = \sin^{-1}\left(-\frac{1}{8} \pm \frac{\sqrt{33}}{8}\right)$, so the corresponding points are $(-2.06, 5.28)$, and $(6.56, .634)$.

12.3.29 Suppose $2\cos\theta = 1 + \cos\theta$. Then $\cos\theta = 1$, so this occurs for $\theta = 0$ and $\theta = 2\pi$. At those values, $r = 2$, so the curves intersect at the polar point $(2, 0)$. The curves also intersect when $r = 0$, which occurs for $\theta = \pi/2$ and $\theta = 3\pi/2$ for the first curve and $\theta = \pi$ for the second.

12.3.30 Suppose $1 - \sin\theta = 1 + \cos\theta$, or $\tan\theta = -1$. Then $\theta = 3\pi/4$ or $\theta = 7\pi/4$. So the curves intersect at the polar points $(1 + \sqrt{2}/2, 7\pi/4)$ and $(1 - \sqrt{2}/2, 3\pi/4)$. They also intersect at the pole $(0, 0)$, which occurs for the first curve at $\pi/2$ and for the second curve at π .

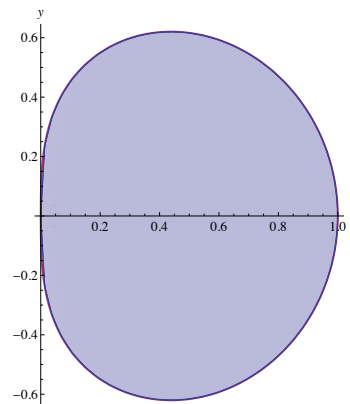
12.3.31 $2\sin 2\theta = 1$ implies that $\sin 2\theta = \frac{1}{2}$, so $2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6},$ and $\frac{17\pi}{6}$, and therefore $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12},$ and $\frac{17\pi}{12}$. Using symmetry, we also see that they intersect at $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12},$ and $\frac{23\pi}{12}$. So the curves intersect at the points

$$\left(1, \frac{\pi}{12}\right), \left(1, \frac{5\pi}{12}\right), \left(1, \frac{7\pi}{12}\right), \left(1, \frac{11\pi}{12}\right), \left(1, \frac{13\pi}{12}\right), \left(1, \frac{17\pi}{12}\right), \left(1, \frac{19\pi}{12}\right), \left(1, \frac{23\pi}{12}\right).$$

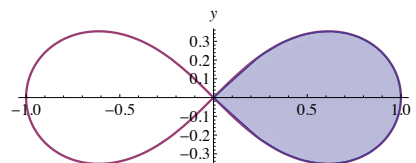
12.3.32 $\cos 2\theta = \sin 2\theta$ implies that $\tan 2\theta = 1$, so $2\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$, so $\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$. Using these and the symmetry of the curve, we find intersection points

$$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{5\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{7\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{9\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{11\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{13\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{15\pi}{8}\right).$$

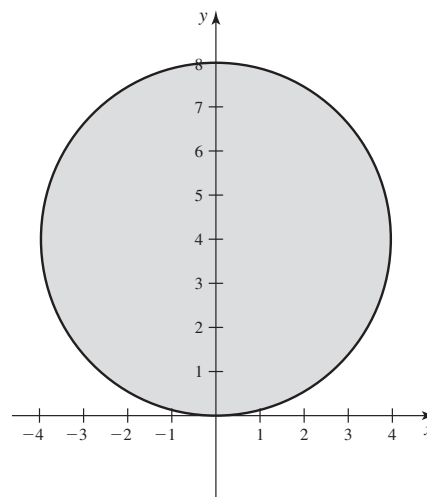
$$12.3.33 \quad A = 2 \cdot \frac{1}{2} \int_0^{\pi/2} \cos \theta \, d\theta = \sin \theta \Big|_0^{\pi/2} = 1.$$



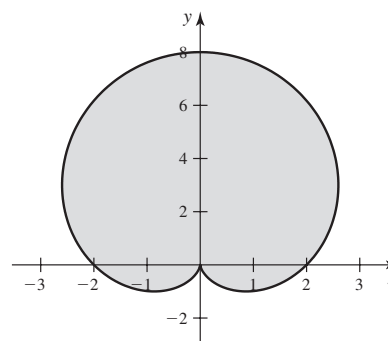
$$12.3.34 \quad A = 2 \cdot \frac{1}{2} \int_0^{\pi/4} \cos 2\theta \, d\theta = \frac{1}{2} (\sin 2\theta) \Big|_0^{\pi/4} = \frac{1}{2}.$$



$$12.3.35 \quad \begin{aligned} A &= \frac{1}{2} \int_0^{\pi} (8 \sin \theta)^2 \, d\theta = \\ 32 \int_0^{\pi} \sin^2 \theta \, d\theta &= 32 \int_0^{\pi} \frac{1 - \cos 2\theta}{2} \, d\theta = \\ 32 \left(\frac{1}{2} \theta - \frac{\sin \theta \cos \theta}{2} \right) \Big|_0^{\pi} &= 16\pi. \end{aligned}$$

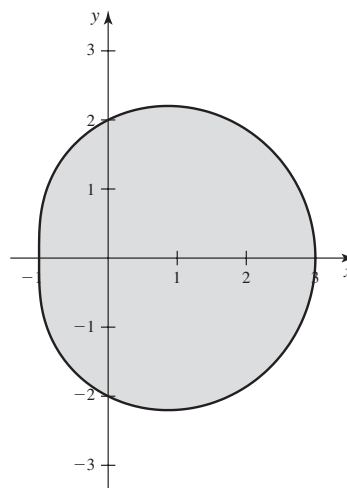


$$12.3.36 \quad \begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin \theta)^2 \, d\theta \\ &= 8 \int_0^{2\pi} (1 + 2 \sin \theta + \sin^2 \theta) \, d\theta \\ &= 8 \left(\theta - 2 \cos \theta + \frac{1}{2} \theta - \frac{\sin \theta \cos \theta}{2} \right) \Big|_0^{2\pi} = \\ 24\pi. \end{aligned}$$



Using symmetry, we have $\frac{1}{2} \cdot 2 \int_0^\pi (2 + \cos \theta)^2 d\theta = \int_0^\pi (4 + 4 \cos \theta + \cos^2 \theta) d\theta =$

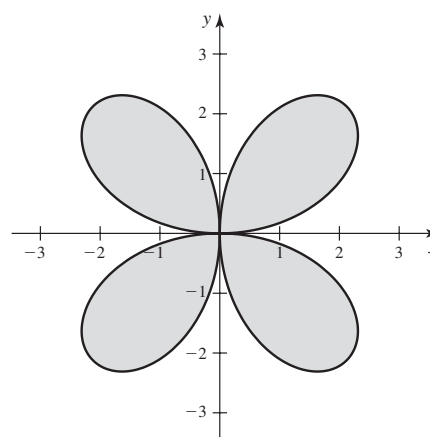
12.3.37 $\left(4\theta + 4 \sin \theta + \frac{1}{2}\theta + \frac{\sin \theta \cos \theta}{2} \right) \Big|_0^\pi = \frac{9\pi}{2}.$



Because there are 4 symmetric leaves, we compute the area of 1/2 of one of the leaves, and then multiply by 8 to get the total area. We have

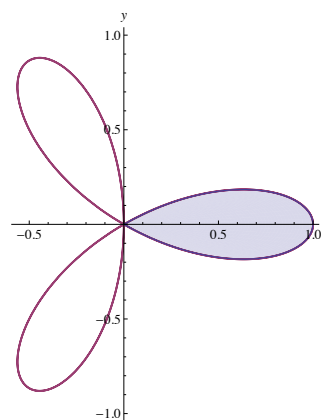
12.3.38 $\frac{1}{2} \int_0^{\pi/4} 9 \sin^2(2\theta) d\theta = \frac{9}{2} \int_0^{\pi/4} \sin^2(2\theta) d\theta =$

$\frac{9}{4} \left(\theta - \frac{\sin 2\theta \cos 2\theta}{2} \right) \Big|_0^{\pi/4} = \frac{9\pi}{16}.$ So the total area is $8 \cdot \frac{9\pi}{16} = \frac{9\pi}{2}.$



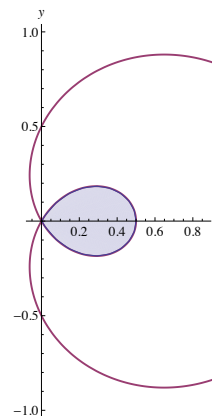
$2 \cdot \frac{1}{2} \int_0^{\pi/6} \cos^2 3\theta d\theta = \int_0^{\pi/6} \frac{1 + \cos 6\theta}{2} d\theta =$

12.3.39 $\left(\theta/2 + \frac{\sin 6\theta}{6} \right) \Big|_0^{\pi/6} = \frac{\pi}{12}.$



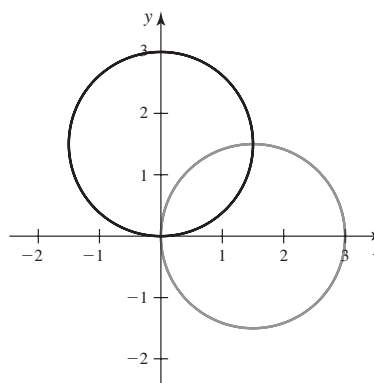
12.3.40

$$\begin{aligned}
 2 \cdot \frac{1}{2} \int_0^{\pi/3} (\cos \theta - 1/2)^2 d\theta &= \int_0^{\pi/3} (\cos^2 \theta - \\
 \cos \theta + 1/4) d\theta &= \int_0^{\pi/3} (\cos(2\theta)/2 - \cos \theta + \\
 3/4) d\theta &= (\sin(2\theta)/4 - \sin \theta + 3\theta/4) \Big|_0^{\pi/3} = \\
 \sqrt{3}/8 - \sqrt{3}/2 + \pi/4 &= \pi/4 - \frac{3\sqrt{3}}{8}.
 \end{aligned}$$



12.3.41

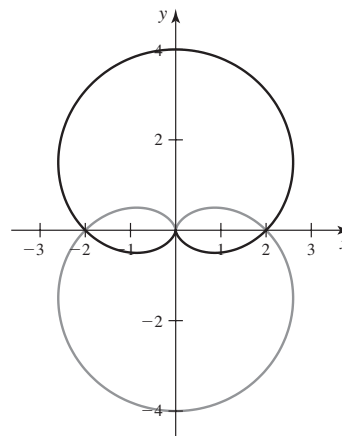
- a. These curves intersect when $\sin \theta = \cos \theta$, which occurs at $\theta = \pi/4$ and $\theta = 5\pi/4$, and when $r = 0$ which occurs for $\theta = 0$ and $\theta = \pi$ for the first curve and $\theta = \pi/2$ and $\theta = 3\pi/2$ for the second curve. Only two of these intersection points are unique: the origin and the point $(3\sqrt{2}/2, \pi/4) = (-3\sqrt{2}/2, 5\pi/4)$.



- b. By symmetry, we need to compute the area inside $r = 3 \sin \theta$ between 0 and $\pi/4$ and then double that result. We have $2 \cdot \frac{1}{2} \int_0^{\pi/4} 9 \sin^2 \theta d\theta = \frac{9}{2} \int_0^{\pi/4} (1 - \cos(2\theta)) d\theta = \frac{9}{2} (\theta - (1/2) \sin(2\theta)) \Big|_0^{\pi/4} = \frac{9}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{9}{8} (\pi - 2)$.

12.3.42

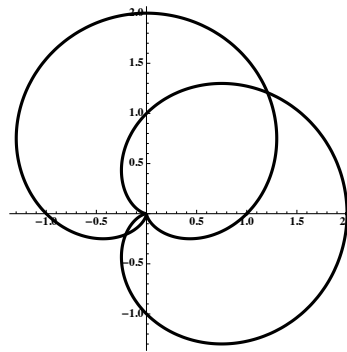
- a. The curves intersect where $2 + 2 \sin \theta = 2 - 2 \sin \theta$, which occurs when $\sin \theta = 0$. The curves also intersect at the origin, which occurs for the first curve at $\theta = 3\pi/2$ and for the second curve at $\pi/2$. The only points of intersection are the origin, $(2, 0)$ and $(2, \pi)$.



- b. By symmetry, we need to compute the area of the region inside $r = 2 - 2 \sin \theta$ between 0 and $\pi/2$ and then quadruple it. We have $4 \cdot \frac{1}{2} \int_0^{\pi/2} (2 - 2 \sin \theta)^2 d\theta = 8 \int_0^{\pi/2} (1 - 2 \sin \theta + \sin^2 \theta) d\theta = 8 \int_0^{\pi/2} ((3/2) - 2 \sin \theta - (1/2) \cos 2\theta) d\theta = (12\theta + 16 \cos \theta - 2 \sin(2\theta)) \Big|_0^{\pi/2} = 6\pi + 0 - 0 - (0 + 16 - 0) = 6\pi - 16$.

12.3.43

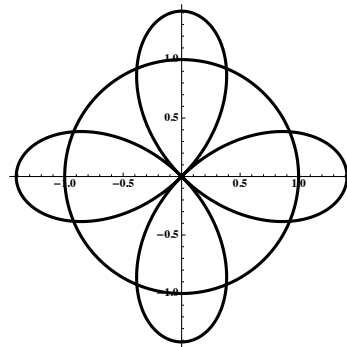
- The curves intersect when $\sin \theta = \cos \theta$, which occurs for $\theta = \pi/4$ and $\theta = 5\pi/4$. The corresponding points are $\left(\frac{2+\sqrt{2}}{2}, \frac{\pi}{4}\right)$ and $\left(\frac{2-\sqrt{2}}{2}, \frac{5\pi}{4}\right)$. They also intersect at the pole: the first curve is at the pole at $(0, \pi)$ and the other at $(0, 3\pi/2)$.



- b. By symmetry, we can compute the area between $\pi/4$ and $5\pi/4$ inside $r = 1 + \cos \theta$ and then double it. This will include both the bigger and smaller enclosed regions. We have $2 \cdot \frac{1}{2} \int_{\pi/4}^{5\pi/4} (1 + \cos \theta)^2 d\theta = \int_{\pi/4}^{5\pi/4} (1 + 2 \cos \theta + (1/2)(1 + \cos(2\theta))) d\theta = \int_{\pi/4}^{5\pi/4} ((3/2) + 2 \cos \theta + (1/2)(\cos 2\theta)) d\theta = (3\theta/2 + 2 \sin \theta + (1/4) \sin 2\theta) \Big|_{\pi/4}^{5\pi/4} = \left(\frac{15\pi}{8} - \sqrt{2} + \frac{1}{4}\right) - \left(\frac{3\pi}{8} + \sqrt{2} + \frac{1}{4}\right) = \frac{3\pi}{2} - 2\sqrt{2}$.

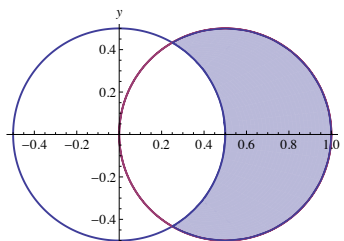
12.3.44

- These curves intersect when $\cos(2\theta) = \frac{\sqrt{2}}{2}$, which occurs when $2\theta = \pi/4, 7\pi/4, 9\pi/4, 15\pi/4$, so for $\theta = \pi/8, 7\pi/8, 9\pi/8, 15\pi/8$. The intersection points are thus $(1, \pi/8), (1, 3\pi/8), (1, 5\pi/8), (1, 7\pi/8), (1, 9\pi/8), (1, 11\pi/8), (1, 13\pi/8)$, and $(1, 15\pi/8)$.

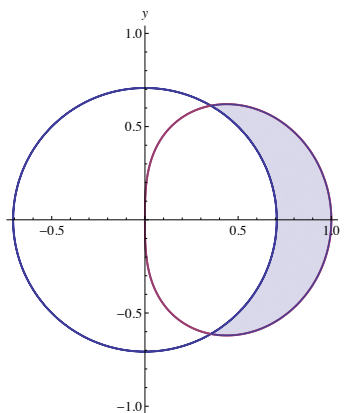


- b. By symmetry, we can compute the area between 0 and $\pi/8$ within the circle $r = 1$ and add it to the area between $\pi/8$ and $\pi/4$ within the curve $\sqrt{2} \cos 2\theta$ and then multiply this by 8. The area within the circle between 0 and $\pi/8$ is $(1/16)$ th the area of the circle, so this area is $\frac{\pi}{16}$. The area between $\pi/8$ and $\pi/4$ within $\sqrt{2} \cos 2\theta$ is given by $\frac{1}{2} \int_{\pi/8}^{\pi/4} 2 \cos^2 2\theta d\theta = \frac{1}{2} \int_{\pi/8}^{\pi/4} (1 + \cos 4\theta) d\theta = \frac{1}{2} (\theta + (1/4) \sin 4\theta) \Big|_{\pi/8}^{\pi/4} = \frac{1}{2} \left(\frac{\pi}{4} + 0 - \left(\frac{\pi}{8} + \frac{1}{4}\right)\right) = \frac{\pi}{16} - \frac{1}{8}$. Adding this to the previously computed area gives that $1/8$ of the total area is $\frac{\pi}{16} - \frac{1}{8} + \frac{\pi}{16} = \frac{\pi}{8} - \frac{1}{8}$. Thus the total area we are seeking is $\pi - 1$.

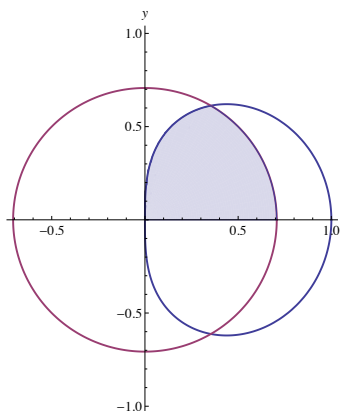
The area is given by $2 \cdot \frac{1}{2} \int_0^{\pi/3} (\cos^2 \theta - (1/2)^2) d\theta = \int_0^{\pi/3} (\cos(2\theta)/2 + \frac{1}{4}) d\theta = (\sin(2\theta)/4 + \theta/4) \Big|_0^{\pi/3} = \frac{\sqrt{3}}{8} + \frac{\pi}{12}.$



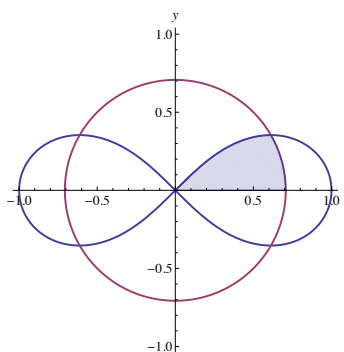
We have already computed the area inside $\sqrt{\cos \theta}$ to be 1. Now we must take away the portion of the circle with radius $1/\sqrt{2}$ between $\theta = -\pi/3$ and $\theta = \pi/3$. This is $1/3$ of a circle, so the area being removed is $(1/3)\pi(1/2) = \pi/6$. We must also remove the area of the regions inside $\sqrt{\cos \theta}$ between $-\pi/2$ and $-\pi/3$ and $\pi/3$ and $\pi/2$. These have area $2 \cdot \frac{1}{2} \int_{\pi/3}^{\pi/2} \cos \theta d\theta = (\sin \theta) \Big|_{\pi/3}^{\pi/2} = 1 - \sqrt{3}/2$. So the area is $1 - (\pi/6 + (1 - \sqrt{3}/2)) = \sqrt{3}/2 - \pi/6$.



The region inside the circle between 0 and $\pi/3$ is $1/6$ the area of a circle of radius $1/\sqrt{2}$ so it has area $(1/6)\pi(1/2) = \pi/12$. The rest of the area is represented by $\frac{1}{2} \int_{\pi/3}^{\pi} \cos \theta d\theta = \frac{1}{2} (\sin \theta) \Big|_{\pi/3}^{\pi} = \frac{1}{2} (1 - \sqrt{3}/2)$. The total area is therefore $\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$.

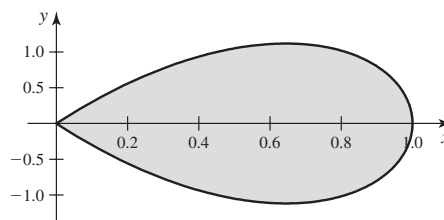


The region inside the circle between 0 and $\pi/6$ is $1/12$ the area of a circle of radius $1/\sqrt{2}$ so it has area $(1/12)\pi(1/2) = \pi/24$. The rest of the area is represented by $\frac{1}{2} \int_{\pi/6}^{\pi/4} \cos 2\theta d\theta = \frac{1}{4} (\sin 2\theta) \Big|_{\pi/6}^{\pi/4} = \frac{1}{4} (1 - \sqrt{3}/2)$. The total area is thus $\frac{\pi}{24} + \frac{1}{4} - \frac{\sqrt{3}}{8}$.



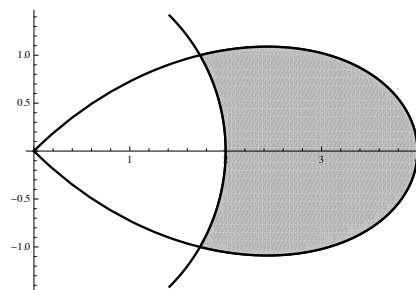
12.3.49

Using symmetry, we compute the area of $1/2$ of one leaf, and then double it. We have $A = \frac{1}{2} \int_0^{\pi/10} \cos^2(5\theta) d\theta = \frac{1}{10} \int_0^{\pi/2} \cos^2 u du = \frac{1}{10} \left(\frac{1}{2}u + \frac{\cos u \sin u}{2} \right) \Big|_0^{\pi/2} = \frac{\pi}{40}$. So the area of one leaf is $2 \cdot \frac{\pi}{40} = \frac{\pi}{20}$.



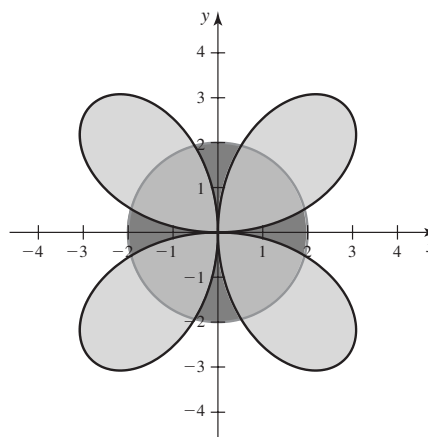
12.3.50

The curves intersect where $4 \cos 2\theta = 2$, or $\theta = \pi/6$. By symmetry, we can compute the area of $1/2$ of the tip of one leaf, and then multiply by 8. The area of $1/2$ of the tip of one leaf is given by $\frac{1}{2} \int_0^{\pi/6} (4 \cos(2\theta))^2 - 4 d\theta = \int_0^{\pi/6} (8 \cos^2(2\theta) - 4) d\theta = (4\theta + \sin(4\theta) - 2\theta) \Big|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$. Thus the total area desired is $8 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{8\pi}{3} + 4\sqrt{3}$.



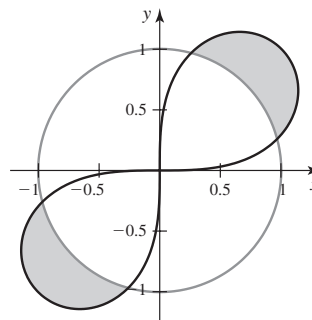
12.3.51

Note that the area inside one leaf of the rose but outside the circle is given by $\frac{1}{2} \int_{\pi/12}^{5\pi/12} (16 \sin^2(2\theta) - 4) d\theta = (2\theta - \sin(4\theta)) \Big|_{\pi/12}^{5\pi/12} = \sqrt{3} + \frac{2\pi}{3}$. Also, the area inside one leaf of the rose is $\frac{1}{2} \int_0^{\pi/2} 16 \sin^2(2\theta) d\theta = (4\theta - \sin(4\theta)) \Big|_0^{\pi/2} = 2\pi$. Thus the area inside one leaf of the rose and inside the circle must be $2\pi - (\sqrt{3} + \frac{2\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$, and the total area inside the rose and inside the circle must be $4(\frac{4\pi}{3} - \sqrt{3}) = \frac{16\pi}{3} - 4\sqrt{3}$.



The curves intersect for $2 \sin(2\theta) = 1$, which occurs in the first quadrant at $\theta = \pi/12$ and $\theta = 5\pi/12$. So one half of the total desired

12.3.52 area is given by $\frac{1}{2} \int_{\pi/12}^{5\pi/12} (2 \sin(2\theta) - 1) d\theta = \frac{1}{2} (-\cos(2\theta) - \theta) \Big|_{\pi/12}^{5\pi/12} = -\frac{1}{2} \left(-\sqrt{3} + \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$. So the total desired area is $\sqrt{3} - \frac{\pi}{3}$.



12.3.53 The circles intersect for $\theta = \pi/6$ and $\theta = 5\pi/6$.

The area inside $r = 2 \sin \theta$ but outside of $r = 1$ would be given by $\frac{1}{2} \int_{\pi/6}^{5\pi/6} (4 \sin^2 \theta - 1) d\theta = \frac{1}{2} (x - \sin(2x)) \Big|_{\pi/6}^{5\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$. The total area of $r = 2 \sin \theta$ is π . Thus, the area inside both circles is $\pi - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

12.3.54 The inner loop is traced from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. So the area is given by $\frac{1}{2} \int_{2\pi/3}^{4\pi/3} (2 + 4 \cos \theta)^2 d\theta = \int_{2\pi/3}^{4\pi/3} (2 + 8 \cos \theta + 8 \cos^2 \theta) d\theta = (2\theta + 8 \sin \theta + 4\theta + 2 \sin(2\theta)) \Big|_{2\pi/3}^{4\pi/3} = 4\pi - 6\sqrt{3}$.

12.3.55 The inner loop is traced out between $\theta = \pi/6$ and $\theta = 5\pi/6$, so its area is given by $\frac{1}{2} \int_{\pi/6}^{5\pi/6} (3 - 6 \sin \theta)^2 d\theta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (9 - 36 \sin \theta + 36 \sin^2 \theta) d\theta = \frac{3}{2} (3\theta + 12 \cos \theta + 6\theta - 3 \sin(2\theta)) \Big|_{\pi/6}^{5\pi/6} = 9\pi - \frac{27\sqrt{3}}{2}$.

We can determine the area inside the outer loop by using symmetry and doubling the area of the region traced out between $5\pi/6$ and $3\pi/2$. Thus the area inside the outer region is $2 \cdot \frac{1}{2} \int_{5\pi/6}^{3\pi/2} (3 - 6 \sin \theta)^2 d\theta = 3 (3\theta + 12 \cos \theta + 6\theta - 3 \sin(2\theta)) \Big|_{5\pi/6}^{3\pi/2} = 18\pi + \frac{27\sqrt{3}}{2}$. So the area outside the inner loop and inside the outer loop is $18\pi + \frac{27\sqrt{3}}{2} - \left(9\pi - \frac{27\sqrt{3}}{2} \right) = 9\pi + 27\sqrt{3}$.

12.3.56 The curves intersect at $\theta = \pi/3$, and using symmetry, the area we seek is $2 \cdot \frac{1}{2} \int_0^{\pi/3} (1 + \cos \theta)^2 d\theta + 2 \cdot \frac{1}{2} \int_{\pi/3}^{\pi/2} (3 \cos \theta)^2 d\theta = \int_0^{\pi/3} (1 + 2 \cos \theta + \cos^2 \theta) d\theta + \int_{\pi/3}^{\pi/2} 9 \cos^2 \theta d\theta = \left(\theta + 2 \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \Big|_0^{\pi/3} + \left(\frac{9\theta}{2} + \frac{9 \sin 2\theta}{4} \right) \Big|_{\pi/3}^{\pi/2} = \frac{\pi}{2} + 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} + \frac{9\pi}{4} - \left(\frac{3\pi}{2} + \frac{9\sqrt{3}}{8} \right) = \frac{5\pi}{4}$.

12.3.57 One half of the area is given by $\frac{1}{2} \int_0^{\pi/2} 6 \sin 2\theta d\theta = -\frac{3}{2} \cos 2\theta \Big|_0^{\pi/2} = 3$, so the total area is 6.

12.3.58 By symmetry, we can compute the area between $\theta = 5\pi/6$ and $\theta = 3\pi/2$ and double it. Thus, the total area we seek is given by $\int_{5\pi/6}^{3\pi/2} (2 - 4\sin\theta)^2 d\theta = \int_{5\pi/6}^{3\pi/2} (4 - 16\sin\theta + 16\sin^2\theta) d\theta =$
 $(4\theta + 16\cos\theta + 8\theta - 4\sin(2\theta)) \Big|_{5\pi/6}^{3\pi/2} = 6\sqrt{3} + 8\pi.$

12.3.59 The area is given by

$$\frac{1}{2} \int_0^{2\pi} (4 - 2\cos\theta)^2 d\theta = \int_0^{2\pi} (8 - 8\cos\theta + 2\cos^2\theta) d\theta = \left(8\theta - 8\sin\theta + \theta + \frac{1}{2}\sin(2\theta) \right) \Big|_0^{2\pi} = 18\pi.$$

12.3.60 The area of one half of one leaf is $\frac{1}{2} \int_0^{\pi/6} 4 \cdot \cos^2(3\theta) d\theta = \left(\theta + \frac{\sin(6\theta)}{6} \right) \Big|_0^{\pi/6} = \frac{\pi}{6}.$ So the area of all 6 half-leaves is $\pi.$

12.3.61 Solving $4 - 2\cos\theta = 2 + 2\cos\theta$, we have $\cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}.$

The area of region A is

$$\int_0^{\pi/3} \frac{1}{2} ((2 + 2\cos\theta)^2 - (4 - 2\cos\theta)^2) d\theta = \frac{1}{2} \int_0^{\pi/3} (24\cos t - 12) dt = \frac{1}{2} (24\sin t - 12t) \Big|_0^{\pi/3} = 6\sqrt{3} - 2\pi.$$

The area of region B is

$$\begin{aligned} & \int_0^{\pi/3} \frac{1}{2} (4 - 2\cos\theta)^2 d\theta + \int_{\pi/3}^{\pi} \frac{1}{2} (2 + 2\cos\theta)^2 d\theta \\ &= \int_0^{\pi/3} (8 - 8\cos\theta + 2\cos^2\theta) d\theta + \int_{\pi/3}^{\pi} (2 + 4\cos\theta + 2\cos^2\theta) d\theta \\ &= \int_0^{\pi/3} (9 - 8\cos\theta + \cos 2\theta) d\theta + \int_{\pi/3}^{\pi} (3 + 4\cos\theta + \cos 2\theta) d\theta \\ &= \left(9\theta - 8\sin\theta + \frac{1}{2}\sin 2\theta \right) \Big|_0^{\pi/3} + \left(3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta \right) \Big|_{\pi/3}^{\pi} \\ &= 3\pi - \frac{15\sqrt{3}}{4} + 2\pi - \frac{9\sqrt{3}}{4} \\ &= 5\pi - 6\sqrt{3}. \end{aligned}$$

The area of region C is

$$\int_{\pi/3}^{\pi} \frac{1}{2} ((4 - 2\cos\theta)^2 - (2 + 2\cos\theta)^2) d\theta = \int_{\pi/3}^{\pi} (6 - 12\cos\theta) d\theta = (6\theta - 12\sin\theta) \Big|_{\pi/3}^{\pi} = 4\pi + 6\sqrt{3}.$$

12.3.62 Solving $4 + \cos 2\theta = 3\cos 2\theta$ implies that $\cos 2\theta = 2$, which has no solutions. However, observe that if $\theta = \frac{\pi}{2}$, then $r = 4 + \cos 2\theta = 3$, and if $\theta = \frac{\pi}{2}$, $r = 3\cos 2\theta = -3$. So the curves intersect at the points $(3, \pm\frac{\pi}{2})$. Because the intersection points of the two curves correspond to different values of θ , we have to be careful when setting up the area integral. From the figure, we can see that the shaded area equals the area in the first quadrant that is inside the curve $r = 4 + \cos 2\theta$ minus the area of one leaf of $r = 3\cos 2\theta$.

This equals

$$\begin{aligned}
 & \int_0^{\pi/2} \frac{1}{2} (4 + \cos 2\theta)^2 d\theta - 2 \int_0^{\pi/4} \frac{1}{2} (3 \cos 2\theta)^2 d\theta \\
 &= \int_0^{\pi/2} \left(8 + 4 \cos 2\theta + \frac{1}{4} + \frac{\cos 4\theta}{4} \right) d\theta - 9 \int_0^{\pi/4} \frac{1 + \cos 4\theta}{2} d\theta \\
 &= \left(\frac{33\theta}{4} + 2 \sin 2\theta + \frac{\sin 4\theta}{16} \right) \Big|_0^{\pi/2} - 9 \left(\frac{\theta}{2} + \frac{\sin 4\theta}{8} \right) \Big|_0^{\pi/4} \\
 &= \frac{33\pi}{8} - \frac{9\pi}{8} = 3\pi.
 \end{aligned}$$

12.3.63 Note that the diameter of the circle is a , and that the complete circle is traversed for $0 \leq t \leq \pi$.

$$L = \int_0^\pi \sqrt{(a \sin \theta)^2 + (a \cos \theta)^2} d\theta = \int_0^\pi a d\theta = \pi a.$$

12.3.64 $L = \int_0^{2\pi} \sqrt{(2 - 2 \sin \theta)^2 + 4 \cos^2 \theta} d\theta = \int_0^{2\pi} \sqrt{8 - 8 \sin \theta} d\theta$. By symmetry, this is

$$2\sqrt{8} \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin \theta} \cdot \frac{\sqrt{1 + \sin \theta}}{\sqrt{1 + \sin \theta}} d\theta = 2\sqrt{8} \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{\sqrt{1 + \sin \theta}} d\theta.$$

Let $u = 1 + \sin \theta$ so that $du = \cos \theta d\theta$. Then we have $L = 2\sqrt{8} \int_0^2 u^{-1/2} du = 4\sqrt{8} \cdot \sqrt{u} \Big|_0^2 = 4\sqrt{8} \cdot \sqrt{2} = 16$.

12.3.65 $L = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} d\theta = \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta$. Let $u = \theta^2 + 4$, so that $du = 2\theta d\theta$. Substituting gives

$$\frac{1}{2} \int_4^{4\pi^2+4} \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} \left(u^{3/2} \right) \Big|_4^{4\pi^2+4} = \frac{1}{3} \left(8(\pi^2 + 1)^{3/2} - 8 \right) = \frac{8}{3} \left((\pi^2 + 1)^{3/2} - 1 \right).$$

12.3.66 $L = \int_0^{2\pi n} \sqrt{e^{2\theta} + e^{2\theta}} d\theta = \sqrt{2} \int_0^{2\pi n} e^\theta d\theta = \sqrt{2} \cdot e^\theta \Big|_0^{2\pi n} = \sqrt{2}(e^{2\pi n} - 1)$.

12.3.67

$$\begin{aligned}
 L &= 2 \int_{-\pi/2}^{\pi/2} \sqrt{(4 + 4 \sin \theta)^2 + (4 \cos \theta)^2} d\theta = 8 \int_{-\pi/2}^{\pi/2} \sqrt{2 + 2 \sin \theta} d\theta = 8\sqrt{2} \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin \theta} \cdot \frac{\sqrt{1 - \sin \theta}}{\sqrt{1 - \sin \theta}} d\theta \\
 &= 8\sqrt{2} \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{\sqrt{1 - \sin \theta}} d\theta.
 \end{aligned}$$

Let $u = 1 - \sin \theta$, so that $du = -\cos \theta d\theta$. Then we have

$$L = 8\sqrt{2} \int_0^2 u^{-1/2} du = 16\sqrt{2} (\sqrt{2} - 0) = 32.$$

12.3.68 $L = \int_0^6 \sqrt{(4\theta^2)^2 + 64\theta^2} d\theta = \int_0^6 4\theta \sqrt{\theta^2 + 4} d\theta$. Let $u = \theta^2 + 4$ so that $du = 2\theta d\theta$. Then $L =$

$$2 \int_4^{40} u^{1/2} du = \frac{4}{3} \cdot u^{3/2} \Big|_4^{40} = \frac{4}{3} (40^{3/2} - 8) = \frac{32}{3} (10\sqrt{10} - 1).$$

12.3.69 $L = \int_0^{\ln 8} \sqrt{4e^{4\theta} + 16e^{4\theta}} d\theta = 2\sqrt{5} \int_0^{\ln 8} e^{2\theta} d\theta = \sqrt{5} \cdot e^{2\theta} \Big|_0^{\ln 8} = \sqrt{5}(64 - 1) = 63\sqrt{5}$.

12.3.70

$$\begin{aligned}
 L &= \int_0^{\pi/2} \sqrt{\frac{2}{(1+\cos\theta)^2} + \left(\frac{\sqrt{2}\sin\theta}{(1+\cos\theta)^2}\right)^2} d\theta = \int_0^{\pi/2} \sqrt{\frac{2(1+\cos\theta)^2 + 2\sin^2\theta}{(1+\cos\theta)^4}} d\theta \\
 &= \int_0^{\pi/2} \sqrt{\frac{4+4\cos\theta}{(1+\cos\theta)^4}} d\theta = 2 \int_0^{\pi/2} (1+\cos\theta)^{-3/2} d\theta.
 \end{aligned}$$

Now note that $(\sqrt{1+\cos\theta})^{-3} = (\sqrt{2}\cos(\theta/2))^{-3}$, so $L = \frac{2}{2\sqrt{2}} \int_0^{\pi/2} \sec^3(\theta/2) d\theta = \frac{2}{\sqrt{2}} \int_0^{\pi/4} \sec^3 u du = \frac{1}{\sqrt{2}} (\sec u \tan u + \ln |\sec u + \tan u|) \Big|_0^{\pi/4} = \frac{\sqrt{2}}{2} (\sqrt{2} + \ln(\sqrt{2} + 1)) = 1 + \frac{\sqrt{2}}{2} \ln(\sqrt{2} + 1).$

$$\begin{aligned}
 \text{12.3.71 } L &= \int_0^{\pi/2} \sqrt{\sin^6(\theta/3) + \sin^4(\theta/3) \cos^2(\theta/3)} d\theta = \int_0^{\pi/2} \sin^2(\theta/3) d\theta = \frac{1}{2} \int_0^{\pi/2} 1 - \cos(2\theta/3) d\theta = \\
 &\frac{1}{2} \left(\theta - \frac{3}{2} \sin(2\theta/3) \right) \Big|_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - \frac{3\sqrt{3}}{4} \right) = \frac{2\pi - 3\sqrt{3}}{8}.
 \end{aligned}$$

$$\text{12.3.72 } \int_0^{\pi} \sqrt{4\cos^2(3\theta) + 36\sin^2(3\theta)} d\theta = \int_0^{\pi} 2\sqrt{1+8\sin^2\theta} d\theta \approx 13.36.$$

$$\text{12.3.73 } L = \int_0^{2\pi} \sqrt{(4-2\cos\theta)^2 + 4\sin^2\theta} d\theta = \int_0^{2\pi} \sqrt{20-16\cos\theta} d\theta = 26.73.$$

12.3.74 Using symmetry, we are seeking 4 times the curve traversed for $0 \leq \theta \leq \pi/4$. Thus, $L = 4 \int_0^{\pi/4} \sqrt{6\sin 2\theta + 6\cot 2\theta \cos 2\theta} d\theta = 12.85.$

12.3.75

a. False. The area is given by $\frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta.$

b. False. The slope is given by $\frac{dy}{dx}$, which can be computed using the formula

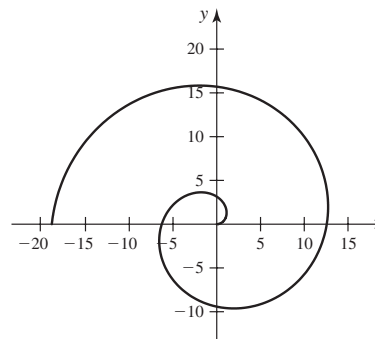
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

c. True. For example, see number 23 above.

12.3.76 An equation of the unit circle in polar coordinates is $r = 1$ and the equation the vertical line in polar coordinates is $r \cos \theta = \frac{\sqrt{2}}{2}$, or $r = \frac{\sqrt{2}}{2} \sec \theta$. So the area is $2 \int_0^{\pi/4} \frac{1}{2} \left(1 - \frac{1}{2} \sec^2 \theta \right) d\theta = \left(\theta - \frac{1}{2} \tan \theta \right) \Big|_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2}.$

12.3.77

The first horizontal tangent line is at the origin. The next is at approximately (4.0576, 2.0288), and the third at approximately (9.8262, 4.9131). The first vertical tangent line is at approximately (1.7206, 0.8603), the next is at about (6.8512, 3.4256), and the next at approximately (12.8746, 6.4373).



12.3.78

a. $L = \int_0^{\sqrt{8}} \sqrt{16\theta^2 + 16} d\theta = 4 \int_0^{\sqrt{8}} \sqrt{\theta^2 + 1} d\theta \approx 20.5.$

b. $L(\theta) = 4 \int_0^{\theta} \sqrt{\theta^2 + 1} d\theta = 2(\theta\sqrt{1 + \theta^2} + \ln(\theta + \sqrt{\theta^2 + 1})).$

c. By the Fundamental Theorem of Calculus, $L'(\theta) = 4\sqrt{\theta^2 + 1} > 0$. $L''(\theta) = \frac{4\theta}{\sqrt{1 + \theta^2}} \geq 0$ if $\theta \geq 0$. The arc length is increasing at an increasing rate.

12.3.79 $r'(\theta) = -ae^{-a\theta}$. Thus,

$$\begin{aligned} L &= \int_0^{\infty} \sqrt{e^{-2a\theta} + a^2 e^{-2a\theta}} d\theta = \sqrt{1 + a^2} \int_0^{\infty} e^{-a\theta} d\theta \\ &= \lim_{b \rightarrow \infty} \sqrt{1 + a^2} \int_0^b e^{-a\theta} d\theta = \frac{\sqrt{1 + a^2}}{-a} \lim_{b \rightarrow \infty} e^{-a\theta} \Big|_0^b \\ &= \frac{\sqrt{1 + a^2}}{-a} (0 - 1) = \frac{\sqrt{1 + a^2}}{a}. \end{aligned}$$

12.3.80

a. The area of one half of one leaf is $\frac{1}{2} \int_0^{\pi/(4m)} \cos^2(2m\theta) d\theta = \left(\frac{\theta}{4} + \frac{\sin(4m\theta)}{16m} \right) \Big|_0^{\pi/(4m)} = \frac{\pi}{16m}$. So the area of all $8m$ half-leaves is $\frac{\pi}{2}$.

b. The area of one half of one leaf is

$$\frac{1}{2} \int_0^{\pi/(4m+2)} \cos^2((2m+1)\theta) d\theta = \left(\frac{\theta}{4} + \frac{\sin(2(2m+1)\theta)}{4(4m+2)} \right) \Big|_0^{\pi/(4m+2)} = \frac{\pi}{8 \cdot (2m+1)}.$$

So the area of all $2(2m+1)$ half-leaves is $\frac{\pi}{4}$.

12.3.81

a. $A_n = \frac{1}{2} \int_{(2n-2)\pi}^{(2n-1)\pi} e^{-2\theta} d\theta - \frac{1}{2} \int_{2n\pi}^{(2n+1)\pi} e^{-2\theta} d\theta = -\frac{1}{4} e^{-(4n-2)\pi} + \frac{1}{4} e^{-(4n-4)\pi} + \frac{1}{4} e^{-(4n+2)\pi} - \frac{1}{4} e^{-4n\pi}.$

b. Each term tends to 0 as $n \rightarrow \infty$ so $\lim_{n \rightarrow \infty} A_n = 0$.

c. $\frac{A_{n+1}}{A_n} = \frac{e^{-(4n+2)\pi} + e^{-(4n)\pi} + e^{-(4n+6)\pi} - e^{-(4n+4)\pi}}{e^{-(4n-2)\pi} + e^{-(4n-4)\pi} + e^{-(4n+2)\pi} - e^{-4n\pi}} = e^{-4\pi}$, so $\lim_{n \rightarrow \infty} \frac{A_{n+1}}{A_n} = e^{-4\pi}.$

12.3.82

- a. The slope of the line tangent to $r = f(\theta)$ at P is $\left. \frac{dy}{dx} \right|_P$. Also, the slope of a line intersecting the x -axis at an angle α is $\tan \alpha$. (Note that in the picture, $\tan(\pi - \alpha) = -\tan(\alpha) = \frac{\text{rise}}{-\text{run}} = -\text{slope of the tangent line.}$)
- b. Draw a vertical line through P and let Q be the point where this line intersects the x -axis. Then in triangle OPQ we see $\tan \theta = \frac{y}{x}$.
- c. Note that

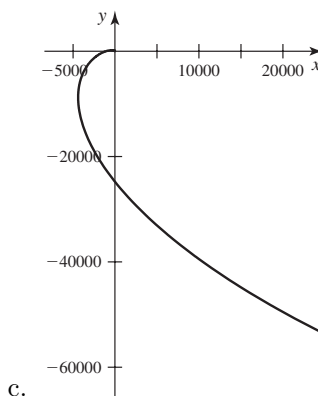
$$\frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} = \frac{\tan \theta + \frac{f(\theta)}{f'(\theta)}}{1 - \frac{f(\theta)}{f'(\theta)} \tan \theta} = \tan \alpha.$$

Because $\alpha = \varphi + \theta$ and $\tan(\varphi + \theta) = \frac{\tan \theta + \tan \varphi}{1 - \tan \varphi \tan \theta}$, we see that $\tan \varphi = \frac{f(\theta)}{f'(\theta)}$.

- d. l is parallel to the x -axis when $\frac{dy}{dx} = 0$, or when $f'(\theta) \sin \theta + f(\theta) \cos \theta = 0$, hence if $\tan \theta = -\frac{f(\theta)}{f'(\theta)}$.
- e. l is parallel to the y -axis when $\frac{dx}{dy} = 0$, which occurs when $f'(\theta) \cos \theta - f(\theta) \sin \theta = 0$, hence if $\tan \theta = \frac{f(\theta)}{f'(\theta)}$.

12.3.83

- a. If $\cot \varphi = \frac{f'(\theta)}{f(\theta)}$ is constant for all θ , then $\varphi = \cot^{-1} \left(\frac{f'(\theta)}{f(\theta)} \right)$ is constant. Then $\frac{d}{d\theta} \ln(f(\theta)) = \frac{1}{f(\theta)} \cdot f'(\theta) = \cot \varphi$ is constant.
- b. If $f(\theta) = Ce^{k\theta}$, then $\cot \varphi = \frac{f'(\theta)}{f(\theta)} = \frac{kCe^{k\theta}}{Ce^{k\theta}} = k$.



12.3.84 If $r = f(\theta)$, then parametrically we have $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$. Note that $\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta$ and $\frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta$. Thus

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(f'(\theta) \cos \theta - f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta + f(\theta) \cos \theta)^2},$$

which can be written

$$\sqrt{(f'(\theta)\cos\theta)^2 + (f'(\theta)\sin\theta)^2 + (f(\theta)\sin\theta)^2 + (f(\theta)\cos\theta)^2} = \sqrt{(f'(\theta))^2 + (f(\theta))^2}.$$

Thus $L = \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2} d\theta$ as desired.

12.3.85 Suppose that the goat is tethered at the origin, and that the center of the corral is $(1, \pi)$. The circle that the goat can graze is $r = a$, and the corral is given by $r = -2\cos\theta$. The intersection occurs for $\theta = \cos^{-1}(-a/2)$.

The area grazed by the goat is twice the area of the sector of the circle $r = a$ between $\cos^{-1}(-a/2)$ and π , plus twice the area of the circle $r = -2\cos\theta$ between $\pi/2$ and $\cos^{-1}(-a/2)$. Thus we need to compute

$$A = \int_{\cos^{-1}(-a/2)}^{\pi} a^2 d\theta + \int_{\pi/2}^{\cos^{-1}(-a/2)} 4\cos^2\theta d\theta = a^2\pi - a^2\cos^{-1}(-a/2) + (2\cos\theta\sin\theta + 2\theta) \Big|_{\pi/2}^{\cos^{-1}(-a/2)} =$$

$$a^2(\pi - \cos^{-1}(-a/2)) - \pi - \frac{1}{2}a\sqrt{4-a^2} + 2\cos^{-1}(-a/2).$$

Note that $\pi - \cos^{-1}(-a/2) = \cos^{-1}(a/2)$, so this can be written as $(a^2 - 2)\cos^{-1}(a/2) + \pi - \frac{1}{2}a\sqrt{4-a^2}$. Note that for $a = 0$ this is 0, and for $a = 2$, this is π , as desired.

12.3.86 Imagine that the boundary of the concrete slab is the fence from the previous problem. Then the area the goat could graze in the previous problem becomes the area it can't graze in this problem. If the slab weren't there, the goat could graze a region of area πa^2 . Thus, the goat can graze a region of area $\pi a^2 - \left((a^2 - 2)\cos^{-1}(a/2) + \pi - \frac{1}{2}a\sqrt{4-a^2}\right) = \pi(a^2 - 1) + \frac{1}{2}a\sqrt{4-a^2} + (2 - a^2)\cos^{-1}(a/2)$. If $a = 0$, this quantity is 0, while if $a = 2$, this quantity is 3π .

12.3.87 Again, suppose that the goat is tethered at the origin, and that the center of the corral is $(1, \pi)$. The equation of the corral fence is given by $r = -2\cos\theta$. Note that to the right of the vertical line $\theta = \pi/2$, the goat can graze a half-circle of area $\pi a^2/2$. Also, there is a region in the 2nd quadrant and one in the 3rd quadrant of equal size that can also be grazed. Let this region have area A , so that the total area grazed will then be $\frac{\pi a^2}{2} + 2A$.

Imagine that the goat is walking "west" from the polar point $(a, \pi/2)$, and is keeping the rope taut until his whole rope is along the fence in the third quadrant. Let φ be the central angle from the origin to the polar point $(1, \pi)$ to the point on the fence that the goat's rope is touching as he makes this walk. When the goat is at $(a, \pi/2)$, we have $\varphi = 0$. When the goat is all the way to the fence, we have $\varphi = a$. Then length of the rope not along the fence is $a - \varphi$. Thus, the value of A is $\frac{1}{2} \int_0^a (a - \varphi)^2 d\varphi = \frac{1}{2} \left(a^2\varphi - a\varphi^2 + \frac{\varphi^3}{3} \right) \Big|_0^a = \frac{a^3}{6}$.

Thus, the goat can graze a region of area $\frac{\pi a^2}{2} + \frac{a^3}{3}$.

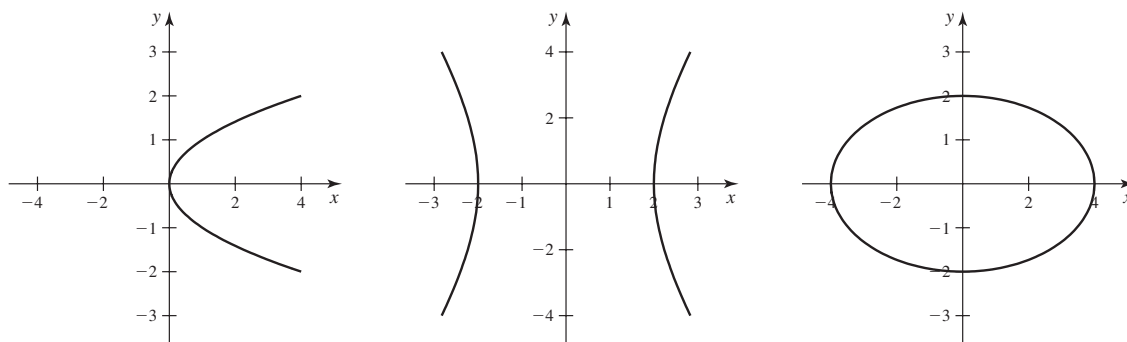
12.4 Conic Sections

12.4.1 A parabola is the set of points in the plane which are equidistant from a given fixed point and a given fixed line.

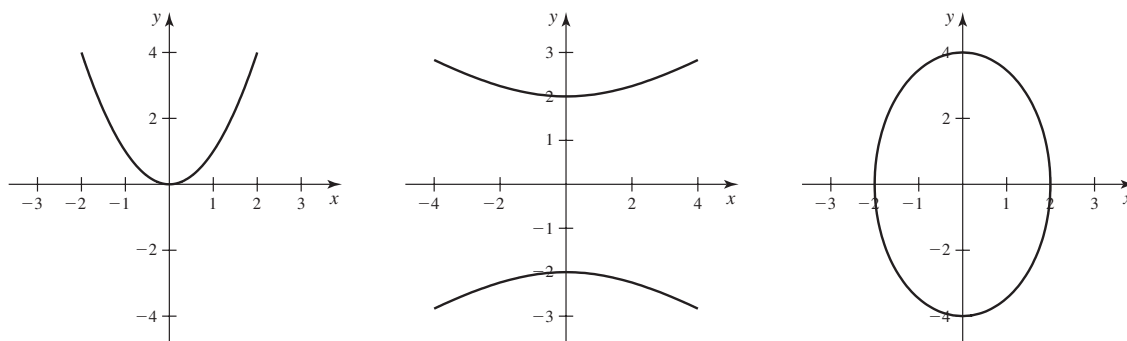
12.4.2 An ellipse is the set of points in the plane with the property that the sum of the distances from the point to two given fixed points is a given constant.

12.4.3 A hyperbola is the set of points in the plane with the property that the difference of the distances from the point to two given fixed points is a given constant.

12.4.4



12.4.5



12.4.6 $x^2 = 4py$, where $p < 0$.

12.4.7 $\left(\frac{x}{a}\right)^2 + \frac{y^2}{a^2 - c^2} = 1$.

12.4.8 $\left(\frac{y}{a}\right)^2 - \frac{x^2}{c^2 - a^2} = 1$.

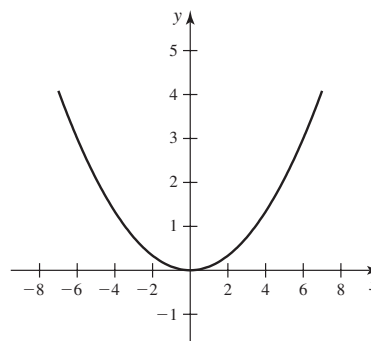
12.4.9 The foci for both are $(\pm ae, 0)$.

12.4.10 This is given by $r = \frac{ed}{1 + e \cos \theta}$, $-\pi < \theta < \pi$.

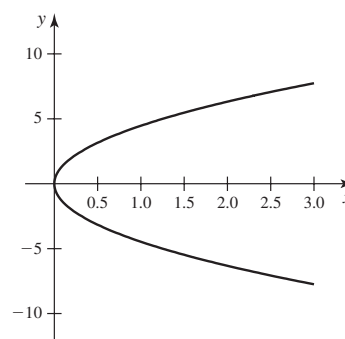
12.4.11 The asymptotes are $y = -\frac{b}{a} \cdot x$ and $y = \frac{b}{a} \cdot x$.

12.4.12 If $e = 1$, the conic section is a parabola. If $e > 1$, it is a hyperbola. If $0 < e < 1$, it is an ellipse.

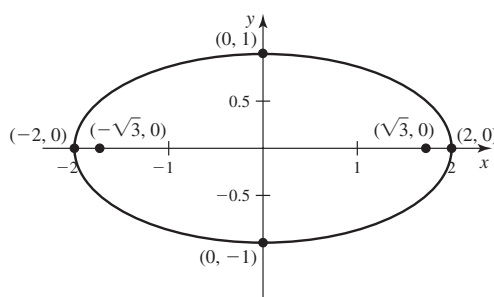
- 12.4.13** This is a parabola. Directrix: $y = -3$. Focus: $(0, 3)$.



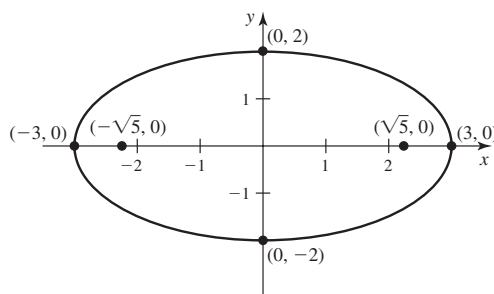
- 12.4.14** This is a parabola. Directrix: $x = -5$. Focus: $(5, 0)$.



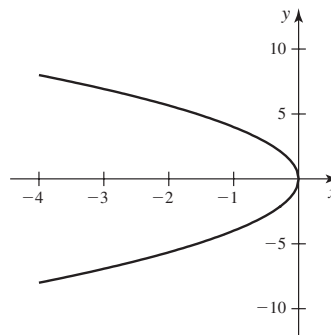
- 12.4.15** This is an ellipse. Vertices are $(\pm 2, 0)$, and the foci are $(\pm\sqrt{3}, 0)$. The major axis has length 4 and the minor axis has length 2.



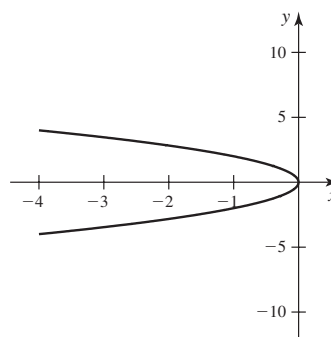
- 12.4.16** This is an ellipse. Vertices are $(\pm 3, 0)$, and the foci are $(\pm\sqrt{5}, 0)$. The major axis has length 6 and the minor axis has length 4.



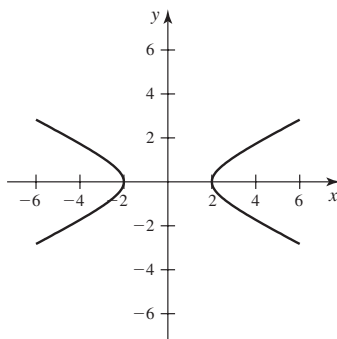
- 12.4.17** This is a parabola. Directrix: $x = 4$. Focus: $(-4, 0)$.



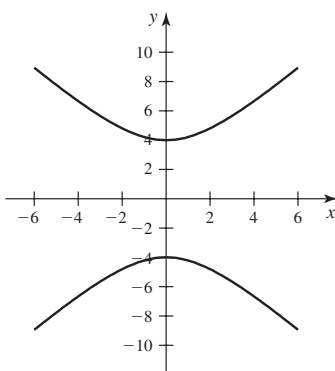
- 12.4.18** This is a parabola. Directrix: $x = 1$. Focus: $(-1, 0)$.



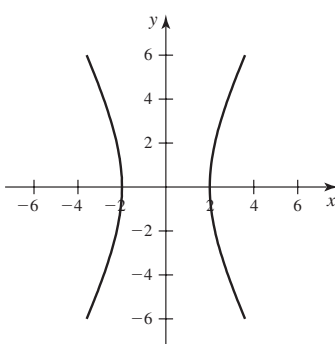
- 12.4.19** This is a hyperbola. The vertices are $(\pm 2, 0)$, and the foci are $(\pm\sqrt{5}, 0)$. The asymptotes are $y = \frac{\pm 1}{2} \cdot x$.



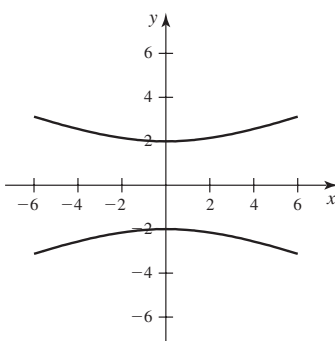
- 12.4.20** This is a hyperbola. The vertices are $(0, \pm 4)$, and the foci are $(0, \pm 5)$. The asymptotes are $y = \frac{\pm 4}{3} \cdot x$.



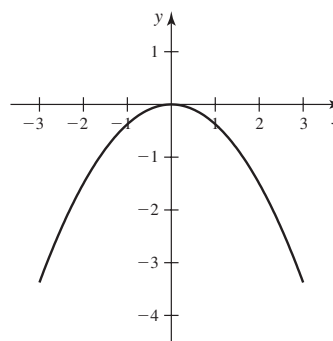
12.4.21 This is a hyperbola. The vertices are $(\pm 2, 0)$, and the foci are $(\pm 2\sqrt{5}, 0)$. The asymptotes are $y = \pm 2x$.



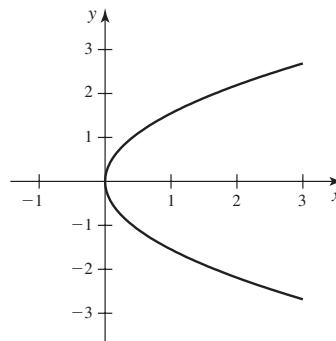
12.4.22 This is a hyperbola. The vertices are $(0, \pm 2)$, and the foci are $(0, \pm 29)$. The asymptotes are $y = \frac{\pm 2}{5} \cdot x$.



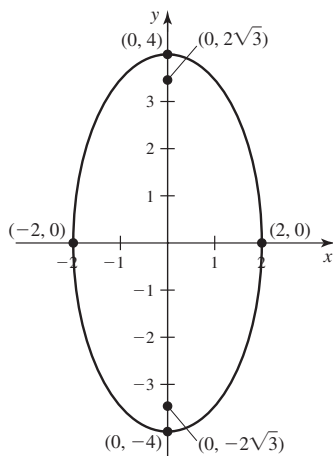
12.4.23 This is a parabola. Directrix: $y = \frac{2}{3}$. Focus: $\left(0, -\frac{2}{3}\right)$.



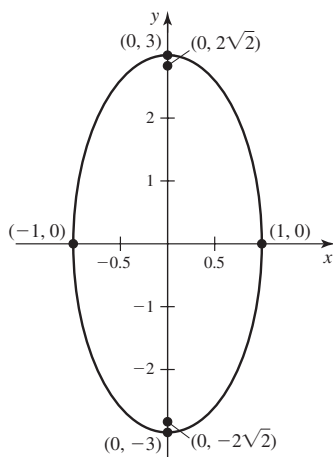
- 12.4.24** This is a parabola. Directrix: $x = -\frac{3}{5}$. Focus: $\left(\frac{3}{5}, 0\right)$.



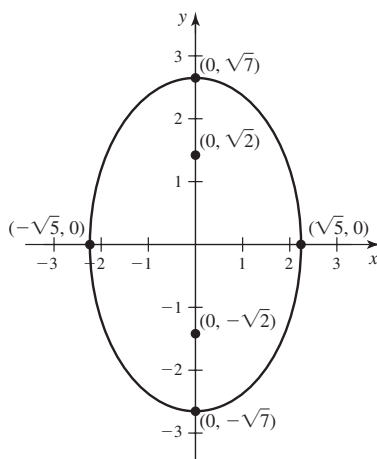
- 12.4.25** This is an ellipse. Vertices are $(0, \pm 4)$, and the foci are $(0, \pm 2\sqrt{3})$. The major axis has length 8 and the minor axis has length 4.



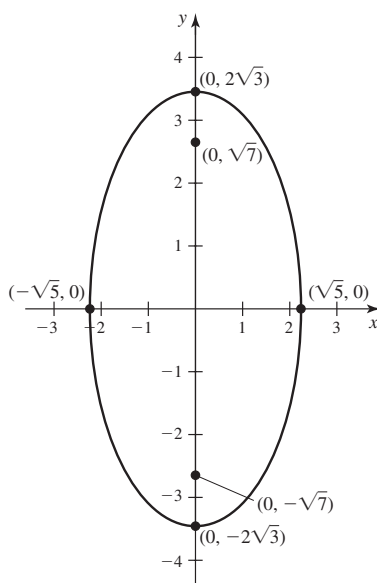
- 12.4.26** This is an ellipse. Vertices are $(0, \pm 3)$, and the foci are $(0, \pm 2\sqrt{2})$. The major axis has length 6 and the minor axis has length 2.



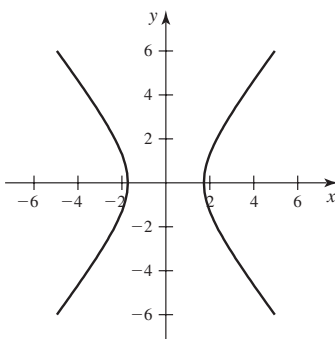
- 12.4.27** This is an ellipse. Vertices are $(0, \pm\sqrt{7})$, and the foci are $(0, \pm\sqrt{2})$. The major axis has length $2\sqrt{7}$ and the minor axis has length $2\sqrt{5}$.



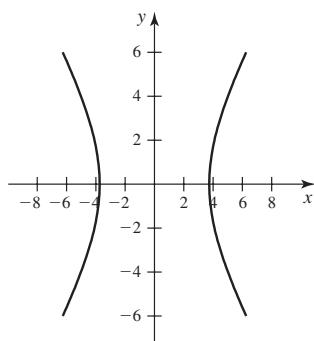
12.4.28 This is an ellipse. Vertices are $(0, \pm 2\sqrt{3})$, and the foci are $(0, \pm \sqrt{7})$. The major axis has length $4\sqrt{3}$ and the minor axis has length $2\sqrt{5}$.



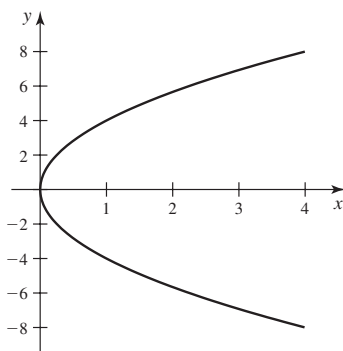
12.4.29 This is a hyperbola. The vertices are $(\pm\sqrt{3}, 0)$, and the foci are $(\pm 2\sqrt{2}, 0)$. The asymptotes are $y = \pm \sqrt{\frac{5}{3}} \cdot x$.



12.4.30 This is a hyperbola. The vertices are $(\pm\sqrt{14}, 0)$, and the foci are $(\pm\sqrt{34}, 0)$. The asymptotes are $y = \pm \sqrt{\frac{10}{7}} \cdot x$.

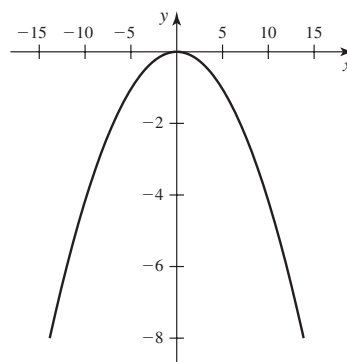


12.4.31



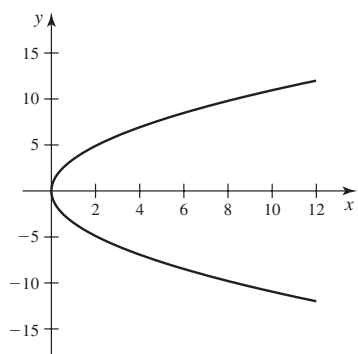
$$y^2 = 16x.$$

12.4.32



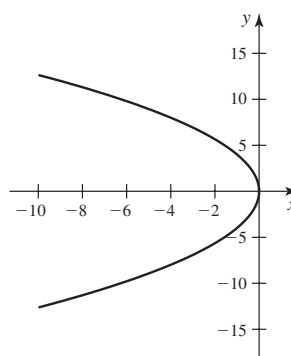
$$x^2 = -24y.$$

12.4.33



$$y^2 = 12x.$$

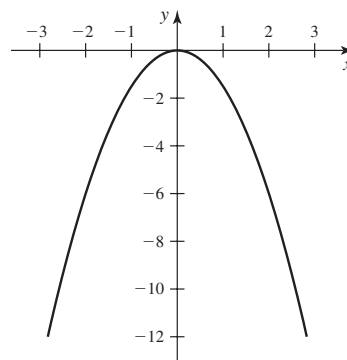
12.4.34



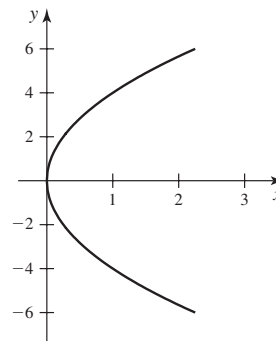
$$y^2 = -16x.$$

12.4.35

$$x^2 = 4py \text{ and } 4 = 4p(-6), \text{ so } p = -\frac{1}{6} \text{ and } x^2 = -\frac{2}{3} \cdot y.$$



12.4.36 $y^2 = 4px$ and $(-4)^2 = 4p(1)$, so $p = 4$ and $y^2 = 16x$.



12.4.37 Because the vertex is $(-1, 0)$ and the parabola is symmetric about the x -axis, we have $y^2 = 4p(x+1)$ and because the directrix is one unit left of the vertex, we obtain $p = 1$ and $y^2 = 4(x+1)$.

12.4.38 Because the vertex is $(0, 2)$ and the parabola is symmetric about the y -axis, we have $x^2 = 4p(y-2)$ and because the directrix is 2 units above the vertex, we obtain $p = -2$ and $x^2 = -8(y-2)$.

12.4.39 $a = 4$, and $b = 3$, so the equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

12.4.40 $a = 6$, and $a^2 = b^2 + c^2$ where $c = 4$, so $b^2 = 36 - 16 = 20$, and the equation is $\frac{x^2}{36} + \frac{y^2}{20} = 1$.

12.4.41 We have $a = 4$ and $c = 6$, so $b^2 = c^2 - a^2 = 20$, so the equation is $\frac{x^2}{16} - \frac{y^2}{20} = 1$.

12.4.42 We have $a = 1$, so the equation is of the form $x^2 - \frac{y^2}{b^2} = 1$. Because $(5/3, 8)$ is on the curve, we have $\frac{25}{9} - \frac{64}{b^2} = 1$, so $b = 6$. The equation is $x^2 - \frac{y^2}{36} = 1$.

12.4.43 $a = 5$, and the equation is of the form $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$. Because $(4, \frac{3}{5})$ is on the curve, we have $\frac{16}{25} + \frac{9}{25b^2} = 1$, so $b = 1$. The equation is $\frac{x^2}{25} + y^2 = 1$.

12.4.44 $a = 10$, and the equation is of the form $\frac{y^2}{100} + \frac{x^2}{b^2} = 1$, and because $(\sqrt{3}/2, 5)$ is on the curve, we have $\frac{1}{4} + \frac{3}{4b^2} = 1$, so $b = 1$. The equation is $x^2 + \frac{y^2}{100} = 1$.

12.4.45 We have $a = 2$, and because the asymptotes are $y = \frac{\pm bx}{a}$, we have that $b = 3$, so the equation is $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

12.4.46 We have $a = 2$ and because the asymptotes are $y = \frac{\pm a}{b} \cdot x$, we have $b = 1$, and the equation is $\frac{y^2}{4} - x^2 = 1$.

12.4.47 $a = 3$ and $b = 2$, so the equation is $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

12.4.48 $a = 10$ and $b = 8$, so $\frac{x^2}{100} + \frac{y^2}{64} = 1$.

12.4.49 We have $a = 4$ and $c = 5$, so $b^2 = 25 - 16 = 9$, so $b = 3$ and the equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

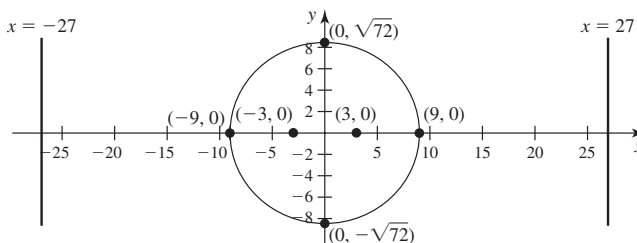
12.4.50 We have $a = 6$ and $c = 10$, and $b^2 = 100 - 36 = 64$, so $b = 8$, and the equation is $\frac{y^2}{36} - \frac{x^2}{64} = 1$.

12.4.51

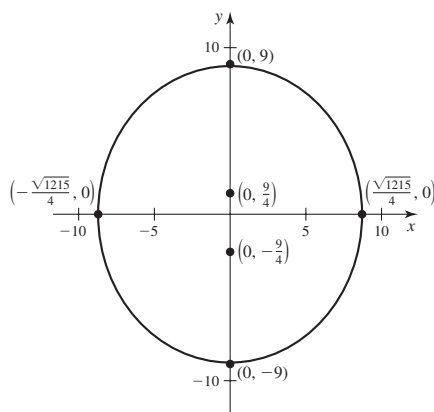
- True. Note that if $x = 0$, the equation becomes $-y^2 = 9$, which has no solution.
- True. The slopes of the tangent lines range continuously from $-\infty$ to 0 to ∞ and then back through 0 to $-\infty$ again.
- True. Given c and d , one can compute a , b , and e . See the summary after Theorem 10.3.
- True. The vertex is exactly halfway between the focus and the directrix.

12.4.52 We have a vertex at $(0, 0)$ and the parabola passes through $(640, 152)$, so $152 = a(640)^2$, so $a = \frac{19}{51200} \approx 0.000371$. Thus, $y = \frac{19}{51200}x^2$, and the guy wire has length $L = \frac{19}{51200}(500^2) = \frac{11875}{128} \approx 92.77$ meters.

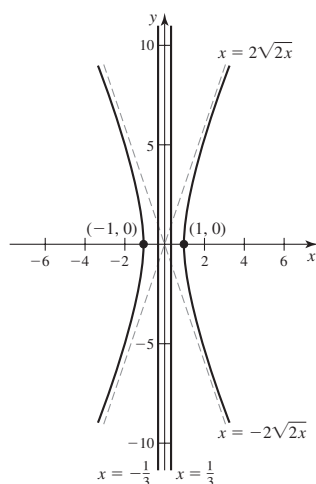
12.4.53 We have $a = 9$ and $e = \frac{1}{3}$, so $c = ae = 3$, and $b^2 = a^2 - c^2 = 72$, so the equation is $\frac{x^2}{81} + \frac{y^2}{72} = 1$.



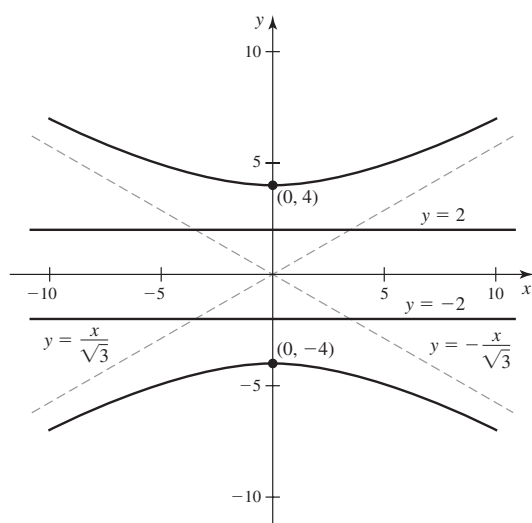
12.4.54 We have $a = 9$ and $e = \frac{1}{4}$, so $c = ae = \frac{9}{4}$, and $b^2 = a^2 - c^2 = 81 - \frac{81}{16} = \frac{1215}{16}$. Thus the equation is $\frac{16x^2}{1215} + \frac{y^2}{81} = 1$.



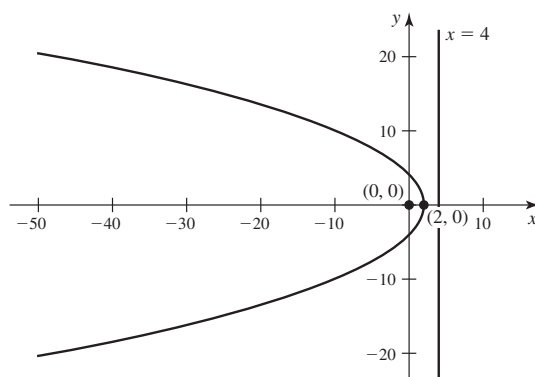
12.4.55 We have $a = 1$ and $e = 3$, so $c = ae = 3$ and $b^2 = c^2 - a^2 = 9 - 1 = 8$. Thus, the equation is $x^2 - \frac{y^2}{8} = 1$.



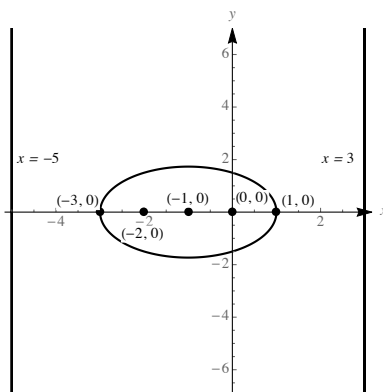
12.4.56 We have $a = 4$ and $e = 2$, so $c = ae = 8$ and $b^2 = c^2 - a^2 = 64 - 16 = 48$. Thus, the equation is $\frac{y^2}{16} - \frac{x^2}{48} = 1$.



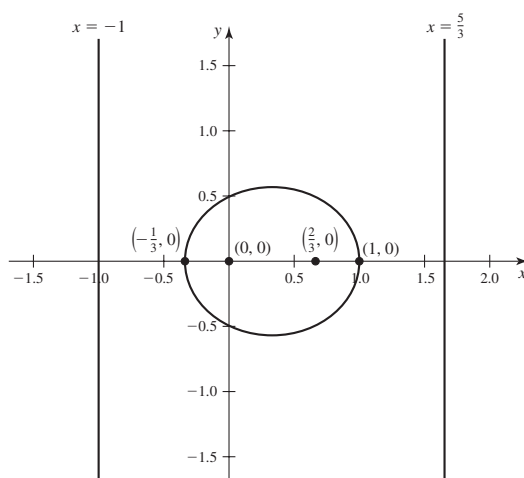
12.4.57 The vertex is $(2, 0)$. The focus is $(0, 0)$, and the directrix is the line $x = 4$.



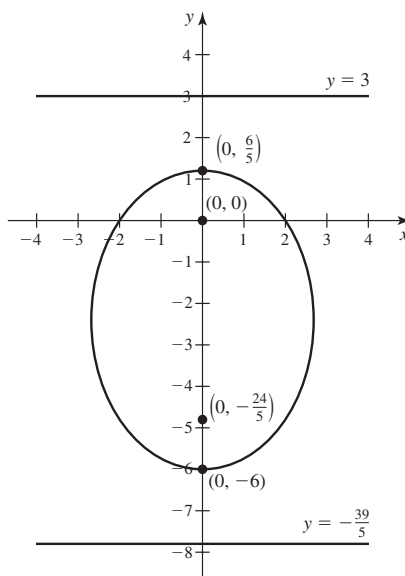
12.4.58 The vertices are $(1, 0)$ and $(-3, 0)$. The center is $(-1, 0)$. The foci are $(0, 0)$ and $(-2, 0)$.



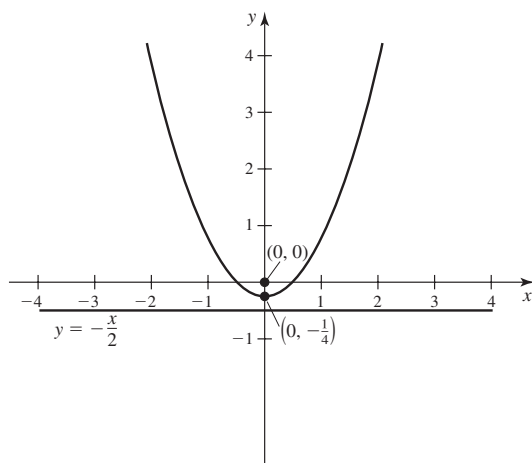
12.4.59 The vertices are $(1, 0)$ and $(-1/3, 0)$. The center is $(1/3, 0)$. The directrices are $x = -1$ and $x = 5/3$.



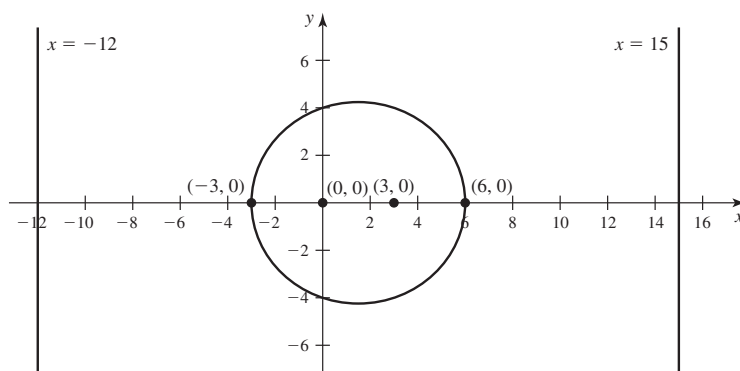
12.4.60 The vertices are $(0, 6/5)$ and $(0, -6)$. The center is $(0, -12/5)$. The foci are $(0, 0)$ and $(-24/5, 0)$.



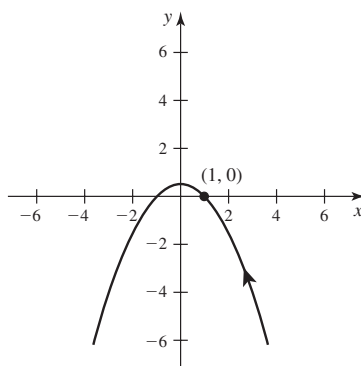
12.4.61 The vertex is $(0, -1/4)$, and the focus is $(0, 0)$. The directrix is the line $y = -\frac{1}{2}$.



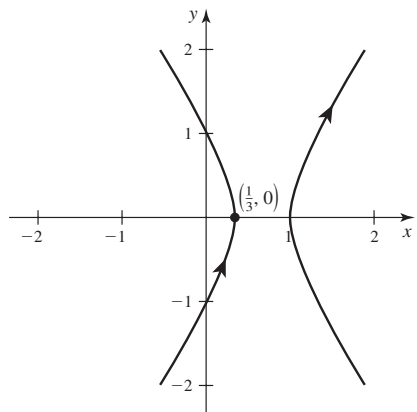
12.4.62 The vertices are $(6, 0)$ and $(-3, 0)$. The center is $(3/2, 0)$. The foci are $(0, 0)$ and $(3, 0)$.



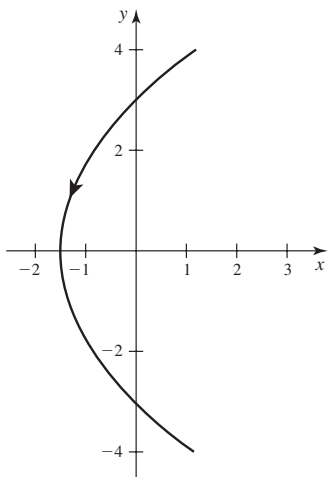
12.4.63 The parabola starts at $(1, 0)$ and goes through quadrants I, II, and III for $\theta \in [0, 3\pi/2]$. It then approaches $(1, 0)$ by traveling through quadrant IV for $\theta \in (3\pi/2, 2\pi)$.



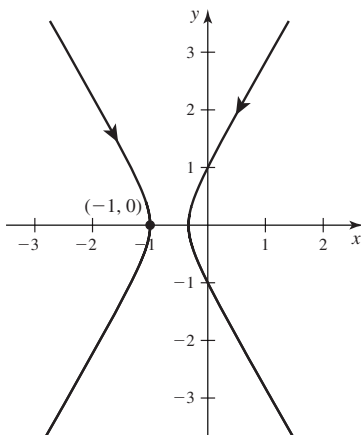
12.4.64 Note that the value of r for $\theta = 0$ is $1/3$. As θ proceeds to $\pi/2$, the curve is traced in the first quadrant and approaches the polar point $(1, \pi/2)$. From $\pi/2$ to π , the curve approaches the asymptote, and then appears along the asymptote in the fourth quadrant and heads toward the polar point $(-1, \pi)$. From π to 2π , the curve approaches the asymptote in the first quadrant, and then reappears in the third quadrant along the asymptote, and heads toward the point $(1/3, 2\pi)$.



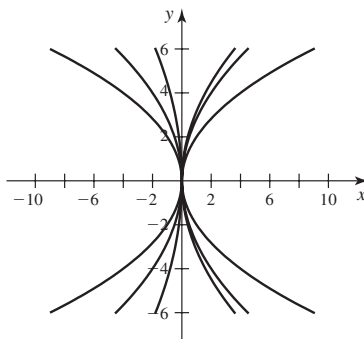
12.4.65 The parabola begins in the first quadrant and passes through the points $(0, 3)$ and then $(-3/2, 0)$ and $(0, -3)$ as θ ranges from 0 to 2π .



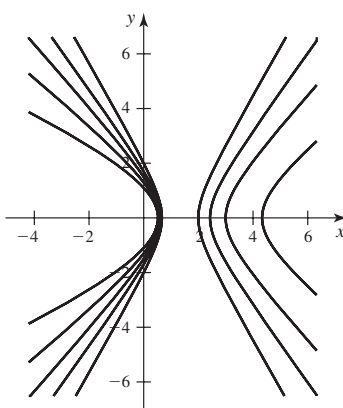
12.4.66 As θ ranges from 0 to $\pi/3$, the branch of the hyperbola in quadrant III starts at the point $(-1, 0)$, and approaches the asymptote (note that $r \rightarrow \infty$ as $\theta \rightarrow \pi/3^-$.) As θ takes on the values from $\pi/3$ to $\pi/2$, the portion of the parabola in quadrant I appears and heads toward the point $(0, 1)$. For θ ranging from $\pi/2$ to $3\pi/2$, the curve ranges from $(0, 1)$ to $(-1/3, 0)$ to $(0, -1)$. From $\theta = 3\pi/2$ to $\theta = 5\pi/3$, the curve approaches the asymptote in quadrant IV, and approaches the point $(-1, 0)$. From $5\pi/3$ to 2π , the curve reappears along the asymptote in quadrant II, and approaches the point $(-1, 0)$.



12.4.67 For negative p , the parabola opens to the left and for positive p it opens to the right. As p increases to 0, the parabola opens wider and as p decreases (for $p > 0$), it gets narrower.



12.4.68 As e gets larger, the vertices move closer to each other.



12.4.69 Differentiating gives $2x = -6y'$, so at $(-6, -6)$ we obtain $-12 = -6y'$, so $y' = 2$. Thus $y - (-6) = 2(x - (-6))$, or $y = 2x + 6$ is the equation of the tangent line.

12.4.70 Using implicit differentiation, we have $2yy' = 8$, and at the point $(8, -8)$, we have $y' = -\frac{1}{2}$. So $y - (-8) = -\frac{1}{2}(x - 8)$, or $y = -\frac{1}{2}x - 4$ is the equation of the tangent line.

12.4.71 Differentiating implicitly, we have $2yy' - \frac{x}{32} = 0$, so at $(6, -5/4)$ we have $-\frac{5}{2}y' - \frac{3}{16} = 0$, so $y' = -\frac{3}{40}$. The equation of the tangent line is $y + \frac{5}{4} = -\frac{3}{40}(x - 6)$, or $y = -\frac{3x}{40} - \frac{4}{5}$.

12.4.72 We have

$$\frac{dy}{dx} = \frac{-\frac{\cos \theta \sin \theta}{(1 + \sin \theta)^2} + \frac{\cos \theta}{1 + \sin \theta}}{-\frac{\cos^2 \theta}{(1 + \sin \theta)^2} - \frac{\sin \theta}{1 + \sin \theta}} = \frac{\sin \theta - 1}{\cos \theta}.$$

At $\theta = \frac{\pi}{6}$ we have $y' = \frac{(1/2) - 1}{\sqrt{3}/2} = -\frac{\sqrt{3}}{3}$. The equation of the tangent line is therefore $y - \frac{1}{3} = -\frac{\sqrt{3}}{3}(x - \frac{\sqrt{3}}{3})$, or $y = -\frac{\sqrt{3}}{3}x + \frac{2}{3}$.

12.4.73 Using implicit differentiation, we have $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$, or $y' = -\frac{b^2x}{a^2y}$. If (x_0, y_0) is the point of tangency, then $-\frac{b^2x_0}{a^2y_0} = \frac{y - y_0}{x - x_0}$, so $\frac{x_0(x - x_0)}{a^2} = -\frac{y_0(y - y_0)}{b^2}$, so $\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$.

12.4.74 Using implicit differentiation, we have $\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$, so $y' = \frac{b^2x}{a^2y}$. At (x_0, y_0) , we have $y' = \frac{b^2x_0}{a^2y_0}$, so the tangent line is given by $y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$, or $\frac{y(y - y_0)}{b^2} = \frac{x_0(x - x_0)}{a^2}$, or $\frac{y_0y}{b^2} - \frac{x_0x}{a^2} = \frac{y_0^2}{b^2} - \frac{x_0^2}{a^2} = -1$, so $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$.

12.4.75 We have a hyperbola with focal point at the origin and directrix $y = -2$. Furthermore $P = (0, -4/3)$ is a vertex. Thus, $e = \frac{|PF|}{|PL|} = \frac{4/3}{2/3} = 2$, and $r(\theta) = \frac{2(2)}{1 - 2\sin\theta} = \frac{4}{1 - 2\sin\theta}$.

12.4.76 We have an ellipse with focus at the origin and directrix $x = 2$. Because $(2/3, 0)$ is a vertex, $e = \frac{|PF|}{|PL|} = \frac{2/3}{4/3} = \frac{1}{2}$ and $r(\theta) = \frac{\frac{1}{2} \cdot 2}{1 + \frac{1}{2}\cos\theta} = \frac{2}{2 + \cos\theta}$.

12.4.77 The points on the intersection of the two circles are a distance of $2a + r$ from F_1 and a distance of r from F_2 . So for P an intersection point, we have $|PF_1| - |PF_2| = 2a$ for all r , and the set of all such points form a hyperbola with foci F_1 and F_2 .

12.4.78

a. Making use of the substitution $\frac{x}{\sqrt{2}} = \sin t$, we have

$$\begin{aligned} A &= \int_0^1 \left(\sqrt{1 - \frac{x^2}{2}} - \frac{x^2}{\sqrt{2}} \right) dx = \int_0^{\pi/4} \sqrt{1 - \sin^2 t} \sqrt{2} \cos t dt - \frac{1}{\sqrt{2}} \int_0^1 x^2 dx \\ &= \sqrt{2} \cdot \frac{1}{2} (\cos x \sin x + x) \Big|_0^{\pi/4} - \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{2} (\sin(\pi/4) \cos(\pi/4) + \frac{\pi}{4}) - \frac{1}{3\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \cdot \left(\frac{1}{2} + \frac{\pi}{4} \right) - \frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{12} + \frac{\sqrt{2}\pi}{8}. \end{aligned}$$

b. About the x -axis, we obtain $\pi \int_0^1 \left(1 - \frac{x^2}{2} - \frac{x^4}{2} \right) dx = \pi \left(x - \frac{x^3}{6} - \frac{x^5}{10} \right) \Big|_0^1 = \frac{11\pi}{15} \approx 2.304$.

About the y -axis, we obtain

$$\begin{aligned} 2\pi \int_0^1 \left(x\sqrt{1 - \frac{x^2}{2}} - \frac{x^2}{\sqrt{2}} \right) dx &= 2\pi \int_{1/2}^1 u^{1/2} du - \frac{1}{\sqrt{2}} \int_0^1 x^3 dx \\ &= 2\pi \left(\frac{2}{3} u^{3/2} \right) \Big|_{1/2}^1 - \left(\frac{1}{4\sqrt{2}} x^4 \right) \Big|_0^1 = 2\pi \left(\frac{2}{3} - \frac{2}{6\sqrt{2}} - \frac{1}{4\sqrt{2}} \right) \approx 1.597. \end{aligned}$$

So the volume about the x -axis is greater.

12.4.79 $V_x = \pi \int_{-a}^a \left(b^2 - \frac{b^2x^2}{a^2} \right) dx = \pi b^2 \int_{-a}^a \left(1 - \frac{x^2}{a^2} \right) dx = \pi b^2 \left(x - \frac{x^3}{3a^2} \right) \Big|_{-a}^a = \frac{4\pi b^2 a}{3}.$

$$V_y = \pi \int_{-b}^b \left(a^2 - \frac{a^2y^2}{b^2} \right) dy = \pi a^2 \int_{-b}^b \left(1 - \frac{y^2}{b^2} \right) dy = \pi a^2 \left(y - \frac{y^3}{3b^2} \right) \Big|_{-b}^b = \frac{4\pi a^2 b}{3}.$$

These are different if $a \neq b$. In the case $a = b$, both volumes give $\frac{4\pi a^3}{3}$, the volume of a sphere.

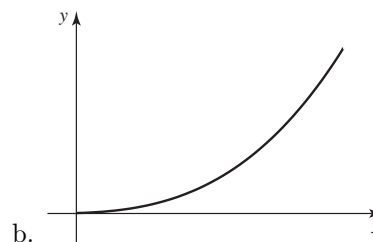
12.4.80

- a. The focus is at $c = \sqrt{a^2 + b^2}$. We have $A = 2b \int_a^c \sqrt{\frac{x^2}{a^2} - 1} dx = \frac{2b}{a} \int_a^c \sqrt{x^2 - a^2} dx$. Using either the substitution $x = a \sec \theta$ (or a table of integrals), we have

$$A = \frac{2b}{a} \cdot \left(\frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln \left(2 \left(\sqrt{x^2 - a^2} + x \right) \right) \right) \Big|_a^c$$

so

$$A = ab \ln(a) - ab \ln(\sqrt{a^2 + b^2} + b) + \frac{\sqrt{a^2 + b^2}}{a} \cdot b^2.$$



12.4.81

a.

$$\begin{aligned} V_x &= \pi \int_a^c \left(\sqrt{\frac{b^2 x^2}{a^2} - b^2} \right)^2 dx = \pi \int_a^c \left(\frac{b^2 x^2}{a^2} - b^2 \right) dx = \pi b^2 \left(\frac{x^3}{3a^2} - x \right) \Big|_a^c \\ &= \pi b^2 \left(\frac{c^3}{3a^2} - c - \frac{a}{3} + a \right) = \frac{\pi b^2}{3a^2} (c^3 - 3ca^2 + 2a^3) = \frac{\pi b^2}{3a^2} (a - c)^2 (2a + c). \end{aligned}$$

$$\text{b. } V_y = 2 \cdot 2\pi \int_a^c a^2 b \sqrt{\frac{x^2}{a^2} - 1} dx = 2\pi \int_0^{b^2/a^2} a^2 b \sqrt{u} du = 2\pi a^2 b \left(\frac{2}{3} u^{3/2} \right) \Big|_0^{b^2/a^2} = 2\pi a^2 b \frac{2b^3}{3a^3} = \frac{4\pi b^4}{3a}.$$

12.4.82

$$V_R = 2\pi \int_0^{\sqrt{h/a}} x(h - ax^2) dx = 2\pi \int_0^{\sqrt{h/a}} (xh - ax^3) dx = 2\pi \left(\frac{hx^2}{2} - \frac{ax^4}{4} \right) \Big|_0^{\sqrt{h/a}} = 2\pi \left(\frac{h^2}{2a} - \frac{h^2}{4a} \right) = \frac{\pi h^2}{2a}.$$

The cone has height h and radius $\sqrt{h/a}$, so $V_c = \frac{1}{3}\pi \left(\sqrt{\frac{h}{a}} \right)^2 \cdot h = \frac{\pi h^2}{3a}$, and $V_R = \frac{3}{2} \cdot \frac{\pi h^2}{3a} = \frac{\pi h^2}{2a}$.

12.4.83

- a. The slope of a line making an angle θ with the horizontal is $\tan \theta$. The slope of the tangent line at (x_0, y_0) is $y' = \frac{x}{2p}$, so $y' = \frac{x_0}{2p}$, so $\tan \theta = \frac{x_0}{2p}$.

- b. The distance from $(0, y_0)$ to $(0, p)$ is $p - y_0$, and $\tan \varphi = \frac{\text{opposite}}{\text{adjacent}} = \frac{p - y_0}{x_0}$.

- c. Because l is perpendicular to $y = y_0$, we have $\alpha + \theta = \pi/2$, or $\alpha = \frac{\pi}{2} - \theta$, so $\tan \alpha = \cot \theta = \frac{2p}{x_0}$.

- d. $\tan \beta = \tan(\theta + \varphi) = \frac{\frac{x_0}{2p} + \frac{p - y_0}{x_0}}{1 - \frac{p - y_0}{2p}} = \frac{\frac{x_0^2 + 2p^2 - 2py_0}{2px_0}}{\frac{x_0(p + y_0)}{2px_0}}$. Now because $x_0^2 = 4py_0$, we obtain $\tan \beta =$

$$\begin{aligned} &\frac{4py_0 + 2p^2 - 2py_0}{x_0(p + y_0)} \\ &= \frac{2p(p + y_0)}{x_0(p + y_0)} = \frac{2p}{x_0}. \end{aligned}$$

- e. Because α and β are acute, we have that $\tan \alpha = \tan \beta$, so $\alpha = \beta$.

12.4.84

a. $e = \frac{|PF|}{|PL|}$, so $r = |PF| = e|PL| = e|-d - r \cos \theta|$, or $r = e(d + r \cos \theta)$. Solving for r yields

$$r = \frac{ed}{1 - e \cos \theta}.$$

b. $r = |PF| = e|PL|$, or $r = e(d - r \sin \theta)$. Solving for r yields $r = \frac{ed}{1 + e \sin \theta}$.

c. $r = |PF| = e|PL|$, or $r = e(-d - r \sin \theta)$. So $r = e(d + r \sin \theta)$, and solving for r yields $r = \frac{ed}{1 - e \sin \theta}$.

12.4.85 Assume the two fixed points are at $(c, 0)$ and $(-c, 0)$. Let P be the point $(0, b)$, and note that P is equidistant from the two given points, so we must have $b^2 + c^2 = a^2$ by the Pythagorean theorem. Now let $Q = (u, 0)$ be on the ellipse for $u > c$. Then $u - c + (c + u) = 2a$, so $u = a$. Now let $R = (x, y)$ be an arbitrary point on the ellipse (assume $x > 0$ and $y > 0$ – the other cases are similar.) Using the triangles formed between the foci, R , and the projection of R onto the x -axis, we have $\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(c - x)^2 + y^2}$. Squaring both sides gives $(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2$. Isolating the root gives $\sqrt{(c - x)^2 + y^2} = \frac{1}{4a}((c - x)^2 + y^2 - (c + x)^2 - y^2 + 4a^2)$, so $\sqrt{(c - x)^2 + y^2} = a - \frac{c}{a}x$. Squaring again yields $(c - x)^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$, so $c^2 - 2cx + x^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$, or $x^2 \left(1 - \frac{c^2}{a^2}\right) + y^2 = a^2 - c^2$. Thus $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$, which can be written $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, because $b^2 = a^2 - c^2$.

12.4.86 The intersection points of the branches with the x -axis are at $(-a, 0)$ and $(a, 0)$ because the distances to $(c, 0)$ and $(-c, 0)$ are $c + a$ and $c - a$, so the difference is $\pm 2a$. Consider one point on the right branch (the left branch will follow by a similar argument.) Let the distance from $(-c, 0)$ to (x, y) be u and the distance from $(c, 0)$ to (x, y) be v . Then $u = \sqrt{(c + x)^2 + y^2}$ and $v = \sqrt{(c - x)^2 + y^2}$, and because $u - v = 2a$, we have $\sqrt{(c + x)^2 + y^2} = 2a + \sqrt{(c - x)^2 + y^2}$. Squaring gives $(c + x)^2 + y^2 = 4a^2 + 4a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2$. Isolating the root gives $\sqrt{(c - x)^2 + y^2} = \frac{1}{4a}((c + x)^2 + y^2 - (c - x)^2 - y^2 - 4a^2)$, so $\sqrt{(c - x)^2 + y^2} = -a + \frac{c}{a}x$.

Squaring again yields $(c - x)^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$, so $c^2 - 2cx + x^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$, or $x^2 \left(1 - \frac{c^2}{a^2}\right) + y^2 = a^2 - c^2$. Thus $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$, which can be written $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$.

12.4.87 Let the parabola be symmetric about the y -axis with vertex at the origin. Let the circle have radius r and be centered at $(r + a, 0)$, and let the line be $y = -a$. The distance from the point $P(x, y)$ to the line is $u = y + a$. The distance from the point P to the circle is $v = \sqrt{x^2 + (r + a - y)^2} - r$. Setting $u = v$ yields $y + a = \sqrt{x^2 + (r + a - y)^2} - r$, so $y + r + a = \sqrt{x^2 + (r + a - y)^2}$, and squaring gives $y^2 + 2(r + a)y + (r + a)^2 = x^2 + (r + a - y)^2$, so $y^2 + 2(r + a)y + (r + a)^2 = x^2 + (r + a)^2 - 2(r + a)y + y^2$, and thus $4(r + a)y = x^2$, so $y = \frac{1}{4(r + a)}x^2$, the equation of a parabola.

12.4.88 With focus at the origin, the cartesian equation of an ellipse with the second focus at $(-2c, 0)$ and major axis length $2a$, minor axis length $2b$ is $\frac{(x + c)^2}{a^2} + \frac{y^2}{b^2} = 1$. Using $c = ae$ and polar coordinates yields $\frac{(r \cos \theta + ae)^2}{a^2} + \frac{r^2 \sin^2 \theta}{a^2(1 - e^2)} = 1$. Thus, $(1 - e^2)(r^2 \cos^2 \theta + 2aer \cos \theta + a^2e^2) + r^2 \sin^2 \theta = a^2(1 - e^2)$, so $r^2 - e^2r^2 \cos^2 \theta + 2ae(1 - e^2)r \cos \theta + (1 - e^2)a^2e^2 = a^2(1 - e^2)$. Gathering like terms gives $(1 - e^2 \cos^2 \theta)r^2 + 2ae(1 - e^2) \cos \theta \cdot r - a^2(1 - e^2)^2 = 0$. Using the quadratic formula, we have

$$r = \frac{-2ae(1 - e^2) \cos \theta + \sqrt{4a^2e^2(1 - e^2)^2 \cos^2 \theta + 4a^2(1 - e^2)^2(1 - e^2 \cos^2 \theta)}}{2(1 - e^2 \cos^2 \theta)}.$$

This can be written as

$$r = \frac{-2ae(1-e^2)\cos\theta + 2a(1-e^2)}{2(1-e^2\cos^2\theta)} = \frac{a(1-e^2)(-e\cos\theta + 1)}{(1-e\cos\theta)(1+e\cos\theta)} = \frac{a(1-e^2)}{1+e\cos\theta}.$$

12.4.89 Let the hyperbolas be centered at the origin with equations $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{B^2} - \frac{x^2}{A^2} = 1$ and eccentricities $e = \frac{c}{a}$ and $E = \frac{C}{B}$, respectively. Because the hyperbolas share a set of asymptotes $A = ra$ and $B = rb$ for some $r > 0$, and

$$\begin{aligned} C^2 &= A^2 + B^2 = (ra)^2 + (rb)^2 \\ &= r^2(a^2 + b^2) = r^2c^2. \end{aligned}$$

Then we have

$$\begin{aligned} e^{-2} + E^{-2} &= \left(\frac{c}{a}\right)^{-2} + \left(\frac{C}{B}\right)^{-2} = \frac{a^2}{c^2} + \frac{B^2}{C^2} \\ &= \frac{a^2}{c^2} + \frac{r^2b^2}{r^2c^2} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1. \end{aligned}$$

12.4.90 The focal chord of slope $m \neq 0$ has equation $y = m(x - p)$. Because $y^2 = 4px$, the focal chord and the parabola intersect for $(mx - mp)^2 = 4px$, which occurs (via the quadratic formula) at $x = \frac{(m^2 + 2 \pm 2\sqrt{m^2 + 1})p}{m^2}$. The corresponding y -values are $\frac{(m^2 + 2 \pm 2\sqrt{m^2 + 1})p}{m} - mp$. Now $y' = \frac{2p}{y}$, so $y' = \frac{m}{1 \pm \sqrt{m^2 + 1}}$ at the two points. The product of these two values of y' is -1 , so the two lines are perpendicular. If we call the intersection points found above (x_0, y_0) and (x_1, y_1) , then the two lines intersect for

$$\frac{m}{1 + \sqrt{m^2 + 1}}(x - x_0) + y_0 = \frac{m}{1 - \sqrt{m^2 + 1}}(x - x_1) + y_1,$$

which when solved for x gives $x = -p$, so the two lines meet on the directrix.

Note that in the case of a vertical chord, we have $(x_0, y_0) = (p, 2p)$ and $(x_1, y_1) = (p, -2p)$, and thus the slopes of the tangent lines are 1 and -1 , so their product is still -1 and thus they are perpendicular. Then the tangent lines meet when $1(x - p) + 2p = -1(x - p) - 2p$, which occurs when $x = -p$, so they still meet on the directrix.

12.4.91 The latus rectum L intersects the parabola at $x = p$, $y = \pm 2p$. The distance between any point $P(x, y)$ on the parabola to the left of L and L is $p - x$. The distance from F to P is $\sqrt{(x - p)^2 + y^2} = \sqrt{x^2 - 2px + p^2 + 4px} = \sqrt{x^2 + 2px + p^2} = x + p$ (because both x and p are positive.) Thus $D + |FP| = p - x + x + p = 2p$.

12.4.92 Because the latus rectum intersects the parabola at $(p, 2p)$ and $(p, -2p)$, its length is $4|p|$.

12.4.93 Let P be a point on the intersection of the latus rectum and the ellipse. The length of the latus rectum is twice the distance from P to the focus. Let l be the length from P to the focus, and let L be the distance from P to the other focal point. Then $l + L = 2a$, so $L^2 = 4c^2 + l^2$, and thus $(2a - l)^2 = 4c^2 + l^2$, and solving for l yields $l = a - \frac{c^2}{a}$. Because $c^2 = a^2 - b^2$, this can be written as $l = a - \frac{a^2 - b^2}{a} = a - (a - \frac{b^2}{a}) = \frac{b^2}{a}$.

The length of the latus rectum is therefore $\frac{2b^2}{a}$. Now because $e = \frac{c}{a}$, we have $\sqrt{1 - e^2} = \sqrt{1 - \frac{a^2 - b^2}{a^2}} = \sqrt{\frac{b^2}{a^2}} = \frac{b}{a}$. The length of the latus rectum can thus also be written as $2b \cdot \frac{b}{a} = 2b\sqrt{1 - e^2}$.

12.4.94 Let P be a point on the intersection of the latus rectum and the hyperbola. The length of the latus rectum is twice the distance from P to the focus. Let l be the length from P to the focus, and let L be the

distance from P to the other focal point. Then $L - l = 2a$, so $L^2 = 4c^2 + l^2$, and thus $(2a + l)^2 = 4c^2 + l^2$, and solving for l yields $l = \frac{c^2}{a} - a$. Because $c^2 = a^2 + b^2$, this can be written as $l = \frac{a^2 + b^2}{a} - a = \frac{b^2}{a}$. The length of the latus rectum is therefore $\frac{2b^2}{a}$. Now because $e = \frac{c}{a}$, we have $\sqrt{e^2 - 1} = \sqrt{\frac{a^2 + b^2}{a^2} - 1} = \sqrt{\frac{b^2}{a^2}} = \frac{b}{a}$. The length of the latus rectum can thus also be written as $2b \cdot \frac{b}{a} = 2b\sqrt{e^2 - 1}$.

12.4.95 Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$ and let the equation of the hyperbola be $\frac{x^2}{r^2} - \frac{y^2}{c^2 - r^2} = 1$. Let (x_0, y_0) be a point of intersection. By evaluating both equations at the point of intersection and subtracting, we obtain the result

$$\frac{x_0^2}{a^2} - \frac{x_0^2}{r^2} + \frac{y_0^2}{a^2 - c^2} + \frac{y_0^2}{c^2 - r^2} = 0,$$

which can be written

$$\frac{r_0^2 x_0^2 - a^2 x_0^2}{a^2 r^2} + \frac{(c^2 - r^2)y_0^2 + (a^2 - c^2)y_0^2}{(a^2 - c^2)(c^2 - r^2)} = 0.$$

This equation can be rewritten in the form $\frac{x_0^2}{y_0^2} = \frac{a^2 r^2}{(a^2 - c^2)(c^2 - r^2)}$, which we will use later.

Now implicitly differentiating the equation for the ellipse yields $\frac{2x}{a^2} + \frac{2yy'}{a^2 - c^2} = 0$, and thus the slope of the tangent line to the ellipse at (x_0, y_0) is $y'_e = -\frac{x_0}{y_0} \cdot \frac{a^2 - c^2}{a^2}$. Differentiating the equation of the hyperbola gives $\frac{2x}{r^2} - \frac{2yy'}{c^2 - r^2} = 0$, so the slope of the tangent line to the hyperbola at the point of intersection is $y'_h = \frac{x_0}{y_0} \cdot \frac{c^2 - r^2}{r^2}$.

Now consider the product

$$-1 \cdot y'_e \cdot y'_h = \frac{x_0^2}{y_0^2} \cdot \frac{(a^2 - c^2)(c^2 - r^2)}{a^2 r^2}.$$

By the result of the first paragraph, this is equal to 1, and thus the two curves are perpendicular at the point of intersection.

12.4.96 The vertical distance at x_0 is given by $d(x_0) = \frac{bx_0}{a} - \sqrt{\frac{x_0^2 b^2}{a^2} - a^2} = \frac{b}{a} \left(x_0 - \sqrt{x_0^2 - \frac{a^4}{b^2}} \right)$. We have

$$\lim_{x_0 \rightarrow \infty} d(x_0) = \frac{b}{a} \lim_{x_0 \rightarrow \infty} \left(x_0 - \sqrt{x_0^2 - \frac{a^4}{b^2}} \right) = \frac{b}{a} \lim_{x_0 \rightarrow \infty} \left(\frac{x_0^2 - \left(x_0^2 - \frac{a^4}{b^2} \right)}{x_0 + \sqrt{x_0^2 - \frac{a^4}{b^2}}} \right) = 0.$$

12.4.97

- a. The curve and the line intersect when $x^2 - m^2(x^2 - 4x + 4) - 1 = 0$, which occurs for $\frac{2m^2 \pm \sqrt{1 + 3m^2}}{m^2 - 1}$, assuming $m \neq \pm 1$. So there are two solutions in this case – but if $-1 < m < 1$, one of the solutions is negative (the intersection lies on the other branch of the hyperbola.) If $m^2 = 1$, then the equation becomes $4x - 5 = 0$, and there is only the solution $x = \frac{5}{4}$. So there are two intersection points on the right branch exactly for $|m| > 1$. We have $v(m) = \frac{2m^2 + \sqrt{1 + 3m^2}}{m^2 - 1}$ and $u(m) = \frac{2m^2 - \sqrt{1 + 3m^2}}{m^2 - 1}$.

$$\begin{aligned} \text{b. } \lim_{m \rightarrow 1^+} u(m) &= \lim_{m \rightarrow 1^+} u(m) \cdot \frac{2m^2 + \sqrt{1+3m^2}}{2m^2 + \sqrt{1+3m^2}} = \lim_{m \rightarrow 1^+} \frac{4m^4 - 3m^2 - 1}{(m^2 - 1)(2m^2 + \sqrt{1+3m^2})} = \\ &= \lim_{m \rightarrow 1^+} \frac{(m^2 - 1)(4m^2 + 1)}{(m^2 - 1)(2m^2 + \sqrt{1+3m^2})} = \frac{5}{4}. \end{aligned}$$

$$\begin{aligned} \lim_{m \rightarrow 1^+} v(m) &= \lim_{m \rightarrow 1^+} v(m) \cdot \frac{2m^2 - \sqrt{1+3m^2}}{2m^2 - \sqrt{1+3m^2}} = \lim_{m \rightarrow 1^+} \frac{4m^4 - 3m^2 - 1}{(m^2 - 1)(2m^2 - \sqrt{1+3m^2})} = \\ &= \lim_{m \rightarrow 1^+} \frac{(m^2 - 1)(4m^2 + 1)}{(m^2 - 1)(2m^2 - \sqrt{1+3m^2})} = \lim_{m \rightarrow 1^+} \frac{(4m^2 + 1)}{(2m^2 - \sqrt{1+3m^2})} = \infty. \end{aligned}$$

$$\text{c. } \lim_{m \rightarrow \infty} u(m) = \lim_{m \rightarrow \infty} \frac{2 - \sqrt{\frac{1}{m^4} + \frac{3}{m^2}}}{1 - \frac{1}{m^2}} = 2.$$

$$\lim_{m \rightarrow \infty} v(m) = \lim_{m \rightarrow \infty} \frac{2 + \sqrt{\frac{1}{m^4} + \frac{3}{m^2}}}{1 - \frac{1}{m^2}} = 2.$$

d. The expression $\lim_{m \rightarrow \infty} A(m)$ represents the area of the region bounded by the hyperbola and the line $x = 2$. It is given by $2 \int_1^2 \sqrt{x^2 - 1} dx = 2 \left(\frac{x}{2} \sqrt{x^2 - 1} - \frac{1}{2} \ln(x + \sqrt{x^2 - 1}) \right) \Big|_1^2 = 2\sqrt{3} - \ln(2 + \sqrt{3})$.

12.4.98

a. The area of the anvil is $A = 4 \int_0^p \sqrt{1 + y^2} dy = 4 \int_0^{\tan^{-1}(p)} \sec^3 t dt = 2p\sqrt{1 + p^2} + 2 \ln(\sqrt{1 + p^2} + p)$, where this last integral can be evaluated using the techniques of chapter 7 (or a table of integrals.)

The area of R is equal to the area of S when $2 = p\sqrt{1 + p^2} + \ln(\sqrt{1 + p^2} + p)$. Using a CAS, the result is $p \approx 0.8927$.

b. For R to have twice the area of S , we need $4 = p\sqrt{1 + p^2} + \ln(\sqrt{1 + p^2} + p)$, which occurs for $p \approx 1.5279$.

Chapter Twelve Review

1

- False. For example, $x = r \cos t$, $y = r \sin t$ for $0 \leq t \leq 2\pi$ and $x = r \sin t$, $y = r \cos t$ for $0 \leq t \leq 2\pi$ generate the same circle.
- False. Because $e^t > 0$ for all t , this only describes the portion of that line where $x > 0$.
- True. They both describe the point whose cartesian coordinates are $(3 \cos(-3\pi/4), 3 \sin(-3\pi/4)) = (-3 \cos(\pi/4), -3 \sin(\pi/4)) = (-3/\sqrt{2}, -3/\sqrt{2})$.
- False. The given integral counts the inner loop twice.
- True. This follows because the equation $0 - x^2/4 = 1$ has no real solutions.
- True. Note that the given equation can be written as $(x - 1)^2 + 4y^2 = 4$, or $\frac{(x - 1)^2}{4} + y^2 = 1$.

2 $y = 3 + 2 \sin t = 3 + 2(x - 1) = 2x + 1$, which implies that $y = 2x + 1$. This is a line segment starting at $(1, 3)$ and ending at $(2, 5)$. The curve $x = 1 + \sin t$, $y = 3 + 2 \sin t$, for $\pi/2 \leq t \leq \pi$, has the opposite orientation, starting at $(2, 5)$ and ending at $(1, 3)$.

3 Note that $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$. This represents an ellipse generated counterclockwise.

4 Note that $(x+3)^2 + (y-6)^2 = 1$. This is the right half of a circle of radius 1 centered at $(-3, 6)$. It is generated clockwise.

5 $x = y^2$, so $y = \sqrt{x}$. This is a segment of a parabola opening to the right, starting at $(4, 2)$ and ending at $(9, 3)$.

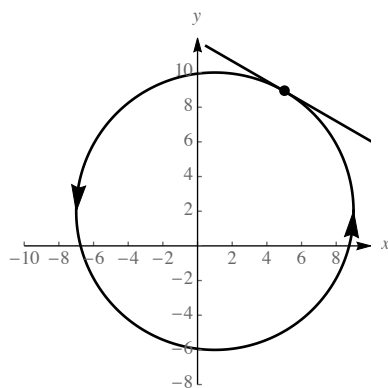
6 $1 + \cot^2 t = \csc^2 t$, so $1 - x = y - 1$, so $y = 2 - x$. This is a line segment starting at $(-1, 3)$ and ending at $(0, 2)$.

7

a. $(x-1)^2 + (y-2)^2 = 64\cos^2 t + 64\sin^2 t = 64$. This is the circle centered at $(1, 2)$ with radius 8.

b. $\left.\frac{dy}{dx}\right|_{t=\pi/3} = -\frac{\cos t}{\sin t}\bigg|_{t=\pi/3} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}.$

c. Note that the point on the curve corresponding to $t = \pi/3$ has coordinates $(5, 2 + 4\sqrt{3}) \approx (5, 8.9)$.

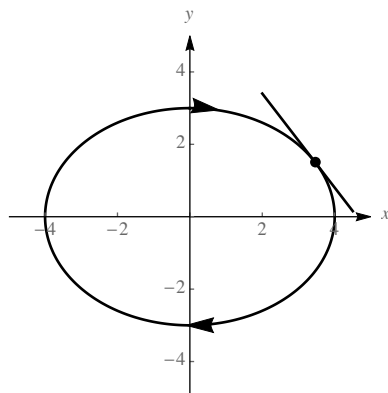


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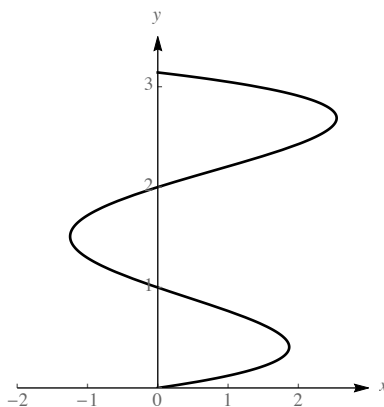
a. $\frac{x^2}{16} + \frac{y^2}{9} = \sin^2 2t + \cos^2 2t = 1$, so this is an ellipse.

b. $\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\frac{6\sin 2t}{8\cos 2t}\bigg|_{t=\pi/6} = -\frac{3\sqrt{3}}{4}.$

c. Note that the point on the curve corresponding to $t = \pi/6$ has coordinates $(2\sqrt{3}, 3/2) \approx (3.5, 1.5)$.



9 Let $y = t$, then $x = 5(t - 1)(t - 2) \sin t$.



10

a. $x = (-y)^2 + 4$, or $x = y^2 + 4$.

b. This is a segment of the upper half of a parabola opening to the right, starting at the point $(8,2)$ and ending at $(4,0)$.

c. The slope at $(5,1)$ is $\left. \frac{dy}{dx} \right|_{t=-1} = -\frac{1}{2t} \Big|_{t=-1} = \frac{1}{2}$.

11

a. $x^2 + (y + 1)^2 = 9 \cos^2(-t) + 9 \sin^2(-t) = 9$, so $x^2 + (y + 1)^2 = 9$.

b. This is the lower half of a circle of radius 3 centered at $(0, -1)$ which starts at $(3, -1)$ and ends at $(-3, -1)$.

c. The slope at $(0, -4)$ is $\left. \frac{dy}{dx} \right|_{t=\pi/2} = -\frac{3 \cos(-t)}{3 \sin(-t)} \Big|_{t=\pi/2} = 0$.

12

a. $y = 16 \ln t = 16x$.

b. This is a line segment starting at $(0,0)$ and ending at $(2,32)$.

c. The slope at $(1,16)$ is the slope of the line segment, which is $\frac{32}{2} = 16$.

13 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}$. At $t = \pi/6$, the slope of the tangent line is $\frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$. So the equation of the tangent line is $y - (1 - \sqrt{3}/2) = (2 + \sqrt{3})(x - (\pi/6 - 1/2))$, or $y = (2 + \sqrt{3})x + \left(2 - \frac{\pi}{3} - \frac{\pi\sqrt{3}}{6}\right)$.

At $t = 2\pi/3$, the slope of the tangent line is $\frac{\sqrt{3}}{3}$, so the equation of the tangent line is $y - \frac{3}{2} = \frac{\sqrt{3}}{3} \left(x - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \right)$, or $y = \frac{x}{\sqrt{3}} + 2 - \frac{2\pi}{3\sqrt{3}}$.

14 Let $y = t$, and $x = t^3 + t + 1$, for $0 \leq t \leq 2$.

15 From P to Q , we use $(x(t), y(t)) = tQ + (1-t)P = (t, t) + (t-1, 0) = (2t-1, t)$. So $x(t) = 2t-1$, $y(t) = t$, $0 \leq t \leq 1$.

From Q to P , we use $(x(t), y(t)) = tP + (1-t)Q = (-t, 0) + (1-t, 1-t) = (1-2t, 1-t)$, for $0 \leq t \leq 1$. Thus $x(t) = 1-2t$, $y(t) = 1-t$, $0 \leq t \leq 1$.

16 $x = t$, $y = t^3 + 2t$, $0 \leq t \leq 2$.

17 $x = 3 \sin t$, $y = 3 \cos t$, $0 \leq t \leq 2\pi$.

18 $x = 3 \cos t$, $y = 2 \sin t$, $-\pi/2 \leq t \leq \pi/2$.

19 The area is $\int_0^1 (t-t^3)(2t) dt = \int_0^1 (2t^2 - 2t^4) dt = \left(\frac{2t^3}{3} - \frac{2t^5}{5} \right) \Big|_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$.

20 The area is $\int_{\pi/2}^0 (\sin 2t)(-\sin t) dt = \int_0^{\pi/2} 2 \sin^2 t \cos t dt = \frac{2 \sin^3 t}{3} \Big|_0^{\pi/2} = \frac{2}{3}$.

21 The volume is $2 \int_0^{\pi/4} 2\pi \cos t \sqrt{\cos^2 t + 4 \sin^2 2t} dt \approx 9.1$.

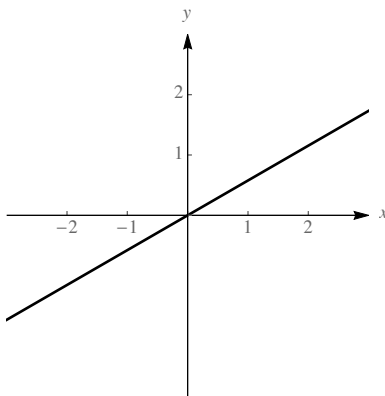
22 The arc length is

$$\begin{aligned} & \int_0^{\pi/3} \sqrt{(2e^{2t} + 3e^{2t} \cos 3t)^2 + (2e^{2t} \cos 3t - 3e^{2t} \sin 3t)^2} dt \\ &= \int_0^{\pi/3} \sqrt{13} e^{2t} dt = \frac{\sqrt{13}}{2} e^{2t} \Big|_0^{\pi/3} = \frac{\sqrt{13}}{2} (e^{2\pi/3} - 1). \end{aligned}$$

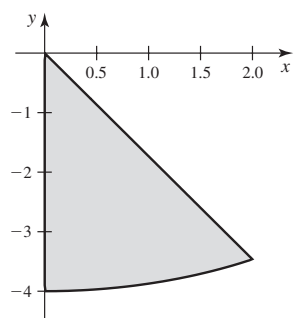
23 The arc length is

$$\begin{aligned} & \int_0^{\pi/4} \sqrt{(-2 \sin 2t)^2 + (2 - 2 \cos 2t)^2} dt = 2\sqrt{2} \int_0^{\pi/4} \sqrt{1 - \cos 2t} dt \\ &= 2\sqrt{2} \int_0^{\pi/4} \sqrt{1 - (1 - 2 \sin^2 t)} dt = 4 \int_0^{\pi/4} \sin t dt = -4 \cos t \Big|_0^{\pi/4} = 4 - 2\sqrt{2}. \end{aligned}$$

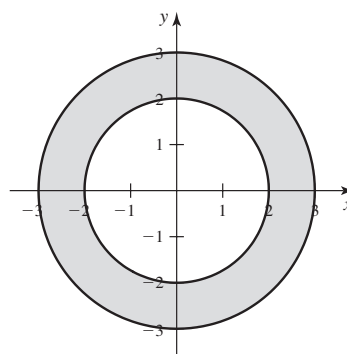
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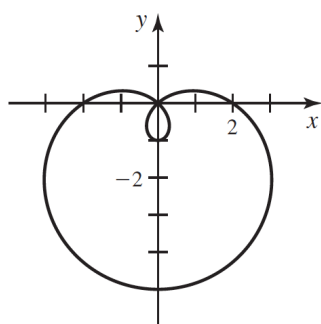
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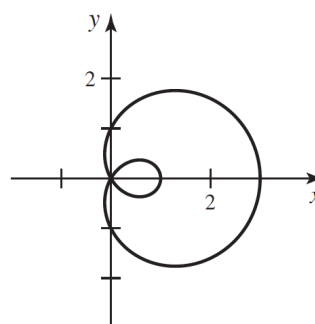
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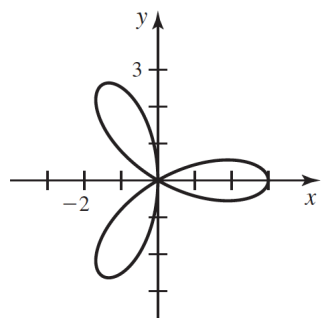
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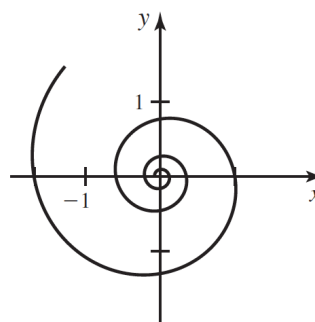
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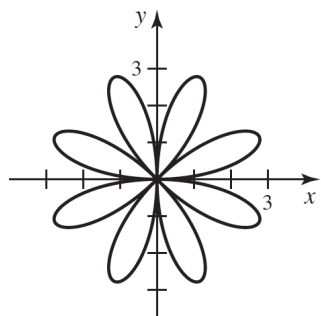
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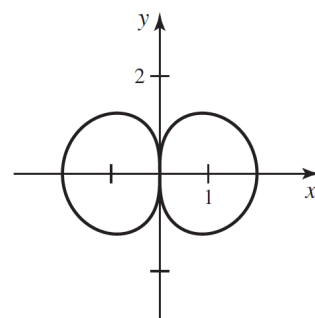
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31



32



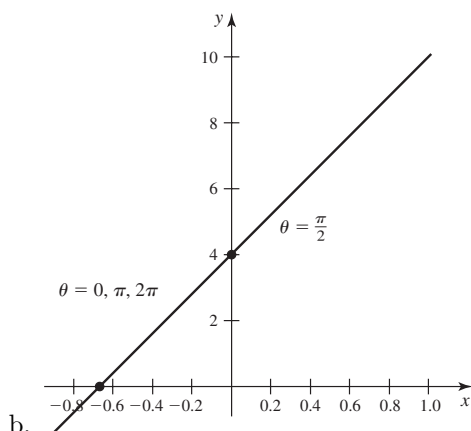
33 Liz should choose the cardioid, which is $r = 1 - \sin \theta$.

34 Jake should send $r^2 = \cos 2\theta$.

35 Letting $x = r \cos \theta$, $y = r \sin \theta$, and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + 2y - 6x = 0$, which can be written as $x^2 - 6x + 9 + y^2 + 2y + 1 = 10$, or $(x - 3)^2 + (y + 1)^2 = 10$, so this is a circle of radius $\sqrt{10}$ centered at $(3, -1)$.

36

a. We can write the equation as $r \sin \theta - 6r \cos \theta = 4$, or $y - 6x = 4$. This is a straight line with slope 6 and y -intercept 4.



c. Note that $\sin \theta - 6 \cos \theta = 0$ for $\theta = \tan^{-1}(6)$. The whole curve can be generated for $\tan^{-1}(6) - \pi < \theta < \tan^{-1}(6) + \pi$.

37 If we let $r = 1 + \cos t$, then $x = r \cos t$ and $y = r \sin t$. The curve $r = 1 + \cos t$ is a cardioid.

38 We have $r \cos \theta = r^2 \sin^2 \theta$, so $r = \cot \theta \csc \theta$. The whole parabola is described by $0 < \theta < \pi$.

39 If $x = r \cos \theta$ and $y = r \sin \theta$, then $(r \cos \theta - 4)^2 + r^2 \sin^2 \theta = 16$, so $r^2 \cos^2 \theta - 8r \cos \theta + 16 + r^2 \sin^2 \theta = 16$, so $r^2 = 8r \cos \theta$, and thus $r = 8 \cos \theta$. The complete circle can be described by $-\pi/2 \leq \theta \leq \pi/2$.

40

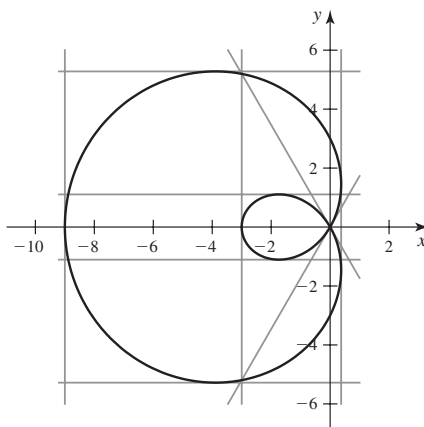
a.
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{6 \sin \theta \cdot \sin \theta + (3 - 6 \cos \theta) \cos \theta}{6 \sin \theta \cos \theta - (3 - 6 \cos \theta) \sin \theta} = \frac{6 - 12 \cos^2 \theta + 3 \cos \theta}{12 \sin \theta \cos \theta - 3 \sin \theta}.$$

This is 0 when $\cos^2 \theta - \frac{1}{4} \cos \theta - \frac{1}{2} = 0$, which (by the quadratic formula) occurs where $\cos \theta = \frac{1}{8} \pm \frac{\sqrt{33}}{8}$, so for $\theta \approx 0.568, 2.206, 4.078$, and 5.715 .

The denominator is 0 when $\sin \theta = 0$ and when $12 \cos \theta - 3 = 0$, or $\theta = \pm \cos^{-1}(1/4)$.

b. The curve is at the origin for $\theta = \pm \pi/3$, and because $\tan \pi/3 = \sqrt{3}$, the tangent lines have the equations $y = \pm \sqrt{3}x$.

c.



41

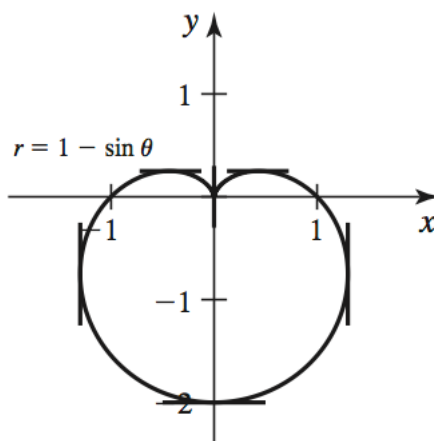
a. The curve intersects the origin at $\theta = \pi/2$, and there is a vertical tangent at $(0, \pi/2)$.

$$\frac{dy}{dx} = \frac{-\cos \theta \sin \theta + (1 - \sin \theta) \cos \theta}{-\cos^2 \theta - (1 - \sin \theta) \sin \theta} = \frac{\cos \theta (1 - 2 \sin \theta)}{\sin^2 \theta - \cos^2 \theta - \sin \theta} = \frac{\cos \theta (1 - 2 \sin \theta)}{2 \sin^2 \theta - \sin \theta - 1}.$$

There are horizontal tangents when $\sin \theta = 1/2$, which occurs for $\theta = \pi/6, 5\pi/6$, and when $\cos \theta = 0$ (but not $\sin \theta = 1$) which occurs at $\theta = 3\pi/2$. So the horizontal tangents are at $(1/2, \pi/6)$, $((1/2, 5\pi/6)$, and $(2, 3\pi/2)$. There are vertical tangents when $2 \sin^2 \theta - \sin \theta - 1 = (2 \sin \theta + 1)(\sin \theta - 1) = 0$, or $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The vertical tangents are thus at $(3/2, 7\pi/6)$, $(3/2, 11\pi/6)$, and $(0, \pi/2)$, as well as the aforementioned $(0, \pi/2)$.

b. This is undefined.

c.



42 $\sqrt{3} = 2 \cos 2\theta$, so $\cos 2\theta = \frac{\sqrt{3}}{2}$, so $2\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$. Thus $\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$. The 4 intersection points are $\left(\sqrt{3}, \frac{\pi}{12}\right), \left(\sqrt{3}, \frac{11\pi}{12}\right), \left(\sqrt{3}, \frac{13\pi}{12}\right), \left(\sqrt{3}, \frac{23\pi}{12}\right)$. Using symmetry, we see that there are also intersection points at $\left(\sqrt{3}, \frac{5\pi}{12}\right), \left(\sqrt{3}, \frac{7\pi}{12}\right), \left(\sqrt{3}, \frac{17\pi}{12}\right), \left(\sqrt{3}, \frac{19\pi}{12}\right)$.

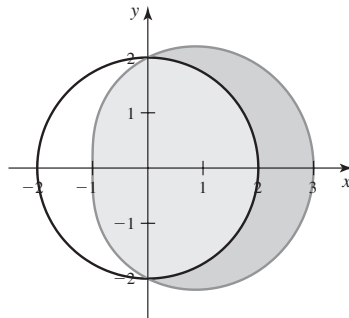
43 $\sqrt{\cos 3t} = \sqrt{\sin 3t}$, so for $t \neq 0$, $\tan 3t = 1$, so $3t = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$. So $t = \frac{\pi}{12}, \frac{9\pi}{12}, \frac{17\pi}{12}$. The curves intersect at $\left(\frac{\pi}{12}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(\frac{17\pi}{12}, \frac{1}{\sqrt{2}}\right)$. Also, they intersect at $(0, 0)$.

44 The area is given by $\int_0^{\pi/4} \frac{1}{2} (9 \sin^2 4\theta) d\theta = \frac{9}{4} \int_0^{\pi/4} (1 - \cos 8\theta) d\theta = \frac{9}{4} \left(\theta - \frac{\sin 8\theta}{8} \right) \Big|_0^{\pi/4} = \frac{9\pi}{16}$.

45 The area is

$$\begin{aligned} 2 \int_{7\pi/6}^{3\pi/2} \frac{1}{2} (1 + 2 \sin t)^2 dt &= \int_{7\pi/6}^{3\pi/2} (1 + 4 \sin t + 4 \sin^2 t) dt \\ &= \int_{7\pi/6}^{3\pi/2} (1 + 4 \sin t + 2 - 2 \cos 2t) dt = (3t - 4 \cos t - \sin 2t) \Big|_{7\pi/6}^{3\pi/2} \\ &= \left(\frac{9\pi}{2} - 0 - 0 \right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2} \right) = \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

The curves intersect at $\theta = \pm\pi/2$. By symmetry, the area is twice the area outside the circle and inside the limaçon between 0 and $\pi/2$. We have $A = 2 \cdot \frac{1}{2} \int_0^{\pi/2} ((2 + \cos \theta)^2 - 2^2) d\theta = \int_0^{\pi/2} (4 \cos \theta + \cos^2 \theta) d\theta = \left(4 \sin \theta + \frac{1}{2} (\cos \theta \sin \theta + \theta) \right) \Big|_0^{\pi/2} = 4 + \frac{\pi}{4}$.



47 $2 = 4 \cos 2\theta$, so $\cos 2\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{6}$. The area is $4 \int_0^{\pi/6} \frac{1}{2} (4 \cos 2\theta - 2) d\theta = 4 \int_0^{\pi/6} (2 \cos 2\theta - 1) d\theta = 4(\sin 2\theta - \theta) \Big|_0^{\pi/6} = 4 \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) = 2\sqrt{3} - \frac{2\pi}{3}$.

48 $\sqrt{2} + \sin \theta = 3 \sin \theta$, so $\sin \theta = \frac{\sqrt{2}}{2}$, so $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. The area is given by

$$\pi \left(\frac{3}{2} \right)^2 - 2 \int_{\pi/4}^{3\pi/4} \frac{1}{2} (9 \sin^2 \theta - (\sqrt{2} + \sin \theta)^2) d\theta.$$

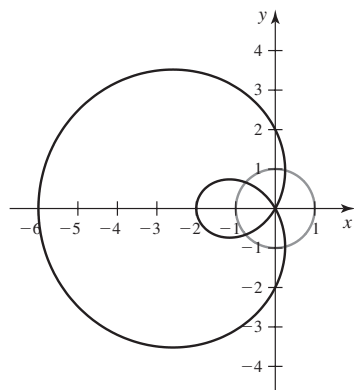
Note that $(9 \sin^2 \theta - (\sqrt{2} + \sin \theta)^2) = -2 - 2\sqrt{2} \sin \theta + 8 \sin^2 \theta = 2 - 2\sqrt{2} \sin \theta - 4 \cos 2\theta$. So our area becomes

$$\begin{aligned} \frac{9\pi}{4} + \int_{\pi/4}^{\pi/2} (4 \cos 2\theta + 2\sqrt{2} \sin \theta - 2) d\theta &= \frac{9\pi}{4} + \left(2 \sin 2\theta - 2\sqrt{2} \cos \theta - 2\theta \right) \Big|_{\pi/4}^{\pi/2} \\ &= \frac{9\pi}{4} + (0 - 0 - \pi) - \left(2 - 2 - \frac{\pi}{2} \right) \\ &= \frac{7\pi}{4}. \end{aligned}$$

49 The area is given by $2 \int_0^{\pi/2} \frac{1}{2} ((1 + \cos \theta)^2 - (1 - \cos \theta)^2) d\theta = 4 \int_0^{\pi/2} \cos \theta d\theta = 4 \sin \theta \Big|_0^{\pi/2} = 4..$

50

a.



There are 4 intersection points.

b. Note that $2 - 4 \cos \theta = 1$ for $\theta = \cos^{-1}(1/4) \approx 1.32$, and $2 - 4 \cos \theta = -1$ for $\theta = \cos^{-1}(3/4) \approx 0.73$. The points of intersection (in polar form) are approximately $(1, 1.32)$, $(1, 2\pi - 1.32) \approx (1, 4.96)$, $(-1, 0.73)$, and $(-1, 2\pi - 0.73) \approx (-1, 5.56)$.

51 $L = \int_0^{2\pi} \sqrt{(3 - 6 \cos \theta)^2 + (6 \sin \theta)^2} d\theta = \int_0^{2\pi} 3\sqrt{5 - 4 \cos(\theta)} d\theta \approx 40.09.$

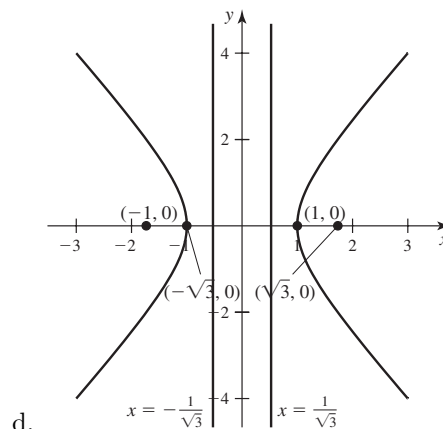
52 $L = \int_0^{2\pi} \sqrt{(3 + 2 \cos \theta)^2 + (-2 \sin \theta)^2} d\theta = \int_0^{2\pi} \sqrt{13 + 12 \cos \theta} d\theta \approx 21.01.$

53

a. This represents a hyperbola with $a = 1$ and $b = \sqrt{2}$.

b. The vertices are $(\pm 1, 0)$, the foci are $(\pm c, 0)$ where $c^2 = a^2 + b^2 = 3$, so they are $(\pm\sqrt{3}, 0)$. The directrices are $x = \frac{\pm a^2}{c} = \frac{\pm 1}{\sqrt{3}}.$

c. The eccentricity is $e = \frac{c}{a} = \sqrt{3}$.

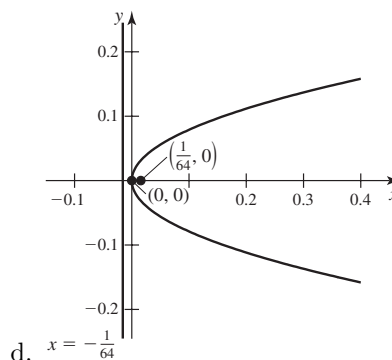


d.

54

a. This represents a parabola.

b. We can write $y^2 = \frac{1}{16}x = 4 \cdot \frac{1}{64}x$, so $(p, 0) = (\frac{1}{64}, 0)$ is the focus, and the directrix is $x = -\frac{1}{64}$. The vertex is $(0, 0)$.



c. $e = 1$, because that is the case for all parabolas.

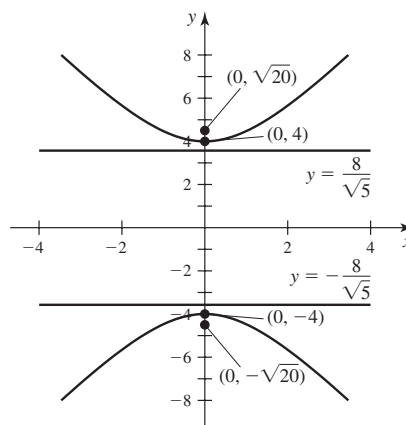
d.

55

a. This can be written as $\frac{y^2}{16} - \frac{x^2}{4} = 1$. It is a hyperbola with $a = 4$ and $b = 2$.

b. The vertices are $(0, \pm 4)$. The foci are $(0, \pm c)$ where $c^2 = a^2 + b^2 = 16 + 4 = 20$, so they are $(0, \pm\sqrt{20})$.
The directrices are $y = \frac{\pm a^2}{c} = \frac{\pm 16}{\sqrt{20}} = \frac{\pm 8}{\sqrt{5}}$.

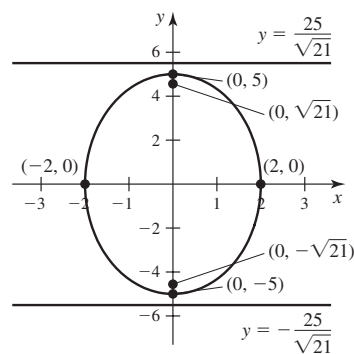
c. The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{20}}{4} = \frac{\sqrt{5}}{2}$.



d.

56

- a. This represents an ellipse with $a = 5$ and $b = 2$.
- b. The vertices are $(0, \pm 5)$. The foci are $(0, \pm c)$ where $c^2 = a^2 - b^2 = 25 - 4 = 21$, so they are $(0, \pm\sqrt{21})$.
The directrices are $y = \frac{\pm a^2}{c} = \frac{\pm 25}{\sqrt{21}}$.



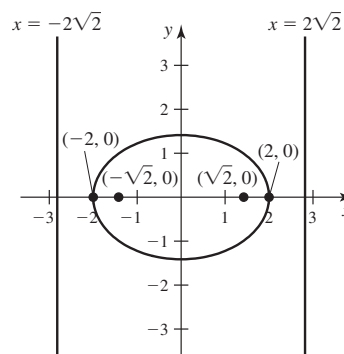
d.

c. The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{21}}{5}$.

57

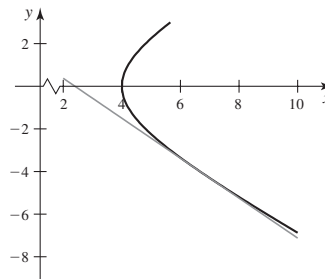
- a. This can be written as $\frac{x^2}{4} + \frac{y^2}{2} = 1$, so it is an ellipse with $a = 2$ and $b = \sqrt{2}$.
- b. The vertices are $(\pm 2, 0)$. The foci are $(\pm c, 0)$ where $c^2 = a^2 - b^2 = 4 - 2 = 2$, so they are $(\pm\sqrt{2}, 0)$.
The directrices are $x = \frac{\pm a^2}{c} = \frac{\pm 4}{\sqrt{2}} = \pm 2\sqrt{2}$.

c. The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{2}}{2}$.

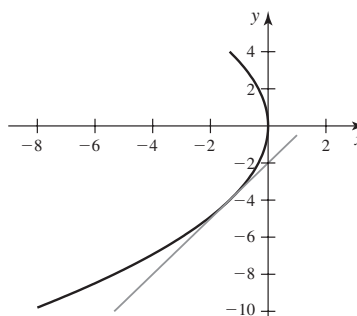


d.

- 58 $\frac{x}{8} - \frac{2y}{9} \cdot \frac{dy}{dx} = 0$, so at the given point,
 $\frac{dy}{dx} = -\frac{15}{16}$. The equation of the tangent
 line is therefore $y + 4 = -\frac{15}{16} \left(x - \frac{20}{3} \right)$, or
 $y = -\frac{15}{16}x + \frac{9}{4}$.



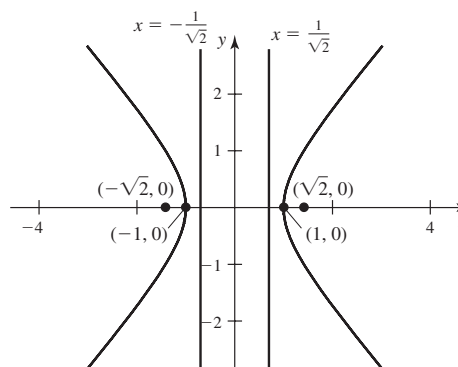
- 59 $2y \frac{dy}{dx} = -12$, so at the point in question,
 $\frac{dy}{dx} = 3/2$. So the equation of the tangent
 line is $y + 4 = \frac{3}{2} \left(x + \frac{4}{3} \right)$, or $y = \frac{3}{2}x - 2$.



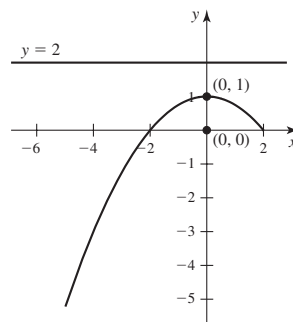
60

- a. Recall that $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, so $r^2 \cos(2\theta) = 1$ becomes $r^2(\cos^2 \theta - \sin^2 \theta) = x^2 - y^2 = 1$. The curve is a hyperbola.
- b. With $a = b = 1$, we have $c^2 = 2$, so the vertices are $(\pm 1, 0)$ and the foci are $(\pm\sqrt{2}, 0)$. The directrices are $x = \pm \frac{a^2}{c} = \pm \frac{1}{\sqrt{2}}$. The eccentricity is $e = \frac{c}{a} = \sqrt{2}$.

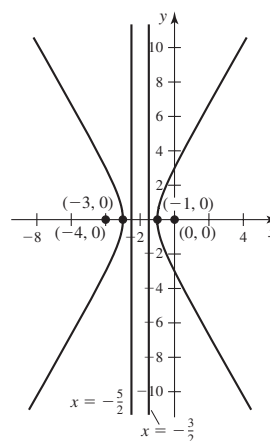
- c. It does not have the form as in Theorem 11.4 because it does not have a focus at the origin.



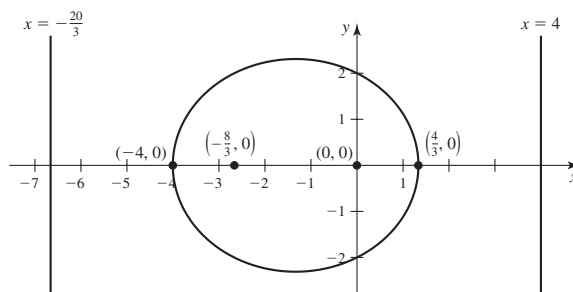
- 61 The eccentricity is 1, and the directrix is $y = 2$. The vertex is $(0, 1)$ and the focus is $(0, 0)$.



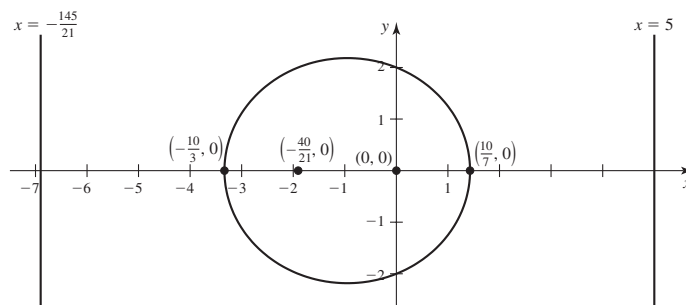
- 62 The eccentricity is 2, and the directrices are $x = -\frac{3}{2}$ and $x = -\frac{5}{2}$. The vertices are $(-1, 0)$ and $(-3, 0)$ and the foci are $(0, 0)$ and $(-4, 0)$.



- 63 The eccentricity is $\frac{1}{2}$, and the directrices are $x = 4$ and $x = -\frac{20}{3}$. The vertices are $(\frac{4}{3}, 0)$ and $(-4, 0)$ and the foci are $(0, 0)$ and $(-\frac{8}{3}, 0)$.

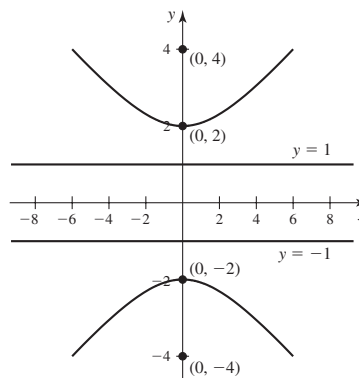


- 64 The eccentricity is $\frac{2}{5}$. The vertices are $(10/7, 0)$ and $(-10/3, 0)$, so the center is $(-20/21, 0)$. The foci are $(0, 0)$ and $(-40/21, 0)$. The directrices are $x = -\frac{145}{21}$ and $x = 5$.



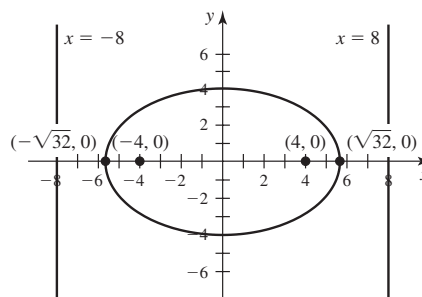
Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 2$

- 65** and because $\frac{a^2}{c} = d = 1$, we have $c = 4$. So $b^2 = 16 - 4 = 12$. The hyperbola has equation $\frac{y^2}{4} - \frac{x^2}{12} = 1$. The eccentricity is $\frac{c}{a} = \frac{4}{2} = 2$.



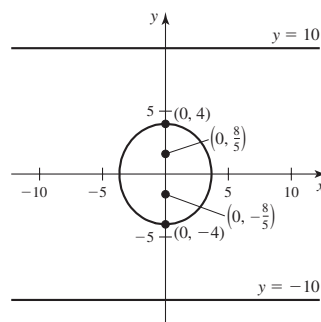
Because the center is halfway between the foci, it is $(0,0)$. We must have $c = 4$ and

- 66** because $\frac{a^2}{c} = d = 8$, we have $a^2 = 32$. So $b^2 = a^2 - c^2 = 32 - 16 = 16$. The ellipse has equation $\frac{x^2}{32} + \frac{y^2}{16} = 1$. The eccentricity is $\frac{c}{a} = \frac{4}{4\sqrt{2}} = \frac{\sqrt{2}}{2}$.



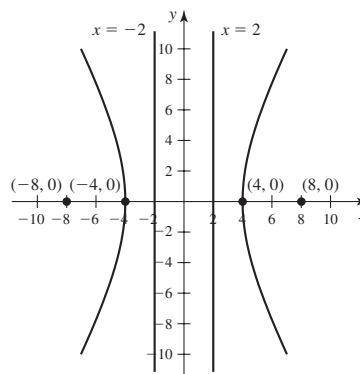
Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 4$

- 67** and because $\frac{a^2}{c} = d = 10$, we have $c = \frac{8}{5}$. So $b^2 = a^2 - c^2 = 16 - \frac{64}{25} = \frac{336}{25}$. The ellipse has equation $\frac{25x^2}{336} + \frac{y^2}{16} = 1$. The eccentricity is $\frac{c}{a} = \frac{8/5}{4} = \frac{2}{5}$.



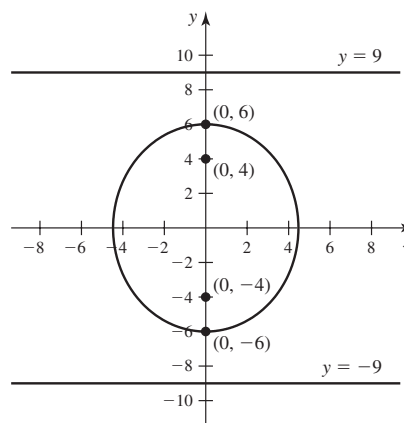
68

Because the center is halfway between the vertices, it is $(0, 0)$. We must have $a = 4$ and because $\frac{a^2}{c} = d = 2$, we have $c = 8$. So $b^2 = 64 - 16 = 48$. The hyperbola has equation $\frac{x^2}{16} - \frac{y^2}{48} = 1$. The eccentricity is $\frac{c}{a} = \frac{8}{4} = 2$.



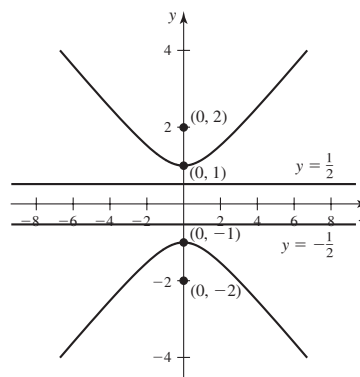
69

We have $a = 6$, $c = 4$ and $e = \frac{c}{a} = \frac{4}{6} = \frac{2}{3}$. Also, $b^2 = a^2 - c^2 = 36 - 16 = 20$, and the equation is $\frac{y^2}{36} + \frac{x^2}{20} = 1$. The vertices are $(\pm 2\sqrt{5}, 0)$. The directrices are $y = \pm \frac{a^2}{c} = \pm 9$.



70

We have $c = 2$, $e = \frac{c}{a} = 2$, so $a = 1$. Also, $b^2 = c^2 - a^2 = 3$, so the equation is $\frac{y^2}{1} - \frac{x^2}{3} = 1$. We have $d = \frac{a^2}{c} = \frac{1}{2}$. The vertices are $(0, \pm 1)$ and the directrices are $y = \pm \frac{1}{2}$.



71 The area of the ellipse in the first quadrant is $\frac{\pi ab}{4}$, so we are seeking θ_0 so that

$$\frac{\pi ab}{8} = \frac{1}{2} \int_0^{\theta_0} \frac{a^2 b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta = \frac{a^2}{2} \int_0^{\theta_0} \frac{\sec^2 \theta}{\frac{a^2}{b^2} \tan^2 \theta + 1} d\theta.$$

Let $u = \frac{a}{b} \tan \theta$ so that $du = \frac{a}{b} \sec^2 \theta d\theta$. Then we have

$$\frac{\pi ab}{8} = \frac{ab}{2} \int_0^{\frac{a}{b} \tan \theta_0} \frac{1}{1+u^2} du = \frac{ab}{2} \tan^{-1}\left(\frac{a}{b} \tan(\theta_0)\right).$$

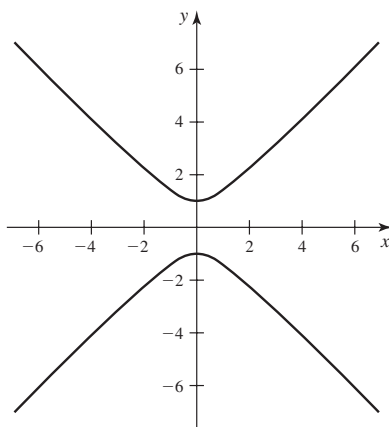
Note that this equation is satisfied when $\tan(\theta_0) = \frac{b}{a}$, because then the expression on the right-hand side of that equation is $\frac{ab}{2} \cdot \frac{\pi}{4} = \frac{\pi ab}{8}$. So the desired value of m is $\tan(\theta_0) = \frac{b}{a}$.

72

- a. The curves are tangent when there is only one point of intersection in the first quadrant. This occurs when $x^2 - p^2 x^4 = 1$ has only one solution. This quadratic-type equation $-p^2(x^2)^2 + (x^2) - 1 = 0$ has solution $x^2 = \frac{1 \pm \sqrt{1-4p^2}}{2p^2}$, and the discriminant $1 - 4p^2$ is 0 for $p = 1/2$.
- b. The two curves intersect for $x^2 = \frac{1 \pm 0}{2p^2} = 2$, so for $x = \sqrt{2}$. The corresponding value for y is $\frac{1}{2}(\sqrt{2})^2 = 1$.
- c. Using the same line of reasoning, we seek the value of p so that $\frac{x^2}{a^2} - \frac{p^2 x^4}{b^2} = 1$, which yields the quadratic-type equation $p^2 a^2 (x^2)^2 - b^2 (x^2) + a^2 b^2 = 0$. The discriminant is 0 when $b^4 - 4p^2 a^4 b^2 = 0$, which occurs when $p = \frac{b}{2a^2}$. The point of intersection is $x = \sqrt{2}a$. The corresponding value of y is $px^2 = b$.

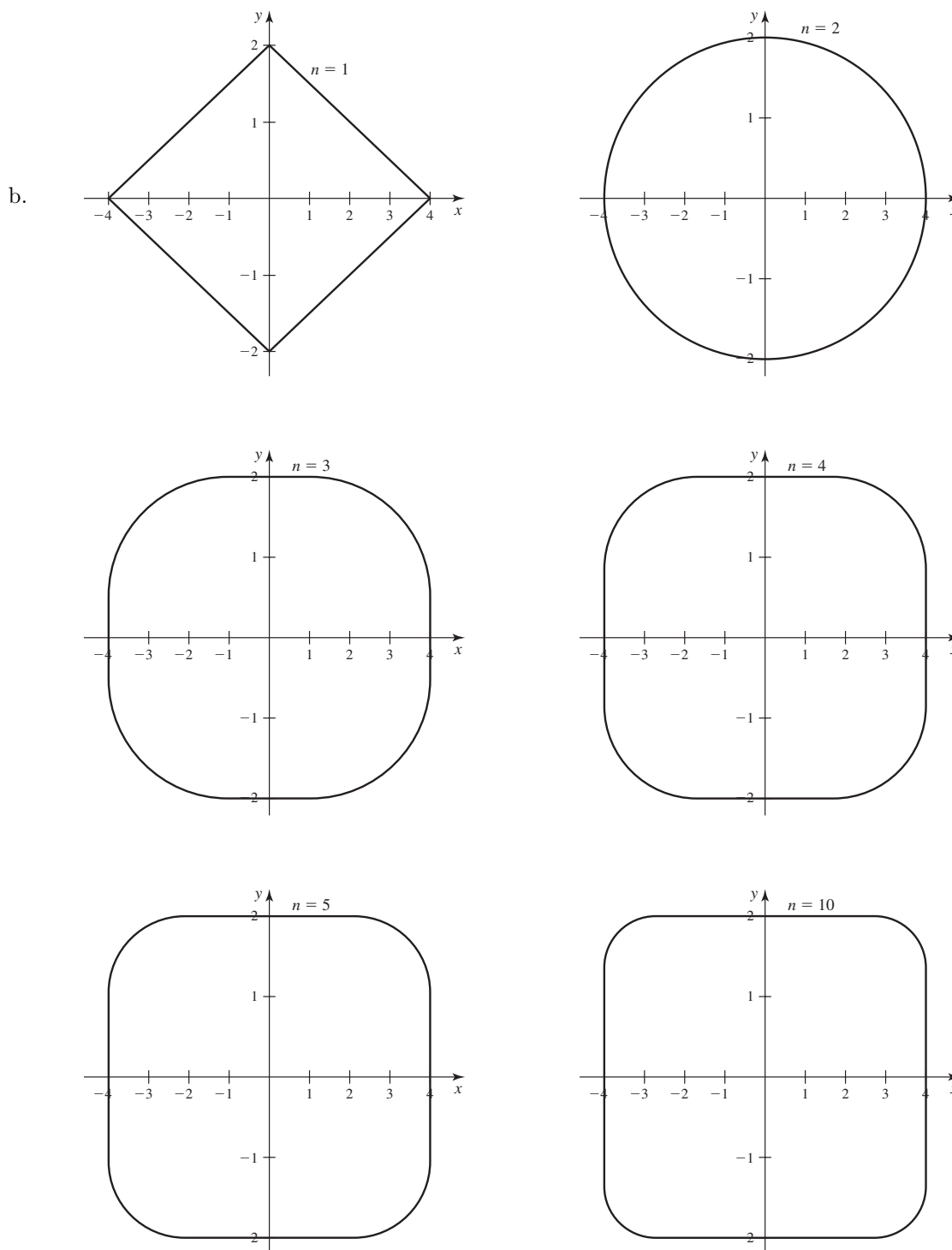
73 Note that $Q = (a \cos \theta, a \sin \theta)$ and $R = (b \cos \theta, b \sin \theta)$, where θ is the angle formed by l and the x -axis. Then $P = (a \sin \theta, b \cos \theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, because it satisfies that equation.

74



75

- a. Let $\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$ Let $x = a \cdot \text{sgn}(\cos t) |\cos(t)|^{2/n}$ and $y = b \cdot \text{sgn}(\sin(t)) |\sin(t)|^{2/n}$.



- c. As n increases from near 0 to near 1, the curves change from star-shaped to a rectangular shape with corners at $(\pm a, 0)$ and $(0, \pm b)$. As n increases from 1 on, the curves become more rectangular with corners at $(\pm a, \pm b)$.

76 Suppose that $a^2 + c^2 = b^2 + d^2$, and that $ab + cd = 0$. Note that

$$x^2 + y^2 = a^2 \cos^2 t + 2ab \sin t \cos t + b^2 \sin^2 t + c^2 \cos^2 t + 2cd \sin t \cos t + d^2 \sin^2 t,$$

which can be rewritten as

$$(a^2 + c^2) \cos^2 t + (b^2 + d^2) \sin^2 t + (2ab + 2cd) \sin t \cos t.$$

Because $b^2 + d^2 = a^2 + c^2$ and because $2ab + 2cd = 0$, we can write this as

$$(a^2 + c^2)(\cos^2 t + \sin^2 t) = R^2,$$

so we have the circle $x^2 + y^2 = R^2$, as desired.

Chapter 13

Vectors and the Geometry of Space

13.1 Vectors in the Plane

13.1.1 The coordinates of a point determine its location, but a given point has no width or breadth, so it has no size or direction. A nonzero vector has size (magnitude) and direction, but it has no location in the sense that it can be translated to a different initial point and be considered the same vector.

13.1.2 A position vector is one with its tail at the origin.

13.1.3 Two vectors are equal if they have the same magnitude and direction. Given a position vector, any translation of that vector to a different initial point yields an equivalent vector. Because there are infinitely many such translations which don't change the given vector's direction or magnitude, there are infinitely many vectors equivalent to the given one.

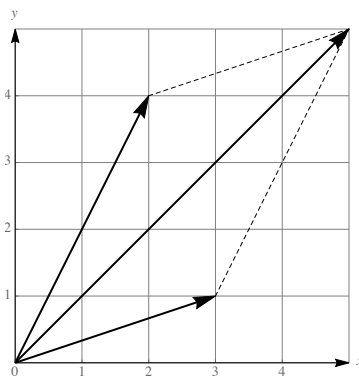
13.1.4 $\overrightarrow{PQ} = \langle 4, 0 \rangle$ and $\overrightarrow{QP} = \langle -4, 0 \rangle$.

13.1.5 $\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$.

13.1.6 $\pm \frac{\langle 2, 3 \rangle}{\sqrt{4+9}} = \pm \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$.

13.1.7 No. $|\langle 1, 1 \rangle| = \sqrt{2} \neq 1$, so $\langle 1, 1 \rangle$ is not a unit vector.

13.1.8 $\langle 3, 1 \rangle + \langle 2, 4 \rangle = \langle 5, 5 \rangle$.



13.1.9 $|\mathbf{v}| = |\langle v_1, v_2 \rangle| = \sqrt{v_1^2 + v_2^2}$.

13.1.10 $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$.

13.1.11 If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are given, then $|\overrightarrow{PQ}| = |\langle q_1 - p_1, q_2 - p_2 \rangle| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$.

13.1.12 The speed is $|\langle 2\sqrt{2}, 2\sqrt{2} \rangle| = \sqrt{8+8} = 4$ ft/s and the heading of the kayak is 45° north of east.

13.1.13 The vectors in choices a, c, and e are all equal to $\overrightarrow{CE}$.

13.1.14 The vectors in choices b, c, and e are equal to $\overrightarrow{BK}$.

13.1.15

- a. $3\mathbf{v}$ b. $2\mathbf{u}$ c. $-3\mathbf{u}$ d. $-2\mathbf{u}$ e. $\mathbf{v}$

13.1.16

- a. $2\mathbf{v}$ b. $-2\mathbf{v}$ c. $3\mathbf{u}$ d. $-5\mathbf{v}$ e. $-3\mathbf{u}$

13.1.17

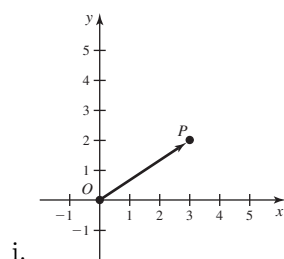
- a. $3\mathbf{u} + 3\mathbf{v}$ b. $\mathbf{u} + 2\mathbf{v}$ c. $2\mathbf{u} + 5\mathbf{v}$ d. $-2\mathbf{u} + 3\mathbf{v}$ e. $3\mathbf{u} + 2\mathbf{v}$
 f. $-3\mathbf{u} - 2\mathbf{v}$ g. $-2\mathbf{u} - 4\mathbf{v}$ h. $\mathbf{u} - 4\mathbf{v}$ i. $-\mathbf{u} - 6\mathbf{v}$

13.1.18

- a. $\mathbf{u} + 3\mathbf{v}$ b. $\mathbf{u} + 3\mathbf{v}$ c. $2\mathbf{u} + 2\mathbf{v}$ d. $2\mathbf{u} - 3\mathbf{v}$ e. $-\mathbf{u} - 2\mathbf{v}$
 f. $4\mathbf{u} + 4\mathbf{v}$ g. $-\mathbf{u} + 2\mathbf{v}$ h. $-\mathbf{u} + 2\mathbf{v}$ i. $2\mathbf{u} - 4\mathbf{v}$

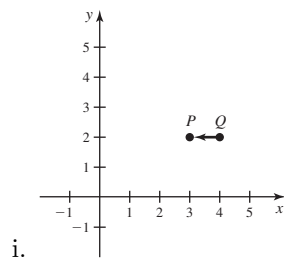
13.1.19

- a. $\overrightarrow{OP}$

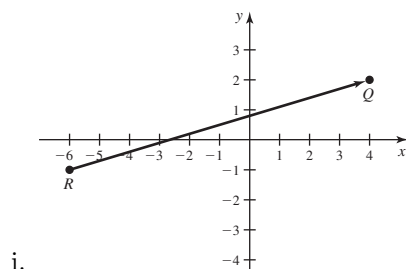


ii. $|3\mathbf{i} + 2\mathbf{j}| = \sqrt{13}$.

- b. $\overrightarrow{QP}$



ii. $|-\mathbf{i} + 0 \cdot \mathbf{j}| = 1$.

c. $\overrightarrow{RQ}$ 

ii. $|10\mathbf{i} + 3\mathbf{j}| = \sqrt{109}.$

13.1.20

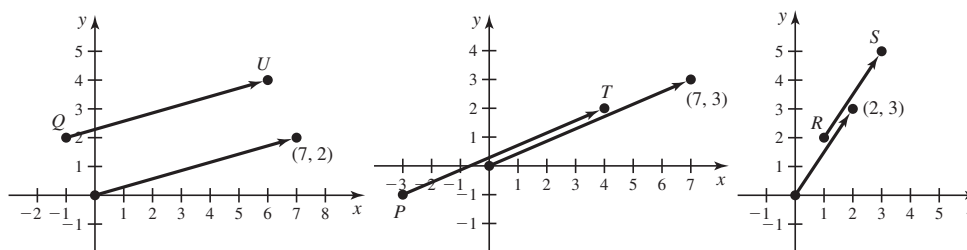
a. $\overrightarrow{AB} = \langle 6, 16 \rangle - \langle -2, 0 \rangle = \langle 8, 16 \rangle.$

b. $\overrightarrow{AC} = \langle 1, 4 \rangle - \langle -2, 0 \rangle = \langle 3, 4 \rangle.$

c. $\overrightarrow{EF} = \langle 3\sqrt{2}, -4\sqrt{2} \rangle - \langle \sqrt{2}, \sqrt{2} \rangle = \langle 2\sqrt{2}, -5\sqrt{2} \rangle.$

d. $\overrightarrow{CD} = \langle 5, 4 \rangle - \langle 1, 4 \rangle = \langle 4, 0 \rangle.$

13.1.21 $\overrightarrow{QU} = \langle 7, 2 \rangle, \overrightarrow{PT} = \langle 7, 3 \rangle, \overrightarrow{RS} = \langle 2, 3 \rangle.$



13.1.22 $\overrightarrow{PQ} = \langle 2, 3 \rangle, \overrightarrow{RS} = \langle 2, 3 \rangle, \text{ and } \overrightarrow{TU} = \langle 2, 2 \rangle, \text{ so } \overrightarrow{PQ} = \overrightarrow{RS}.$

13.1.23 $\overrightarrow{QT} = \langle 5, 0 \rangle, \text{ while } \overrightarrow{SU} = \langle 3, -1 \rangle.$

13.1.24 $\mathbf{u} + \mathbf{v} = \langle 4, -2 \rangle + \langle -4, 6 \rangle = \langle 0, 4 \rangle.$

13.1.25 $\mathbf{w} - \mathbf{u} = \langle 0, 8 \rangle - \langle 4, -2 \rangle = \langle -4, 10 \rangle.$

13.1.26 $2\mathbf{u} + 3\mathbf{v} = 2\langle 4, -2 \rangle + 3\langle -4, 6 \rangle = \langle -4, 14 \rangle.$

13.1.27 $10\mathbf{u} - 3\mathbf{v} + \mathbf{w} = 10\langle 4, -2 \rangle - 3\langle -4, 6 \rangle + \langle 0, 8 \rangle = \langle 52, -30 \rangle.$

13.1.28 $|\mathbf{u} + \mathbf{v} + \mathbf{w}| = |\langle 3, -4 \rangle + \langle 1, 1 \rangle + \langle -1, 0 \rangle| = |\langle 3, -3 \rangle| = 3\sqrt{2}.$

13.1.29 $|-2\mathbf{v}| = |\langle -2, -2 \rangle| = \sqrt{4 + 4} = 2\sqrt{2}.$

13.1.30 Two vectors parallel to $\mathbf{u}$ with magnitude four times that of $\mathbf{u}$ are the vectors $\pm 4\mathbf{u} = \pm 4\langle 3, -4 \rangle = \pm \langle 12, -16 \rangle$. So they are $\langle 12, -16 \rangle$ and $\langle -12, 16 \rangle$.

13.1.31 $|\mathbf{u} - \mathbf{v}| = |\langle 3, -4 \rangle - \langle 1, 1 \rangle| = |\langle 2, -5 \rangle| = \sqrt{29}.$ $|\mathbf{w} - \mathbf{u}| = |\langle -1, 0 \rangle - \langle 3, -4 \rangle| = |\langle -4, 4 \rangle| = \sqrt{32} = 4\sqrt{2}.$ $\mathbf{w} - \mathbf{u}$ has the greater magnitude.

13.1.32 $\frac{\langle -6, 8 \rangle}{10} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle.$

$$13.1.33 \quad \langle -5, 12 \rangle = 13 \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle.$$

$$13.1.34 \quad \overrightarrow{PQ} = \langle 4, -3 \rangle = 5 \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle.$$

$$13.1.35 \quad 6 \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \langle 3, 3\sqrt{3} \rangle.$$

$$13.1.36 \quad -10 \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \langle -6, 8 \rangle.$$

$$13.1.37 \quad \mathbf{u} = 3 \frac{\langle 5, -12 \rangle}{\sqrt{25 + 144}} = \frac{3}{13} \langle 5, -12 \rangle.$$

$$13.1.38 \quad \mathbf{u} = -20 \frac{\langle 6, -8 \rangle}{10} = \langle -12, 16 \rangle.$$

$$13.1.39 \quad 10 \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = 10 \cdot \frac{1}{\sqrt{9+4}} \cdot \langle 3, -2 \rangle = \left\langle \frac{30}{\sqrt{13}}, -\frac{20}{\sqrt{13}} \right\rangle.$$

13.1.40

a. $3 \langle 8, 15 \rangle = \langle 24, 45 \rangle.$

b. $3 \frac{\langle 8, 15 \rangle}{\sqrt{64 + 225}} = 3 \left\langle \frac{8}{17}, \frac{15}{17} \right\rangle = \left\langle \frac{24}{17}, \frac{45}{17} \right\rangle.$

$$13.1.41 \quad \overrightarrow{QR} = \langle 2, 6 \rangle - \langle 3, -4 \rangle = \langle -1, 10 \rangle = -\mathbf{i} + 10\mathbf{j}.$$

$$13.1.42 \quad \overrightarrow{PQ} = \langle 3, -4 \rangle - \langle -4, 1 \rangle = \langle 7, -5 \rangle = 7\mathbf{i} - 5\mathbf{j}.$$

$$13.1.43 \quad \mathbf{u} = \frac{\overrightarrow{PR}}{|\overrightarrow{PR}|} = \frac{\langle 2, 6 \rangle - \langle -4, 1 \rangle}{|\overrightarrow{PR}|} = \frac{\langle 6, 5 \rangle}{\sqrt{36 + 25}} = \left\langle \frac{6}{\sqrt{61}}, \frac{5}{\sqrt{61}} \right\rangle \text{ is one unit vector, while the other is } -\mathbf{u} = \left\langle -6/\sqrt{61}, -5/\sqrt{61} \right\rangle$$

$$13.1.44 \quad \mathbf{u} = \frac{\overrightarrow{QR}}{|\overrightarrow{QR}|} = \frac{\langle -1, 10 \rangle}{\sqrt{101}} = -\frac{1}{\sqrt{101}}\mathbf{i} + \frac{10}{\sqrt{101}}\mathbf{j}.$$

13.1.45 $\overrightarrow{QP} = \langle -4, 1 \rangle - \langle 3, -4 \rangle = \langle -7, 5 \rangle.$ A unit vector parallel to $\overrightarrow{QP}$ is $\frac{1}{\sqrt{74}} \langle -7, 5 \rangle.$ So the desired vectors are $\frac{4}{\sqrt{74}} \langle -7, 5 \rangle$ and $-\frac{4}{\sqrt{74}} \langle -7, 5 \rangle.$

13.1.46 Using the result of the previous problem, the two vectors are $4 \left\langle \frac{6}{\sqrt{61}}, \frac{5}{\sqrt{61}} \right\rangle = \left\langle \frac{24}{\sqrt{61}}, \frac{20}{\sqrt{61}} \right\rangle$ and $-4 \left\langle \frac{6}{\sqrt{61}}, \frac{5}{\sqrt{61}} \right\rangle = \left\langle -\frac{24}{\sqrt{61}}, -\frac{20}{\sqrt{61}} \right\rangle.$

13.1.47

a. Because the magnitude of $\mathbf{v}$ is $\sqrt{36 + 64} = 10$, the two desired vectors are $\langle 6/10, -8/10 \rangle = \langle 3/5, -4/5 \rangle$ and $\langle -3/5, 4/5 \rangle.$

b. If the magnitude of $\mathbf{v}$ is 1, then $\sqrt{\frac{1}{9} + b^2} = 1$, so $b^2 = \frac{8}{9}$, so $b = \pm \frac{2\sqrt{2}}{3}.$

c. If the magnitude of $\mathbf{w}$ is 1, then $\sqrt{a^2 + \frac{a^2}{9}} = 1$, so $\frac{10a^2}{9} = 1$, so $a = \pm \frac{3}{\sqrt{10}}$.

13.1.48 The Cartesian coordinates of P are $(r \cos \theta, r \sin \theta)$ and therefore $\overrightarrow{OP} = \langle r \cos \theta, r \sin \theta \rangle$.

13.1.49 $\left\langle 8 \cos \frac{5\pi}{6}, 8 \sin \frac{5\pi}{6} \right\rangle = \langle -4\sqrt{3}, 4 \rangle$.

13.1.50 $\left\langle 40 \cos \frac{\pi}{4}, 40 \sin \frac{\pi}{4} \right\rangle = \langle 20\sqrt{2}, 20\sqrt{2} \rangle$.

13.1.51 $\left\langle 30 \cos \left(-\frac{\pi}{5}\right), 30 \sin \left(-\frac{\pi}{5}\right) \right\rangle = \langle 15\sqrt{3}, -15 \rangle$.

13.1.52 $\left\langle 100 \cos \left(-\frac{\pi}{4}\right), 100 \sin \left(-\frac{\pi}{4}\right) \right\rangle = \langle 50\sqrt{2}, -50\sqrt{2} \rangle$.

13.1.53

a. $\mathbf{v}_a = \langle -320, 0 \rangle$ and $\mathbf{w} = \langle -40 \cos 45^\circ, -40 \sin 45^\circ \rangle = \left\langle -\frac{40}{\sqrt{2}}, -\frac{40}{\sqrt{2}} \right\rangle = \langle -20\sqrt{2}, -20\sqrt{2} \rangle$. Therefore the plane's vector is given by $\mathbf{v}_g = \mathbf{v}_a + \mathbf{w} = (-320 - 20\sqrt{2})\mathbf{i} - 20\sqrt{2}\mathbf{j}$.

b. The magnitude of $\mathbf{v}_g$ is $\sqrt{(-320 - 20\sqrt{2})^2 + (-20\sqrt{2})^2} \approx 349.43$ miles per hour.

$$\theta = \tan^{-1} \left(\frac{20\sqrt{2}}{320 + 20\sqrt{2}} \right) \approx 0.0810 \text{ radians, or about } 4.64 \text{ degrees south of west.}$$

13.1.54

a. $\mathbf{v}_a = \langle 400, 0 \rangle$; $\mathbf{v}_g = \langle 420 \cos 5^\circ, 420 \sin 5^\circ \rangle \approx \langle 418.4, 36.6 \rangle$; $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_a \approx \langle 418.4, 36.6 \rangle - \langle 400, 0 \rangle = \langle 18.4, 36.6 \rangle$.

b. Wind speed is $|\mathbf{w}| \approx |\langle 18.4, 36.6 \rangle| \approx 41$ mi/hr. The heading is $\tan^{-1}(36.6/18.4) \approx 63.3$ degrees north of east.

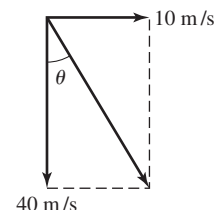
13.1.55 $\mathbf{v}_g = \langle 0, 480 \rangle$ and $\mathbf{w} = \langle 20 \cos(-60^\circ), 20 \sin(-60^\circ) \rangle = \langle 10, -10\sqrt{3} \rangle$. Therefore $\mathbf{v}_a = \mathbf{v}_g - \mathbf{w} = \langle 0, 480 \rangle - \langle 10, -10\sqrt{3} \rangle = \langle -10, 480 + 10\sqrt{3} \rangle$.

$|\mathbf{v}_a| = |\langle -10, 480 + 10\sqrt{3} \rangle| \approx 497.4$ mi/hr; The heading is $180^\circ + \tan^{-1} \left(\frac{480 + 10\sqrt{3}}{-10} \right) \approx 91.15^\circ$ which is about 1.2° west of north.

13.1.56 Let $\mathbf{w} = \langle 0, -10 \rangle$ represent the water, and let $\mathbf{b} = \langle 20, 0 \rangle$ represent the boat relative to the shore. Then the vector $\mathbf{v}$ which represents the boat relative to the water is given by $\mathbf{v} + \mathbf{w} = \mathbf{b}$, so $\mathbf{v} = \mathbf{b} - \mathbf{w} = \langle 20, 10 \rangle$. Note that $|\langle 20, 10 \rangle| = 10\sqrt{5} \approx 22.4$ miles per hour represents the speed of the boat. The direction is the direction of the vector $\langle 2, 1 \rangle$ which is $\tan^{-1}(1/2) \cdot \frac{180}{\pi} \approx 26.6$ degrees north of east.

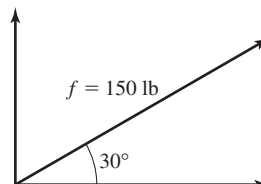
13.1.57 Let $\mathbf{w} = \langle 0, -5 \rangle$ represent the water, and let $\mathbf{v}_w = \langle 40, 0 \rangle$ represent the boat relative to the water. Then relative to the shore we have $\mathbf{v}_s = \mathbf{w} + \mathbf{v}_w = \langle 40, -5 \rangle$, which has magnitude $\sqrt{1625} \approx 40.3$ km/hr.

13.1.58 Let $\mathbf{p}$ represent the vector's terminal velocity vector. $\mathbf{p} = \langle 10, -40 \rangle$, and $|\mathbf{p}| = \sqrt{100 + 1600} = 10\sqrt{17}$. $\theta = \tan^{-1}(10/40) \approx 0.245$ radians, or 14.037 degrees. Thus, the speed is $10\sqrt{17}$ meters per second, and the direction is about 14.037 degrees east of vertical.



13.1.59 Let $\mathbf{u} = \mathbf{i}$ represent the current and $\mathbf{v} = \sqrt{3}\cos(\pi/6)\mathbf{i} + \sqrt{3}\sin(\pi/6)\mathbf{j} = \frac{3}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ represent the boat relative to land. If $\mathbf{w}$ represents the wind, then $\mathbf{u} + \mathbf{w} = \mathbf{v}$, so $\mathbf{w} = \mathbf{v} - \mathbf{u} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$. Then $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$, or 60 degrees. The speed of the wind is 1 meter per second in the direction 60 degrees north of east (or 30 degrees east of north).

13.1.60 $\mathbf{F} = 150\cos(\pi/6)\mathbf{i} + 150\sin(\pi/6)\mathbf{j} = 75\sqrt{3}\mathbf{i} + 75\mathbf{j}$. The horizontal component of the force is $75\sqrt{3}$ pounds, and the vertical component is 75 pounds.



13.1.61

- $\mathbf{F} = 40\cos(\pi/3)\mathbf{i} + 40\sin(\pi/3)\mathbf{j} = 20\mathbf{i} + 20\sqrt{3}\mathbf{j}$, so the horizontal component is 20 and the vertical is $20\sqrt{3}$.
- Yes. If it is 45 degrees, the horizontal component would be $40\cos(\pi/4) = 20\sqrt{2} > 20$.
- No. If it is 45 degrees, the vertical component would be $40\sin(\pi/4) = 20\sqrt{2} < 20\sqrt{3}$.

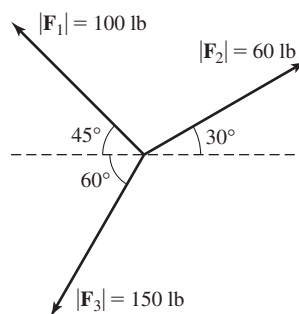
13.1.62 Let $\mathbf{F}_1 = 100\cos(\pi/3)\mathbf{i} + 100\sin(\pi/3)\mathbf{j} = 50\mathbf{i} + 50\sqrt{3}\mathbf{j}$, and let $\mathbf{F}_2 = 60\cos(\pi/6)\mathbf{i} + 60\sin(\pi/6)\mathbf{j} = 30\sqrt{3}\mathbf{i} + 30\mathbf{j}$. Note that $\mathbf{F}_2$ has a greater horizontal component, because $30\sqrt{3} \approx 51.96 > 50$.

13.1.63 Let the magnitude of the force on the two chains be f . Let $\mathbf{F}_1 = (-\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j})f$ and let $\mathbf{F}_2 = (\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j})f$. Then $\mathbf{F}_1 + \mathbf{F}_2 - 500\mathbf{j} = \mathbf{0}$, and solving for f yields $f = 250\sqrt{2}$ pounds.

13.1.64 $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \sqrt{2}(-50\mathbf{i} + 50\mathbf{j}) + (30\sqrt{3}\mathbf{i} + 30\mathbf{j}) + (-75\mathbf{i} - 75\sqrt{3}\mathbf{j}) = (-50\sqrt{2} + 30\sqrt{3} - 75)\mathbf{i} + (50\sqrt{2} + 30 - 75\sqrt{3})\mathbf{j}$. Thus $|\mathbf{F}|$ is about $\sqrt{(-50\sqrt{2} + 30\sqrt{3} - 75)^2 + (50\sqrt{2} + 30 - 75\sqrt{3})^2}$ which is about 98.19 pounds.

$$\theta = \tan^{-1}\left(\frac{50\sqrt{2} + 30 - 75\sqrt{3}}{-50\sqrt{2} + 30\sqrt{3} - 75}\right) \approx$$

17.3 degrees. The magnitude of the net force is about 98 pounds in the direction 17.3 degrees south of west.



13.1.65

- True. This follows because $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = (\mathbf{w} + \mathbf{u}) + \mathbf{v}$ (vector addition is commutative and associative.)
- True. This is because $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- False. For example, if $\mathbf{u} = \langle 3, 4 \rangle$ and $\mathbf{v} = \langle -3, -1 \rangle$, then $|\mathbf{u} + \mathbf{v}| = |\langle 0, 3 \rangle| = 3$, while $|\mathbf{u}| = 5$.
- False. For example, if $\mathbf{u} = \langle 3, 4 \rangle$ and $\mathbf{v} = \langle -1, -4 \rangle$, then $|\mathbf{u} + \mathbf{v}| = |\langle 2, 0 \rangle| = 2$, while $|\mathbf{u}| + |\mathbf{v}| = 5 + \sqrt{17}$.
- False. For example, if $\mathbf{u} = \langle 3, 0 \rangle$ and $\mathbf{v} = \langle 6, 0 \rangle$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel, but have different lengths.
- False. For example, given $A(0, 0)$, $B(3, 4)$, $C(1, 1)$ and $D(4, 5)$, we have $\overrightarrow{AB} = \langle 3, 4 \rangle$ and $\overrightarrow{CD} = \langle 3, 4 \rangle$, but $A \neq C$ and $B \neq D$.

g. False. For example, $\mathbf{u} = \langle 0, 1 \rangle$ and $\mathbf{v} = \langle -1, 0 \rangle$ are perpendicular, but $|\mathbf{u} + \mathbf{v}| = \sqrt{2}$, while $|\mathbf{u}| + |\mathbf{v}| = 2$.

h. True. Suppose $\mathbf{v} = k\mathbf{u}$ with $k > 0$. Then

$$|\mathbf{u} + \mathbf{v}| = |\mathbf{u} + k\mathbf{u}| = |(1+k)\mathbf{u}| = (1+k)|\mathbf{u}| = |\mathbf{u}| + k|\mathbf{u}| = |\mathbf{u}| + |k\mathbf{u}| = |\mathbf{u}| + |\mathbf{v}|.$$

13.1.66 $\overrightarrow{AB} = \langle 3, 6 \rangle$ and $\overrightarrow{CD} = \langle b-a+2, a-b-7 \rangle$. The system of linear equations $-a+b+2=3$, $a-b-7=6$ has no solution, so there are no values for a and b which will make these vectors equal.

13.1.67 $10\langle a, b \rangle = \langle 2, -3 \rangle$, so $10a = 2$, and $a = 1/5$. Also, $10b = -3$, so $b = -3/10$. Thus $\mathbf{x} = \langle 1/5, -3/10 \rangle$.

13.1.68 $2\langle a, b \rangle + \langle 2, -3 \rangle = \langle -4, 1 \rangle$, so $\langle a, b \rangle = \langle -3, 2 \rangle = \mathbf{x}$.

13.1.69 $3\langle a, b \rangle - 4\langle 2, -3 \rangle = \langle -4, 1 \rangle$, so $\langle a, b \rangle = \frac{1}{3}\langle 4, -11 \rangle = \mathbf{x}$.

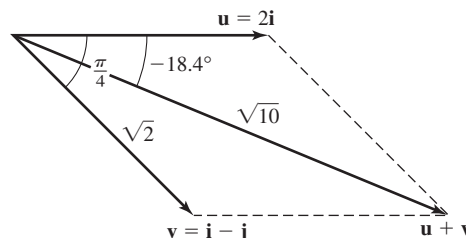
13.1.70 Because $2\mathbf{u} + 3\mathbf{v} = \mathbf{i}$ and $-2(\mathbf{u} - \mathbf{v}) = -2\mathbf{j}$, we can conclude that $3\mathbf{v} + 2\mathbf{v} = \mathbf{i} - 2\mathbf{j}$ (by adding), so $\mathbf{v} = \frac{1}{5}\mathbf{i} - \frac{2}{5}\mathbf{j}$. It then follows that $\mathbf{u} = \mathbf{v} + \mathbf{j} = \frac{1}{5}\mathbf{i} - \frac{2}{5}\mathbf{j} + \mathbf{j} = \frac{1}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$.

13.1.71 $\langle 4, -8 \rangle = 4\mathbf{i} - 8\mathbf{j}$.

13.1.72 Suppose $\langle 4, -8 \rangle = c_1\langle 1, 1 \rangle + c_2\langle -1, 1 \rangle$. Then $c_1 - c_2 = 4$ and $c_1 + c_2 = -8$. Adding these two equations to each other yields $2c_1 = -4$, so $c_1 = -2$. And thus $c_2 = -6$. We have $\langle 4, -8 \rangle = -2\mathbf{u} - 6\mathbf{v}$.

13.1.73 Let $\langle a, b \rangle = c_1\mathbf{u} + c_2\mathbf{v}$. Then $c_1 - c_2 = a$ and $c_1 + c_2 = b$. Adding these two equations to each other yields $2c_1 = a + b$, so $c_1 = \frac{a+b}{2}$. And thus $c_2 = \frac{b-a}{2}$. We have $\langle a, b \rangle = \frac{a+b}{2}\mathbf{u} + \frac{b-a}{2}\mathbf{v}$.

13.1.74 Let $\mathbf{u}$ represent the motion of the ant on the paper, and let $\mathbf{v}$ represent the motion of the paper. $\mathbf{u} + \mathbf{v} = 2\mathbf{i} + (\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - \mathbf{j}$. $|\mathbf{u} + \mathbf{v}| = \sqrt{9+1} = \sqrt{10}$. $\theta = \tan^{-1}(-1/3) = -18.4$ degrees. The ant moves in the direction 18.4 degrees south of east with speed $\sqrt{10}$ miles per hour.



13.1.75

- The sum is $\mathbf{0}$ because each vector has exactly one additive inverse in the set among the 12 vectors.
- The 6:00 vector, because the others cancel in pairs, but this vector remains.
- If we remove the 1:00 through 6:00 vectors, the sum is as large as possible, because all the vectors are pointing toward the left side of the clock. Removing any 6 consecutive vectors gives a sum whose magnitude is as large as possible.
- Let $\mathbf{w}$ be the vector that points from 12:00 toward 6:00 but which has length r equal to the radius of the clock. The sum we are seeking is $12\mathbf{w}$. The sum of the vectors pointing to 1:00 and 11:00 add up to $(2 - \sqrt{3})\mathbf{w}$, the sum of the vectors pointing to 2:00 and 10:00 is $\mathbf{w}$, the vectors pointing to 3:00 and 9:00 add up to $2\mathbf{w}$, the vectors pointing to 4:00 and 8:00 add up to $3\mathbf{w}$, and the vectors pointing to 5:00 and 7:00 add up to $(\sqrt{3} + 2)\mathbf{w}$. Finally, the single vector pointing to 6:00 is $2\mathbf{w}$. The sum of all of these is $12\mathbf{w}$.

13.1.76 Because the ring doesn't move, $\mathbf{F}_3$ is the opposite of $\mathbf{F}_1 + \mathbf{F}_2$, so $\mathbf{F}_3 = \langle 50\sqrt{2} - 30\sqrt{3}, -50\sqrt{2} - 30 \rangle$. Thus $|\mathbf{F}_3| \approx 102.44$ pounds, and the direction is given by $\alpha = \tan^{-1} \left(\frac{50\sqrt{2} + 30}{50\sqrt{2} - 30\sqrt{3}} \right) \approx 79.45$ degrees south of east.

$$\mathbf{13.1.77} \quad \mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \langle v_1, v_2 \rangle + \langle u_1, u_2 \rangle = \mathbf{v} + \mathbf{u}.$$

13.1.78

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle) + \langle w_1, w_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle + \langle w_1, w_2 \rangle \\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2 \rangle = \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2) \rangle \\ &= \langle u_1, u_2 \rangle + \langle v_1 + w_1, v_2 + w_2 \rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w}). \end{aligned}$$

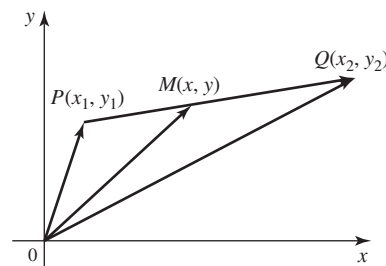
$$\mathbf{13.1.79} \quad a(c\mathbf{v}) = a(c\langle v_1, v_2 \rangle) = a\langle cv_1, cv_2 \rangle = \langle a(cv_1), a(cv_2) \rangle = \langle (ac)v_1, (ac)v_2 \rangle = (ac)\langle v_1, v_2 \rangle = (ac)\mathbf{v}.$$

13.1.80

$$\begin{aligned} a(\mathbf{u} + \mathbf{v}) &= a(\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle) = a\langle u_1 + v_1, u_2 + v_2 \rangle = \langle a(u_1 + v_1), a(u_2 + v_2) \rangle = \langle au_1 + av_1, au_2 + av_2 \rangle \\ &= \langle au_1, au_2 \rangle + \langle av_1, av_2 \rangle = a\mathbf{u} + a\mathbf{v}. \end{aligned}$$

$$\mathbf{13.1.81} \quad (a+c)\mathbf{v} = (a+c)\langle v_1, v_2 \rangle = \langle (a+c)v_1, (a+c)v_2 \rangle = \langle av_1 + cv_1, av_2 + cv_2 \rangle = \langle av_1, av_2 \rangle + \langle cv_1, cv_2 \rangle = a\mathbf{v} + c\mathbf{v}.$$

13.1.82 Let $M(x, y)$ be the midpoint. Because $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$, we have $\langle x, y \rangle = \langle x_1, y_1 \rangle + \frac{1}{2}(\langle x_2, y_2 \rangle - \langle x_1, y_1 \rangle) = \langle x_1, y_1 \rangle + \left\langle \frac{1}{2}x_2, \frac{1}{2}y_2 \right\rangle + \left\langle -\frac{1}{2}x_1, -\frac{1}{2}y_1 \right\rangle = \left\langle \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right\rangle.$



$$\mathbf{13.1.83} \quad |c\mathbf{v}| = |c\langle v_1, v_2 \rangle| = |\langle cv_1, cv_2 \rangle| = \sqrt{(cv_1)^2 + (cv_2)^2} = \sqrt{c^2} \sqrt{v_1^2 + v_2^2} = |c| |\mathbf{v}|.$$

13.1.84 Yes. Because $\overrightarrow{PQ} = \overrightarrow{RS}$, we have that these two vectors are parallel and have the same magnitude. Thus the quadrilateral $RSQP$ is a parallelogram. Hence, $\overrightarrow{PR}$ is parallel to $\overrightarrow{QS}$ and they have the same magnitude, and are thus equal.

13.1.85

- Note that $-6\mathbf{u} = \mathbf{v}$, so $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent. But there is no scalar c so that $c\mathbf{u} = \mathbf{w}$, nor any scalar d so that $d\mathbf{v} = \mathbf{w}$ so $\{\mathbf{u}, \mathbf{w}\}$ is linearly independent and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent.
- Two nonzero vectors are linearly independent when they are not parallel, and are linearly dependent when they are parallel.
- Suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Consider the equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{w}$ for a given vector $\mathbf{w}$. We are seeking a solution for the system of linear equations $c_1u_1 + c_2v_1 = w_1$ and $c_1u_2 + c_2v_2 = w_2$. The solution for this system is given by $c_1 = \frac{1}{u_1v_2 - u_2v_1}(v_2w_1 - v_1w_2)$ and $c_2 = \frac{1}{u_1v_2 - u_2v_1}(-u_2w_1 + u_1w_2)$, provided $u_1v_2 - u_2v_1 \neq 0$. This condition is equivalent to saying that $\mathbf{v}$ is not a multiple of $\mathbf{u}$. Thus a solution to the system of linear equations exists exactly when the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

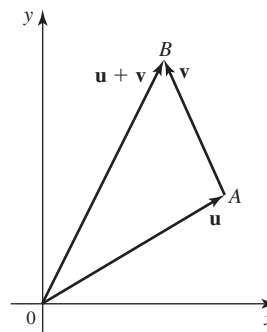
13.1.86 Suppose that $u_1v_1 + u_2v_2 = 0$. Then $\frac{u_2}{u_1} \cdot \frac{v_2}{v_1} = -1$. Let $m_1 = \frac{u_2}{u_1}$, and note that this is the slope of the line containing the vector $\mathbf{u}$. Likewise let $m_2 = \frac{v_2}{v_1}$, and note that this is the slope of the line containing the vector $\mathbf{v}$. Because $m_1 \cdot m_2 = -1$, the vectors are perpendicular.

13.1.87

- a. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, we must have $\frac{a}{2} = \frac{5}{6}$, so $a = \frac{5}{3}$.
- b. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, we must have $2a + 30 = 0$, so $a = -15$.

13.1.88

- a. The triangle rule states that in any triangle, the length of one side is less than or equal to the sum of the lengths of the other two sides. Suppose that $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{AB}$. Then $\mathbf{u} + \mathbf{v} = \overrightarrow{OB}$. The triangle rule applied to triangle OAB assures us that $|\overrightarrow{OB}| \leq |\overrightarrow{OA}| + |\overrightarrow{AB}|$, so $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$.



- b. Equality occurs when $\mathbf{u}$ and $\mathbf{v}$ are parallel and in the same direction, so that $\mathbf{v} = k\mathbf{u}$ where $k > 0$. Then $|\mathbf{u} + \mathbf{v}| = |(1 + k)\mathbf{u}| = (1 + k)|\mathbf{u}| = |\mathbf{u}| + |k\mathbf{u}| = |\mathbf{u}| + |\mathbf{v}|$.

13.2 Vectors in Three Dimensions

13.2.1 Starting at the origin $(0, 0, 0)$, move 3 units in the positive x -direction, 2 units in the negative y -direction, and 1 unit in the positive z -direction, to arrive at the point $(3, -2, 1)$.

13.2.2 Every point in the xz -plane has a y -coordinate of 0.

13.2.3 The plane $x = 4$ is parallel to the yz -plane, but contains all of the points with x -coordinate 4. It is perpendicular to the x -axis.

13.2.4 $\mathbf{u} = \langle 0 - 3, -6 - 5, 3 - (-2) \rangle = \langle -3, -11, 5 \rangle$.

13.2.5 $\mathbf{u} + \mathbf{v} = \langle 3 + 6, 5 + (-5), -7 + 1 \rangle = \langle 9, 0, -6 \rangle$. $3\mathbf{u} - \mathbf{v} = \langle 9, 15, -21 \rangle - \langle 6, -5, 1 \rangle = \langle 3, 20, -22 \rangle$.

13.2.6 $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

13.2.7 Because $\sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14} < \sqrt{0^2 + 0^2 + (-4)^2} = 4$, the point $(0, 0, -4)$ is further from the origin.

13.2.8 $\overrightarrow{PQ} = \langle 1 - (-1), 3 - (-4), -6 - (6) \rangle = \langle 2, 7, -12 \rangle = 2\mathbf{i} + 7\mathbf{j} - 12\mathbf{k}$.

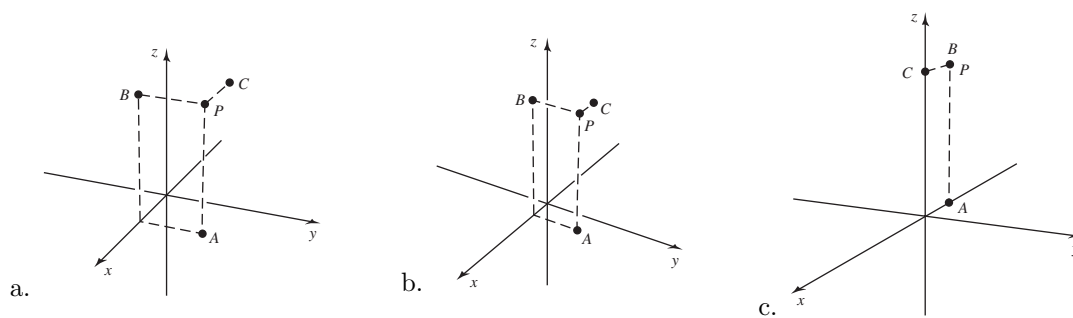
13.2.9 $A(3, 0, 5)$, $B(3, 4, 0)$, $C(0, 4, 5)$.

13.2.10 $A(5, 0, 10)$, $B(0, 8, 0)$, $C(5, 8, 10)$.

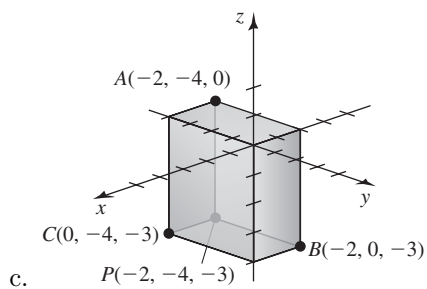
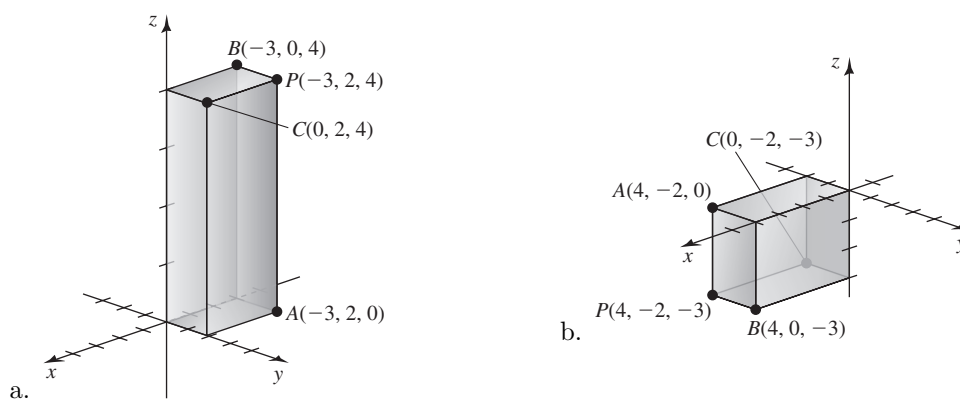
13.2.11 $A(3, -4, 5)$, $B(0, -4, 0)$, $C(0, -4, 5)$.

13.2.12 $A(-3, -3, 0)$, $B(0, 0, 3)$, $C(-3, -3, 3)$.

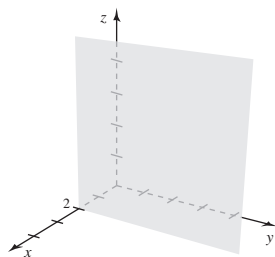
13.2.13



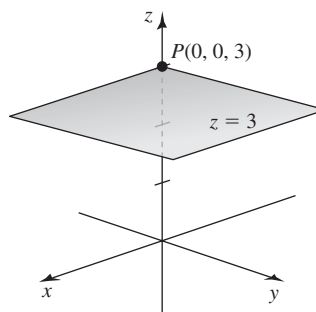
13.2.14



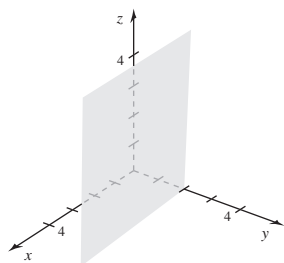
13.2.15



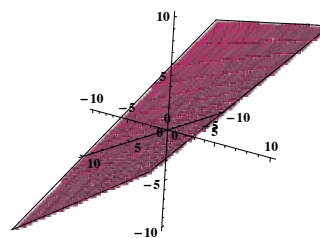
13.2.16



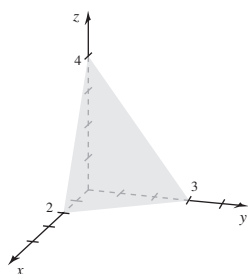
13.2.17



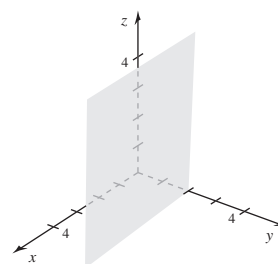
13.2.18



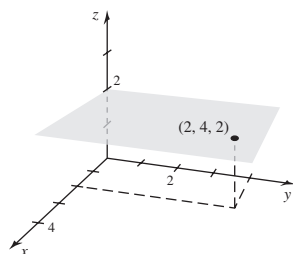
13.2.19



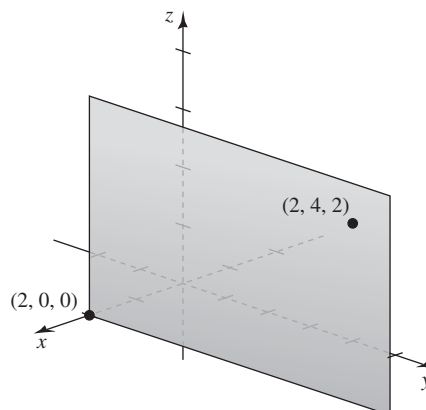
13.2.20



13.2.21

The plane $z = 2$.

13.2.22

The plane $x = 2$.

13.2.23 $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$.

13.2.24 Note that the radius r of this sphere would be $r = \sqrt{(3-1)^2 + (4-2)^2 + 5^2} = \sqrt{33}$. The equation of the sphere is thus $(x-1)^2 + (y-2)^2 + z^2 = 33$.

13.2.25 $(x+2)^2 + y^2 + (z-4)^2 \leq 1$.

13.2.26 Note that the radius of the ball is given by $r = \sqrt{1^2 + (4-(-2))^2 + (8-6)^2} = \sqrt{41}$. The ball is thus described by the inequality $x^2 + (y+2)^2 + (z-6)^2 \leq 41$.

13.2.27 The midpoint of the line segment $\overline{PQ}$ is $\left(\frac{1+2}{2}, \frac{0+3}{2}, \frac{5+9}{2}\right) = (3/2, 3/2, 7)$. The radius of the

sphere is $r = \frac{1}{2}\sqrt{(2-1)^2 + (3-0)^2 + (9-5)^2} = \frac{\sqrt{26}}{2}$. The equation of the sphere is therefore $(x-3/2)^2 + (y-3/2)^2 + (z-7)^2 = \frac{13}{2}$.

13.2.28 The midpoint of the line segment $\overline{PQ}$ is $\left(\frac{-4+0}{2}, \frac{2+2}{2}, \frac{3+7}{2}\right) = (-2, 2, 5)$. The radius of the sphere is $r = \frac{1}{2}\sqrt{(0-(-4))^2 + (2-2)^2 + (7-3)^2} = \sqrt{8}$. The equation of the sphere is therefore $(x+2)^2 + (y-2)^2 + (z-5)^2 = 8$.

13.2.29 This is a sphere centered at $(1, 0, 0)$ of radius 3.

13.2.30 Completing the square, we have $(x+1)^2 + (y^2 - 2y + 1) + z^2 = 25$, which can be written as $(x+1)^2 + (y-1)^2 + z^2 = 5^2$. This is a sphere centered at $(-1, 1, 0)$ with radius 5.

13.2.31 Completing the squares, we have $x^2 + (y^2 - 2y + 1) + (z^2 - 4z + 4) = 4 + 5$, so we have $x^2 + (y-1)^2 + (z-2)^2 = 3^2$, which describes a sphere of radius 3 centered at $(0, 1, 2)$.

13.2.32 Completing the squares, we have $(x^2 - 6x + 9) + (y^2 + 6y + 9) + (z^2 - 8z + 16) = 2 + 9 + 9 + 16 = 36$, so we have $(x-3)^2 + (y+3)^2 + (z-4)^2 = 6^2$, which describes a sphere of radius 6 centered at $(3, -3, 4)$.

13.2.33 Completing the square, we have $x^2 + (y^2 - 14y + 49) + z^2 \geq -13 + 49 = 36$, which can be written as $x^2 + (y-7)^2 + z^2 \geq 6^2$. This is the outside of a ball centered at $(0, 7, 0)$ with radius 6. (Including the sphere itself.)

13.2.34 Completing the square, we have $x^2 + (y^2 - 14y + 49) + z^2 \leq -13 + 49 = 36$, which can be written as $x^2 + (y-7)^2 + z^2 \leq 6^2$. This is a ball centered at $(0, 7, 0)$ with radius 6.

13.2.35 Completing the squares, we have $(x^2 - 8x + 16) + (y^2 - 14y + 49) + (z^2 - 18z + 81) \leq 79 + 16 + 49 + 81 = 225$, which can be written as $(x-4)^2 + (y-7)^2 + (z-9)^2 \leq 15^2$. This is a ball centered at $(4, 7, 9)$ with radius 15.

13.2.36 Completing the squares, we have $(x^2 - 8x + 16) + (y^2 + 14y + 49) + (z^2 - 18z + 81) \geq 65 + 16 + 49 + 81 = 211$, which can be written as $(x-4)^2 + (y+7)^2 + (z-9)^2 \geq 211$. This is the outside of a ball centered at $(4, -7, 9)$ with radius $\sqrt{211}$. (Including the sphere itself.)

13.2.37 Completing the squares, we have $(x^2 - 2x + 1) + (y^2 + 6y + 9) + z^2 = -10 + 1 + 9 = 0$, or $(x-1)^2 + (y+3)^2 + z^2 = 0$. This is the single point $(1, -3, 0)$.

13.2.38 Completing the squares, we have $(x^2 - 4x + 4) + (y^2 + 6y + 9) + z^2 = -14 + 4 + 9 = -1$. This can be written as $(x-2)^2 + (y+3)^2 + z^2 = -1$, and no real numbers satisfy this equations.

13.2.39

- $3\langle 4, -3, 0 \rangle + 2\langle 0, 1, 1 \rangle = \langle 12, -7, 2 \rangle$.
- $4\langle 4, -3, 0 \rangle - \langle 0, 1, 1 \rangle = \langle 16, -13, -1 \rangle$.
- $|\langle 4, -3, 0 \rangle + 3\langle 0, 1, 1 \rangle| = |\langle 4, 0, 3 \rangle| = \sqrt{16 + 0 + 9} = 5$.

13.2.40

- $3\langle -2, -3, 0 \rangle + 2\langle 1, 2, 1 \rangle = \langle -4, -5, 2 \rangle$.
- $4\langle -2, -3, 0 \rangle - \langle 1, 2, 1 \rangle = \langle -9, -14, -1 \rangle$.
- $|\langle -2, -3, 0 \rangle + 3\langle 1, 2, 1 \rangle| = |\langle 1, 3, 3 \rangle| = \sqrt{1 + 9 + 9} = \sqrt{19}$.

13.2.41

- a. $3\langle -2, 1, -2 \rangle + 2\langle 1, 1, 1 \rangle = \langle -4, 5, -4 \rangle$.
 b. $4\langle -2, 1, -2 \rangle - \langle 1, 1, 1 \rangle = \langle -9, 3, -9 \rangle$.
 c. $|\langle -2, 1, -2 \rangle + 3\langle 1, 1, 1 \rangle| = |\langle 1, 4, 1 \rangle| = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}$.

13.2.42

- a. $3\langle -5, 0, 2 \rangle + 2\langle 3, 1, 1 \rangle = \langle -9, 2, 8 \rangle$.
 b. $4\langle -5, 0, 2 \rangle - \langle 3, 1, 1 \rangle = \langle -23, -1, 7 \rangle$.
 c. $|\langle -5, 0, 2 \rangle + 3\langle 3, 1, 1 \rangle| = |\langle 4, 3, 5 \rangle| = \sqrt{16 + 9 + 25} = \sqrt{50} = 5\sqrt{2}$.

13.2.43

- a. $3\langle -7, 11, 8 \rangle + 2\langle 3, -5, -1 \rangle = \langle -15, 23, 22 \rangle$.
 b. $4\langle -7, 11, 8 \rangle - \langle 3, -5, -1 \rangle = \langle -31, 49, 33 \rangle$.
 c. $|\langle -7, 11, 8 \rangle + 3\langle 3, -5, -1 \rangle| = |\langle 2, -4, 5 \rangle| = \sqrt{4 + 16 + 25} = \sqrt{45} = 3\sqrt{5}$.

13.2.44

- a. $3\langle -4, -8\sqrt{3}, 2\sqrt{2} \rangle + 2\langle 2, 3\sqrt{3}, -\sqrt{2} \rangle = \langle -8, -18\sqrt{3}, 4\sqrt{2} \rangle$.
 b. $4\langle -4, -8\sqrt{3}, 2\sqrt{2} \rangle - \langle 2, 3\sqrt{3}, -\sqrt{2} \rangle = \langle -18, -35\sqrt{3}, 9\sqrt{2} \rangle$.
 c. $|\langle -4, -8\sqrt{3}, 2\sqrt{2} \rangle + 3\langle 2, 3\sqrt{3}, -\sqrt{2} \rangle| = |\langle 2, \sqrt{3}, -\sqrt{2} \rangle| = \sqrt{4 + 3 + 2} = \sqrt{9} = 3$.

13.2.45

- a. $\overrightarrow{PQ} = \langle 3 - 1, 11 - 5, 2 - 0 \rangle = \langle 2, 6, 2 \rangle = 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.
 b. $|\langle 2, 6, 2 \rangle| = \sqrt{4 + 36 + 4} = \sqrt{44} = 2\sqrt{11}$.
 c. $\langle 1/\sqrt{11}, 3/\sqrt{11}, 1/\sqrt{11} \rangle$ and $\langle -1/\sqrt{11}, -3/\sqrt{11}, -1/\sqrt{11} \rangle$.

13.2.46

- a. $\overrightarrow{PQ} = \langle 1 - 5, 14 - 11, 13 - 12 \rangle = \langle -4, 3, 1 \rangle = -4\mathbf{i} + 3\mathbf{j} + 1\mathbf{k}$.
 b. $|\langle -4, 3, 1 \rangle| = \sqrt{16 + 9 + 1} = \sqrt{26}$.
 c. $\langle -4/\sqrt{26}, 3/\sqrt{26}, 1/\sqrt{26} \rangle$ and $\langle 4/\sqrt{26}, -3/\sqrt{26}, -1/\sqrt{26} \rangle$.

13.2.47

- a. $\overrightarrow{PQ} = \langle -3 + 3, -4 - 1, 1 - 0 \rangle = \langle 0, -5, 1 \rangle = -5\mathbf{j} + 1\mathbf{k}$.
 b. $|\langle 0, -5, 1 \rangle| = \sqrt{25 + 1} = \sqrt{26}$.
 c. $\langle 0, -5/\sqrt{26}, 1/\sqrt{26} \rangle$ and $\langle 0, 5/\sqrt{26}, -1/\sqrt{26} \rangle$.

13.2.48

- a. $\overrightarrow{PQ} = \langle 3 - 3, 9 - 8, 11 - 12 \rangle = \langle 0, 1, -1 \rangle = \mathbf{j} - \mathbf{k}$.
 b. $|\langle 0, 1, -1 \rangle| = \sqrt{1 + 1} = \sqrt{2}$.
 c. $\langle 0, 1/\sqrt{2}, -1/\sqrt{2} \rangle$ and $\langle 0, -1/\sqrt{2}, 1/\sqrt{2} \rangle$.

13.2.49

- a. $\overrightarrow{PQ} = \langle -2 - 0, 4 - 0, 0 - 2 \rangle = \langle -2, 4, -2 \rangle = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

- b. $|\langle -2, 4, -2 \rangle| = \sqrt{4 + 16 + 4} = 2\sqrt{6}$.
 c. $\langle -1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6} \rangle$ and $\langle 1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6} \rangle$.

13.2.50

- a. $\overrightarrow{PQ} = \langle 1 - a, 1 - b, -1 - c \rangle = (1 - a)\mathbf{i} + (1 - b)\mathbf{j} + (-1 - c)\mathbf{k}$.
 b. $|\langle (1 - a), (1 - b), (-1 - c) \rangle| = \sqrt{(1 - a)^2 + (1 - b)^2 + (-1 - c)^2}$.
 c. $\frac{\pm 1}{\sqrt{(1 - a)^2 + (1 - b)^2 + (-1 - c)^2}} \langle 1 - a, 1 - b, -1 - c \rangle$.

13.2.51

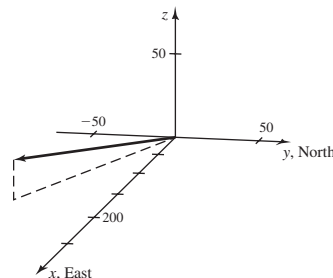
- a. The airplane's velocity vector (without wind) is given by $\langle 0, 20, 0 \rangle$, the wind's is given by $\langle 20, 0, 0 \rangle$ and the downdraft's is $\langle 0, 0, -10 \rangle$. The sum of these is $\langle 20, 20, -10 \rangle$.
 b. The speed is $|\langle 20, 20, -10 \rangle| = \sqrt{400 + 400 + 100} = 30$ mi/hr

13.2.52

- a. The airplane's velocity vector (without wind) is given by $\langle 10, 0, 0 \rangle$, the wind's is given by $\langle 0, -5, 0 \rangle$ and the updraft's is $\langle 0, 0, 5 \rangle$. The sum of these is $\langle 10, -5, 5 \rangle$.
 b. The speed is $|\langle 10, -5, 5 \rangle| = \sqrt{100 + 25 + 25} = 5\sqrt{6} \approx 12.2$ mi/hr

13.2.53

The airplane's velocity is $\mathbf{v}_1 = 250\mathbf{i}$.
 The crosswind is blowing $\mathbf{v}_2 = -25\sqrt{2}\mathbf{i} - 25\sqrt{2}\mathbf{j}$. The updraft is $\mathbf{v}_3 = 30\mathbf{k}$. We have $|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| = |\langle 250 - 25\sqrt{2}, -25\sqrt{2}, 30 \rangle| \approx 219.596$ miles per hour. The direction is sketched in the diagram—it is slightly south of east and upward.



13.2.54 $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 60, 20, 30 \rangle$. $|\mathbf{F}| = 10\sqrt{36 + 4 + 9} = 70$. The direction can be described using the angles α , β , and γ defined by $\alpha = \cos^{-1}(6/7) \approx 31$ degrees, $\beta = \cos^{-1}(20/70) \approx 73.4$ degrees, and $\gamma = \cos^{-1}(3/7) \approx 64.62$ degrees.

13.2.55 The component in the east direction is $(20 \cos 30^\circ)(\cos 45^\circ) = 5\sqrt{6}$ knots. In the north direction, it is $(20 \cos 30^\circ)(\sin 45^\circ) = 5\sqrt{6}$ knots. In the vertical direction, it is $20 \sin 30^\circ = 10$ knots.

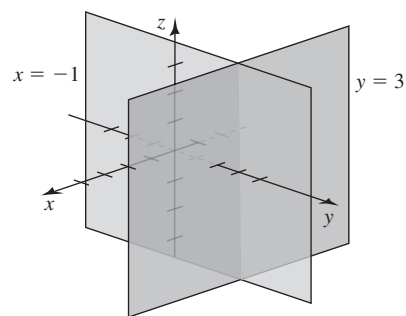
13.2.56 $\mathbf{F}_1 + \mathbf{F}_2 = \langle 10, 6, 3 \rangle + \langle 0, 4, 9 \rangle = \langle 10, 10, 12 \rangle$, so we need $\mathbf{F}_3 = \langle -10, -10, -12 \rangle$.

13.2.57

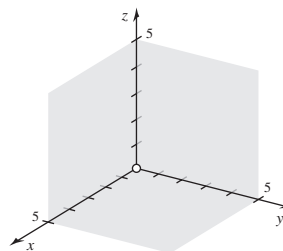
- a. False. For example, let $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$ and $\mathbf{w} = \langle 1, 1, 0 \rangle$. Then both $\mathbf{u}$ and $\mathbf{v}$ make a 45 degree angle with $\mathbf{w}$, but $\mathbf{u} + \mathbf{v} = \mathbf{w}$ makes a zero degree angle with $\mathbf{w}$.
 b. False. For example, $\mathbf{i}$ and $\mathbf{j}$ form a 90 degree angle with $\mathbf{k}$, as does $\mathbf{i} + \mathbf{j}$.
 c. False. $\mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle \neq \langle 0, 0, 0 \rangle$.
 d. True. They intersect at the point $(1, 1, 1)$.

13.2.58

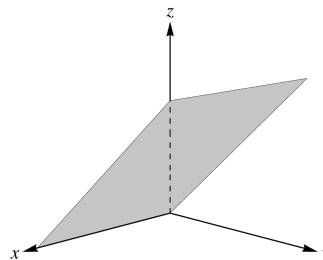
This is the set of all points which lie either on the plane $x = -1$ or the plane $y = 3$ or both.

**13.2.59**

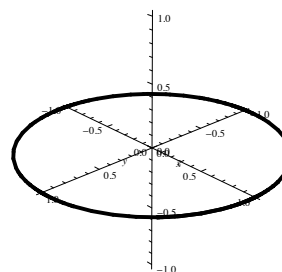
This represents all the points in 3-space, excluding the three axes.

**13.2.60**

This represents the plane which is perpendicular to the yz -plane, and which intersects the yz -plane in the line $y = z$.

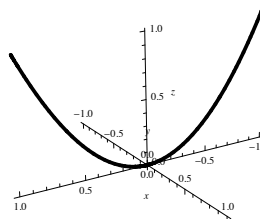
**13.2.61**

This represents a circle of radius 1 centered at $(0, 0, 0)$ in the xy -plane.



13.2.62

This represents a parabola in the xz -plane.



13.2.63 If $z = 1$, then $x^2 + y^2 + 1 = 5$, so $x^2 + y^2 = 4$. We have a circle of radius 2 centered at $(0, 0, 1)$ in the plane $z = 1$.

13.2.64 If $z = 6$ and $x^2 + y^2 + z^2 = 36$, then $x^2 + y^2 = 0$, so $x = 0$ and $y = 0$. This consists of the single point $(0, 0, 6)$.

13.2.65 Planes parallel to the xz -plane have the form $y = c$ for a constant c , so we must have $y = 4$. Thus, we must have $(x - 2)^2 + (z - 1)^2 = 9$ and $y = 4$.

13.2.66 The intersection of the planes $y = -5$ and $z = 1$ is a line parallel to the x -axis that contains the given point.

$$\mathbf{13.2.67} \quad \mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle.$$

$$\mathbf{13.2.68} \quad \mathbf{w} = 10 \frac{\mathbf{w}}{|\mathbf{w}|} = 10 \left\langle \frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{3}}{3} \right\rangle.$$

$$\mathbf{13.2.69} \quad -5 \frac{\mathbf{v}}{|\mathbf{v}|} = -5 \left\langle \frac{3}{4}, -\frac{1}{2}, \frac{\sqrt{3}}{4} \right\rangle = \left\langle -\frac{15}{4}, \frac{5}{2}, -\frac{5\sqrt{3}}{4} \right\rangle.$$

13.2.70 Because the magnitude of $\mathbf{v}$ is $\sqrt{9 + 4 + 36} = 7$, the desired vectors are $\pm 10 \langle 3/7, -2/7, 6/7 \rangle = \pm \langle 30/7, -20/7, 60/7 \rangle$.

13.2.71 Because the magnitude of $\mathbf{v}$ is $\sqrt{36 + 64 + 0} = 10$, the desired vectors are $\pm 20 \langle .6, -.8, 0 \rangle = \pm \langle 12, -16, 0 \rangle$.

13.2.72 Because the magnitude of $\mathbf{v}$ is $\sqrt{1 + 1 + 0} = \sqrt{2}$, the desired vectors are $\pm 3 \langle 1/\sqrt{2}, -1/\sqrt{2}, 0 \rangle$.

13.2.73 Because the magnitude of $\mathbf{v}$ is $\sqrt{1 + 1 + 1} = \sqrt{3}$, the desired vectors are $\pm 3 \langle -1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3} \rangle = \pm \langle -\sqrt{3}, -\sqrt{3}, \sqrt{3} \rangle$.

13.2.74 In order for the points to be collinear, we would require $\langle 4 - 1, 7 - 2, 1 - 3 \rangle = k \langle x - 1, y - 2, 2 - 3 \rangle$. Thus we seek a solution to the system of equations $3 = k(x - 1)$, $5 = k(y - 2)$, $-2 = -k$. Thus $k = 2$, $x = \frac{5}{2}$ and $y = \frac{9}{2}$.

13.2.75

a. Because $\overrightarrow{PQ} = \langle 1, -1, 2 \rangle$ and $\overrightarrow{PR} = \langle 3, -3, 6 \rangle = 3 \langle 1, -1, 2 \rangle$ they are collinear. Q is between P and R .

b. Because $\overrightarrow{PQ} = \langle 4, 8, -8 \rangle$ and $\overrightarrow{PR} = \langle -1, -2, 2 \rangle = -\frac{1}{4} \langle 4, 8, -8 \rangle$ they are collinear. P is between Q and R .

c. Because $\overrightarrow{PQ} = \langle 1, -5, 3 \rangle$ and $\overrightarrow{PR} = \langle 2, -3, 6 \rangle$ are not parallel, the given points are not collinear.

d. Because $\overrightarrow{PQ} = \langle 2, 13, 3 \rangle$ and $\overrightarrow{PR} = \langle -3, -2, -1 \rangle$ are not parallel, the given points are not collinear.

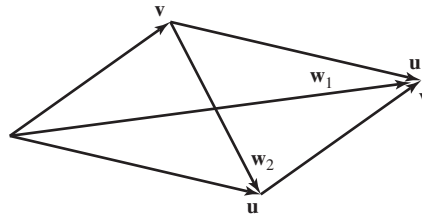
13.2.76 The diagonal of the box has magnitude $\sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$, so the longest rod that will fit in the box has length $\sqrt{29}$ feet.

13.2.77 Let $P(1, -\sqrt{3}, 0)$, $Q(1, \sqrt{3}, 0)$, $R(-2, 0, 0)$, and $S(0, 0, -2\sqrt{3})$ be the given points. Note that $\overrightarrow{PS} = \langle -1, \sqrt{3}, -2\sqrt{3} \rangle$, $\overrightarrow{QS} = \langle -1, -\sqrt{3}, -2\sqrt{3} \rangle$, $\overrightarrow{RS} = \langle 2, 0, -2\sqrt{3} \rangle$. Let $x(\overrightarrow{PS} + \overrightarrow{QS} + \overrightarrow{RS}) = -500\mathbf{k}$, then $-6\sqrt{3}x = -500$, so $x = \frac{250}{3\sqrt{3}}$. Then $x\overrightarrow{PS} = \frac{250}{3\sqrt{3}} \langle -1, \sqrt{3}, -2\sqrt{3} \rangle = \frac{250}{3} \langle -1/\sqrt{3}, 1, -2 \rangle$. $x\overrightarrow{QS} = \frac{250}{3\sqrt{3}} \langle -1, -\sqrt{3}, -2\sqrt{3} \rangle = \frac{250}{3} \langle -1/\sqrt{3}, -1, -2 \rangle$. $x\overrightarrow{RS} = \frac{250}{3\sqrt{3}} \langle 2, 0, -2\sqrt{3} \rangle = \frac{250}{3} \langle 2/\sqrt{3}, 0, -2 \rangle$.

13.2.78 Let $A(2, 0, 0)$, $B(0, 2, 0)$, $C(-2, 0, 0)$, $D(0, -2, 0)$, and $E(0, 0, -4)$ be the given points. Note that $\overrightarrow{AE} = \langle -2, 0, -4 \rangle$, $\overrightarrow{BE} = \langle 0, -2, -4 \rangle$, $\overrightarrow{CE} = \langle 2, 0, -4 \rangle$, $\overrightarrow{DE} = \langle 0, 2, -4 \rangle$. We are seeking x so that $x(\overrightarrow{AE} + \overrightarrow{BE} + \overrightarrow{CE} + \overrightarrow{DE}) = -500\mathbf{k}$. Thus we require $-16x = -500$, so $x = \frac{125}{4}$. Then $x\overrightarrow{AE} = \frac{125}{4} \langle -2, 0, -4 \rangle$. $x\overrightarrow{BE} = \frac{125}{4} \langle 0, -2, -4 \rangle$. $x\overrightarrow{CE} = \frac{125}{4} \langle 2, 0, -4 \rangle$, and $x\overrightarrow{DE} = \frac{125}{4} \langle 0, 2, -4 \rangle$.

13.2.79 Let $R(x, y, z)$ be the fourth vertex. Then perhaps $\overrightarrow{OQ} = \overrightarrow{RP}$, so $\langle 2, 4, 3 \rangle = \langle 1 - x, 4 - y, 6 - z \rangle$, so $x = -1$, $y = 0$, and $z = 3$, so $R(-1, 0, 3)$ is one possible desired vertex. We could also have $\overrightarrow{RP} = -\overrightarrow{OQ}$, in which case $\langle -2, -4, -3 \rangle = \langle 1 - x, 4 - y, 6 - z \rangle$, so $R(3, 8, 9)$ is the other vertex. We could also have $\overrightarrow{OP} = \overrightarrow{RQ}$, so $\langle 1, 4, 6 \rangle = \langle 2 - x, 4 - y, 3 - z \rangle$ and $R(1, 0, -3)$ is the desired point.

13.2.80 Suppose that the parallelogram has vertices $RTDS$, and $\mathbf{u} = \overrightarrow{RS}$ and $\mathbf{v} = \overrightarrow{RT}$ are two sides. Then we also have $\mathbf{u} = \overrightarrow{TD}$, and $\mathbf{v} = \overrightarrow{SD}$. But then $\mathbf{u} + \mathbf{v} = \overrightarrow{RS} + \overrightarrow{SD} = \overrightarrow{RD}$ is one diagonal and $\mathbf{u} - \mathbf{v} = \overrightarrow{TD} + \overrightarrow{DS} = \overrightarrow{TS}$ is the other diagonal.



13.2.81 Let $M(x, y, z)$ be the midpoint. Because $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$, we have

$$\begin{aligned} \langle x, y, z \rangle &= \langle x_1, y_1, z_1 \rangle + \frac{1}{2}(\langle x_2, y_2, z_2 \rangle - \langle x_1, y_1, z_1 \rangle) \\ &= \langle x_1, y_1, z_1 \rangle + \left\langle \frac{1}{2}x_2, \frac{1}{2}y_2, \frac{1}{2}z_2 \right\rangle + \left\langle -\frac{1}{2}x_1, -\frac{1}{2}y_1, -\frac{1}{2}z_1 \right\rangle = \left\langle \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right\rangle. \end{aligned}$$

13.2.82 We complete the squares by adding $a^2 + b^2 + c^2$ to both sides of the given equation, giving

$$(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) + (z^2 - 2cz + c^2) = d + a^2 + b^2 + c^2,$$

which can be written $(x - a)^2 + (y - b)^2 + (z - c)^2 = d + a^2 + b^2 + c^2$. This represents the set of points whose distance from (a, b, c) is $\sqrt{d + a^2 + b^2 + c^2}$, so it represents a sphere as long as $d + a^2 + b^2 + c^2 > 0$. The center is (a, b, c) and the radius is $\sqrt{d + a^2 + b^2 + c^2}$.

13.2.83

a. $\mathbf{u} + \mathbf{v} = -\mathbf{w}$ (by the geometric definition of vector addition), so $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

- b. Let $\mathbf{M}_1 = \overrightarrow{EB}$, $\mathbf{M}_2 = \overrightarrow{FO}$, and $\mathbf{M}_3 = \overrightarrow{GA}$. Consider triangle EAB . We have $\overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BE} = \mathbf{0}$, so $\frac{1}{2}\mathbf{u} + \mathbf{v} = -\overrightarrow{BE} = \overrightarrow{EB} = \mathbf{M}_1$. Using similar arguments, we have $\mathbf{M}_2 = \frac{1}{2}\mathbf{v} + \mathbf{w}$ and $\mathbf{M}_3 = \frac{1}{2}\mathbf{w} + \mathbf{u}$.
- c. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors from O to the points $1/3$ of the way along $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ respectively. Because $-\mathbf{w} = \mathbf{u} + \mathbf{v}$, we have $\frac{\mathbf{u} - \mathbf{w}}{3} = \frac{\mathbf{u}}{3} + \frac{\mathbf{u} + \mathbf{v}}{3} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. Also, $\mathbf{a} = \frac{1}{2}\mathbf{u} + \frac{1}{3}\mathbf{M}_1 = \frac{1}{2}\mathbf{u} + \frac{1}{3}\left(\frac{1}{2}\mathbf{u} + \mathbf{v}\right) = \frac{1}{2}\mathbf{u} + \frac{1}{6}\mathbf{u} + \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. Thus $\mathbf{a} = \frac{\mathbf{u} - \mathbf{w}}{3}$. Also, $\mathbf{b} = -\frac{2}{3}\mathbf{M}_2 = -\frac{2}{3}\left(\frac{1}{2}\mathbf{v} + (-\mathbf{u} - \mathbf{v})\right) = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. We also have $\mathbf{c} = -\frac{1}{2}\mathbf{w} + \frac{1}{3}\mathbf{M}_3 = -\frac{1}{2}\mathbf{w} + \frac{1}{3}\left(\frac{1}{2}\mathbf{w} + \mathbf{u}\right) = \frac{1}{3}\mathbf{u} - \frac{1}{3}(-\mathbf{u} - \mathbf{v}) = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. Thus $\mathbf{a} = \mathbf{b} = \mathbf{c}$.
- d. Because $\mathbf{a} = \mathbf{b} = \mathbf{c}$, the medians all meet at a point that divides each median in a 2:1 ratio.

13.2.84

- a. The coordinates of M_1 are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, 0\right)$, and thus

$$\overrightarrow{RM_1} = \left\langle \frac{x_1 + x_2}{2} - x_3, \frac{y_1 + y_2}{2} - y_3, -z_3 \right\rangle.$$

b.

$$\begin{aligned} \overrightarrow{OZ_1} &= \overrightarrow{OR} + \frac{2}{3}\overrightarrow{RM_1} = \langle x_3, y_3, z_3 \rangle + \left\langle \frac{x_1 + x_2 - 2x_3}{3}, \frac{y_1 + y_2 - 2y_3}{3}, -\frac{2z_3}{3} \right\rangle \\ &= \left\langle \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3} \right\rangle. \end{aligned}$$

- c. The coordinates of M_2 are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_3}{2}\right)$. We have

$$\overrightarrow{PM_2} = \left\langle \frac{x_2 + x_3 - 2x_1}{2}, \frac{y_2 + y_3 - 2y_1}{2}, \frac{z_3}{2} \right\rangle,$$

and

$$\begin{aligned} \overrightarrow{OZ_2} &= \overrightarrow{OP} + \frac{2}{3}\overrightarrow{PM_2} = \langle x_1, y_1, 0 \rangle + \left\langle \frac{x_2 + x_3 - 2x_1}{3}, \frac{y_2 + y_3 - 2y_1}{3}, \frac{z_3}{3} \right\rangle \\ &= \left\langle \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3} \right\rangle. \end{aligned}$$

- d. The coordinates of M_3 are $\left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}, \frac{z_3}{2}\right)$. We have

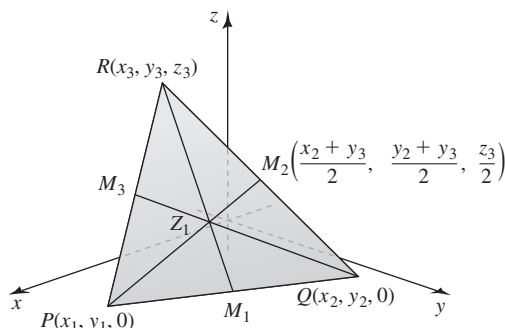
$$\overrightarrow{QM_3} = \left\langle \frac{x_1 + x_3 - 2x_2}{2}, \frac{y_1 + y_3 - 2y_2}{2}, \frac{z_3}{2} \right\rangle,$$

and

$$\begin{aligned} \overrightarrow{OZ_3} &= \overrightarrow{OQ} + \frac{2}{3}\overrightarrow{QM_3} = \langle x_2, y_2, 0 \rangle + \left\langle \frac{x_1 + x_3 - 2x_2}{3}, \frac{y_1 + y_3 - 2y_2}{3}, \frac{z_3}{3} \right\rangle \\ &= \left\langle \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3} \right\rangle. \end{aligned}$$

- e. The medians all intersect at the point $Z_1 = Z_2 = Z_3 = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3}\right)$.

- f. The intersection point is $\left(\frac{2+4+6}{3}, \frac{4+1+3}{3}, \frac{4}{3}\right) = (4, 8/3, 4/3)$.



13.2.85

- a. $\mathbf{u} + \mathbf{v} = \overrightarrow{PR}$ and $\mathbf{w} + \mathbf{x} = \mathbf{x} + \mathbf{w} = \overrightarrow{PR}$, so $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$.
- b. $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} = \mathbf{m} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$.
- c. $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{w} = \mathbf{n} = \frac{1}{2}(\mathbf{x} + \mathbf{w})$.
- d. We have $\mathbf{n} = \frac{1}{2}(\mathbf{x} + \mathbf{w}) = \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \mathbf{m}$.
- e. Because $\mathbf{m}$ and $\mathbf{n}$ are equal, they are parallel. A similar argument will show that the other two sides are parallel as well.

13.2.86 The midpoints are $A\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, 0\right)$, $B\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, 0\right)$, $C\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}, \frac{z_4}{2}\right)$, and $D\left(\frac{x_4+x_1}{2}, \frac{y_4+y_1}{2}, \frac{z_4}{2}\right)$. So

$$\overrightarrow{AB} = \left\langle \frac{x_3-x_1}{2}, \frac{y_3-y_1}{2}, 0 \right\rangle = \overrightarrow{DC},$$

and

$$\overrightarrow{BC} = \left\langle \frac{x_4-x_2}{2}, \frac{y_4-y_2}{2}, \frac{z_4}{2} \right\rangle = \overrightarrow{AD},$$

so quadrilateral $ABCD$ is a parallelogram.

13.3 Dot Products

13.3.1 $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$, where θ is the angle between the two vectors.

13.3.2 If $\mathbf{u} = \langle a, b, c \rangle$ and $\mathbf{v} = \langle r, s, t \rangle$, then $\mathbf{u} \cdot \mathbf{v} = ar + bs + ct$.

13.3.3 $\langle 2, 3, -6 \rangle \cdot \langle 1, -8, 3 \rangle = 2 \cdot 1 + 3 \cdot (-8) + (-6) \cdot 3 = -40$.

13.3.4 Because the angle between a vector and itself is 0, we have $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}| |\mathbf{v}| \cos \theta = |\mathbf{v}| |\mathbf{v}| = |\mathbf{v}|^2$.

13.3.5 Given non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, the angle between them is $\cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}\right)$.

13.3.6 By definition, $\text{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos \theta$, so we have $-2 = 4 \cos \theta$, or $\cos \theta = -\frac{1}{2}$, and so $\theta = \frac{2\pi}{3}$.

$$13.3.7 \quad \text{proj}_{\mathbf{v}} \mathbf{u} = \text{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = -2 \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle = \left\langle -\frac{4}{3}, \frac{2}{3}, \frac{4}{3} \right\rangle.$$

$$13.3.8 \quad \mathbf{u} \cdot \mathbf{v} = 4 + 2 - 6 = 0, \text{ so the vectors are orthogonal.}$$

$$13.3.9 \quad \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \pi = 1 \cdot 1 \cdot -1 = -1.$$

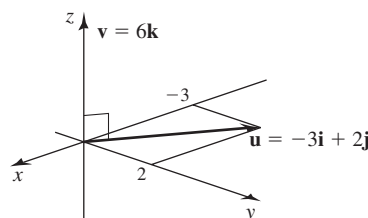
$$13.3.10 \quad \text{The work done by a force } \mathbf{F} \text{ in moving an object along a displacement vector } \mathbf{d} \text{ is } w = \mathbf{F} \cdot \mathbf{d}.$$

$$13.3.11 \quad \text{There are 2.}$$

$$13.3.12 \quad \text{There are infinitely many.}$$

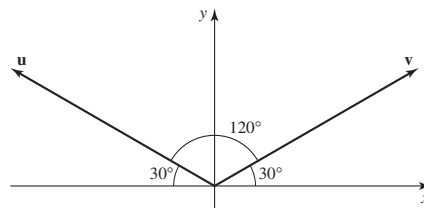
$$13.3.13 \quad \mathbf{u} \cdot \mathbf{v} = 4 \cdot 6 \cdot \cos(\pi/2) = 0.$$

$$13.3.14 \quad \text{Because these vectors are perpendicular, we have } \cos \theta = 0 \text{ where } \theta \text{ is the angle between them, so their dot product is zero.}$$



$$13.3.15 \quad \text{The angle between these vectors is } \pi/4. \text{ Thus, their dot product is } 10 \cdot 10\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 100.$$

$$13.3.16 \quad \text{The angle between these vectors is } 120^\circ, \text{ so their dot product is } |\mathbf{u}| |\mathbf{v}| \cos(120^\circ) = 2 \cdot 2 \cdot -\frac{1}{2} = -2.$$



$$13.3.17 \quad \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta = 1 \cdot 1 \cdot \cos(\pi/3) = 1/2.$$

$$13.3.18 \quad \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta = 1 \cdot 2 \cdot \cos(3\pi/4) = 2 \cdot (-\sqrt{2}/2) = -\sqrt{2}.$$

$$13.3.19 \quad \mathbf{u} \cdot \mathbf{v} = 1 - 1 = 0, \text{ so } \theta = \cos^{-1}(0) = \pi/2.$$

$$13.3.20 \quad \mathbf{u} \cdot \mathbf{v} = -50 + 0 = -50, \text{ so } \theta = \cos^{-1} \left(\frac{-50}{10\sqrt{50}} \right) = \cos^{-1}(-\sqrt{2}/2) = 3\pi/4.$$

$$13.3.21 \quad \mathbf{u} \cdot \mathbf{v} = 1 + 0 = 1, \text{ so } \theta = \cos^{-1} \left(\frac{1}{2} \right) = \pi/3.$$

$$13.3.22 \quad \mathbf{u} \cdot \mathbf{v} = -2 - 2 = -4, \text{ so } \theta = \cos^{-1} \left(-\frac{4}{4} \right) = \pi.$$

$$13.3.23 \quad \mathbf{u} \cdot \mathbf{v} = 4 \cdot 4 + 3 \cdot (-6) = -2. \text{ The angle between the vectors is thus}$$

$$\cos^{-1} \left(-\frac{2}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(-\frac{2}{5 \cdot 2\sqrt{13}} \right) \approx 1.627 \text{ radians.}$$

13.3.24 $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 0 + 4 \cdot 4 + 0 \cdot 5 = 16$. The angle between the vectors is thus

$$\cos^{-1} \left(\frac{16}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{16}{5 \cdot \sqrt{41}} \right) \approx 1.047 \text{ radians.}$$

13.3.25 $\mathbf{u} \cdot \mathbf{v} = -10 + 0 + 12 = 2$. The angle between the vectors is thus

$$\cos^{-1} \left(\frac{2}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{2}{\sqrt{116} \cdot \sqrt{14}} \right) \approx 1.521 \text{ radians.}$$

13.3.26 $\mathbf{u} \cdot \mathbf{v} = -27 - 25 + 2 = -50$. The angle between the vectors is thus

$$\cos^{-1} \left(-\frac{50}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(-\frac{50}{\sqrt{38} \cdot \sqrt{107}} \right) \approx 2.472 \text{ radians.}$$

13.3.27 $\mathbf{u} \cdot \mathbf{v} = 2 + 0 - 6 = -4$. The angle between the vectors is thus

$$\cos^{-1} \left(-\frac{4}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(-\frac{4}{\sqrt{13} \cdot \sqrt{21}} \right) \approx 1.815 \text{ radians.}$$

13.3.28 $\mathbf{u} \cdot \mathbf{v} = 2 + 16 - 12 = 6$. The angle between the vectors is thus

$$\cos^{-1} \left(\frac{6}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{6}{\sqrt{53} \cdot \sqrt{24}} \right) \approx 1.402 \text{ radians.}$$

13.3.29 Note that $\overrightarrow{PQ} = \langle 2, 3, -2 \rangle$, $\overrightarrow{QR} = \langle -4, 0, 3 \rangle$, and $\overrightarrow{RP} = \langle 2, -3, -1 \rangle$, and these have lengths $\sqrt{17}$, 5 and $\sqrt{14}$ respectively. The angle at P measures $\cos^{-1} \left(\frac{3}{\sqrt{17}\sqrt{14}} \right) \approx 78.8$ degrees. The angle at Q measures $\cos^{-1} \left(\frac{14}{5\sqrt{17}} \right) \approx 47.2$ degrees, and the angle at R measures $\cos^{-1} \left(\frac{11}{5\sqrt{14}} \right) \approx 54$ degrees.

13.3.30 Note that $\overrightarrow{PQ} = \langle 1, 11 \rangle$, $\overrightarrow{QR} = \langle -4, -5 \rangle$, and $\overrightarrow{RP} = \langle 3, -6 \rangle$, and these vectors have lengths $\sqrt{122}$, $\sqrt{41}$ and $\sqrt{45}$, respectively. The angle at P measures $\cos^{-1} \left(\frac{63}{\sqrt{122}\sqrt{45}} \right) \approx 31.76$ degrees. The angle at Q measures $\cos^{-1} \left(\frac{59}{\sqrt{122}\sqrt{41}} \right) \approx 33.47$ degrees, and the angle at R measures $\cos^{-1} \left(-\frac{18}{\sqrt{45}\sqrt{41}} \right) \approx 114.78$ degrees.

13.3.31 $\text{proj}_{\mathbf{v}} \mathbf{u} = 3\mathbf{i}$. $\text{scal}_{\mathbf{v}} \mathbf{u} = 3$.

13.3.32 $\text{proj}_{\mathbf{v}} \mathbf{u} = -3\mathbf{i}$. $\text{scal}_{\mathbf{v}} \mathbf{u} = -3$.

13.3.33 $\text{proj}_{\mathbf{v}} \mathbf{u} = 3\mathbf{j}$. $\text{scal}_{\mathbf{v}} \mathbf{u} = 3$.

13.3.34 $\text{proj}_{\mathbf{v}} \mathbf{u} = -\mathbf{v} = \langle -2, -2 \rangle$. $\text{scal}_{\mathbf{v}} \mathbf{u} = -2\sqrt{2}$.

13.3.35 $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{12}{20} \langle -4, 2 \rangle = \langle -12/5, 6/5 \rangle$. $\text{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{12}{\sqrt{20}} = \frac{6}{\sqrt{5}}$.

13.3.36 $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{50}{40} \langle 2, 6 \rangle = \langle 5/2, 15/2 \rangle$. $\text{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{50}{\sqrt{40}} = \frac{25}{\sqrt{10}}$.

13.3.37 $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = -\frac{14}{19} \langle 1, 3, -3 \rangle = \langle -14/19, -42/19, 42/19 \rangle$. $\text{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = -\frac{14}{\sqrt{19}}$.

13.3.38 $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = -\frac{30}{20} \langle 0, 4, -2 \rangle = \langle 0, -6, 3 \rangle$. $\text{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = -\frac{30}{\sqrt{20}} = -3\sqrt{5}$.

$$13.3.39 \quad \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6}{6} \langle -1, 1, -2 \rangle = \langle -1, 1, -2 \rangle. \quad \text{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$

$$13.3.40 \quad \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{0}{24} \langle 2, -4, 2 \rangle = \langle 0, 0, 0 \rangle. \quad \text{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{0}{\sqrt{24}} = 0.$$

$$13.3.41 \quad w = 30 \cdot 50 \cos \pi/6 = 750\sqrt{3} \text{ foot-pounds.}$$

$$13.3.42 \quad w = 10 \cdot 20 \cdot \cos 15^\circ \approx 193.185 \text{ J.}$$

$$13.3.43 \quad w = 10 \cdot 5 \cdot \cos 45^\circ = 25\sqrt{2} \text{ J.}$$

$$13.3.44 \quad w = \langle 4, 3, 2 \rangle \cdot \langle 8, 6, 0 \rangle = 50 \text{ J.}$$

$$13.3.45 \quad w = (40\mathbf{i} + 30\mathbf{j}) \cdot 10\mathbf{i} = 400 \text{ J.}$$

$$13.3.46 \quad w = \langle 2, 4, 1 \rangle \cdot \langle 2, 4, 6 \rangle = 4 + 16 + 6 = 26.$$

$$13.3.47 \quad \text{Parallel to: use } \mathbf{v} = \langle 1/2, -\sqrt{3}/2 \rangle. \quad \text{proj}_{\mathbf{v}} \mathbf{F} = \frac{\mathbf{F} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = 5\sqrt{3} \left\langle 1/2, -\sqrt{3}/2 \right\rangle = \left\langle 5\sqrt{3}/2, -15/2 \right\rangle.$$

Normal to: $\mathbf{N} = \langle 0, -10 \rangle - \langle 5\sqrt{3}/2, -15/2 \rangle = \langle -5\sqrt{3}/2, -5/2 \rangle.$

$$13.3.48 \quad \text{Parallel to: use } \mathbf{v} = \langle 5/\sqrt{41}, -4/\sqrt{41} \rangle.$$

$$\text{proj}_{\mathbf{v}} \mathbf{F} = \frac{\mathbf{F} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{40}{\sqrt{41}} \left\langle 5/\sqrt{41}, 4/\sqrt{41} \right\rangle = \langle 200/41, 160/41 \rangle.$$

$$\text{Normal to: } \mathbf{N} = \langle 0, -10 \rangle - \langle 200/41, 160/41 \rangle = \langle -200/41, -570/41 \rangle.$$

13.3.49 The direction down the plane is $\mathbf{v} = \left\langle \cos\left(-\frac{\pi}{4}\right), \sin\left(-\frac{\pi}{4}\right) \right\rangle = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$ and the gravitational force acting on the block is $\mathbf{F} = \langle 0, 100(9.8) \rangle = \langle 0, -980 \rangle$. The component of the force parallel to the plane is $\text{proj}_{\mathbf{v}} \mathbf{F} = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = 490\sqrt{2} \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle = \langle 490, -490 \rangle$ and the component of the force perpendicular to the plane is $\mathbf{F} - \text{proj}_{\mathbf{v}} \mathbf{F} = \langle 0, -980 \rangle - \langle 490, -490 \rangle = \langle -490, -490 \rangle$.

13.3.50

- We have $|\mathbf{W}_{\text{par}}| = 100 \sin 20^\circ \approx 34.2 \text{ lbs}$ and $|\mathbf{W}_{\text{perp}}| = 100 \cos 20^\circ \approx 94 \text{ lbs}$.
- The value of μ must satisfy $\mu \geq \frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|} \approx 0.36$ in order for there to be no sliding. This condition is met here, so the block does not slide.
- In general, we have $|\mathbf{W}_{\text{par}}| = 100 \sin \theta$ and $|\mathbf{W}_{\text{perp}}| = 100 \cos \theta$. Sliding occurs if $\mu = 0.65 < \frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|} = \tan \theta$. Because $\tan^{-1} 0.65 \approx 33^\circ$, sliding will occur at angles greater than 33 degrees.

13.3.51

- False. One is a vector in the same direction as $\mathbf{u}$ and the other is a vector in the direction of $\mathbf{v}$, so if these vectors aren't in the same direction, they can't be equal.
- True. This follows because $\mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{v} \cdot \mathbf{u} + |\mathbf{v}|^2$, and these are equal if $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude.
- True. Let $\mathbf{u} = \langle a, b, c \rangle$. Then $(\mathbf{u} \cdot \mathbf{i})^2 + (\mathbf{u} \cdot \mathbf{j})^2 + (\mathbf{u} \cdot \mathbf{k})^2 = a^2 + b^2 + c^2 = |\mathbf{u}|^2$.
- False. For example, consider $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \langle 0, 1 \rangle$, and $\mathbf{w} = \langle 2, 0 \rangle$.

e. False. Consider $\langle 1, -1, 0 \rangle$, $\langle 2, -1, -1 \rangle$ and $\langle 3, -2, -1 \rangle$. These are all orthogonal to $\langle 1, 1, 1 \rangle$, but don't all lie in the same line.

f. True. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$, and this can't be zero unless $\mathbf{u} \cdot \mathbf{v} = 0$.

13.3.52 $\mathbf{v} \cdot \mathbf{w} = 4a - 24 + 21 = 0$ implies that $a = \frac{3}{4}$.

13.3.53 $\mathbf{v} \cdot \mathbf{w} = 6 - 10 + 9c = 0$ implies that $c = \frac{4}{9}$.

13.3.54 The two other vectors could be $\langle 0, 1, -1 \rangle$ and $\langle 1, 0, 0 \rangle$.

13.3.55 We must have $4 - 8a + 2b = 0$, so $b = 4a - 2$. These vectors have the form $\langle 1, a, 4a - 2 \rangle$ where a can be any real number.

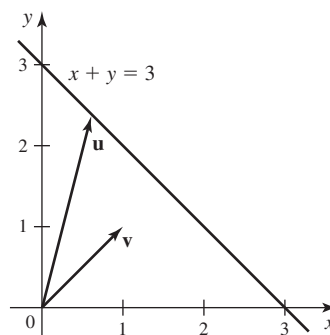
13.3.56 Let $\mathbf{u} = \pm \langle \sqrt{2}/2, \sqrt{2}/2, 0 \rangle$, $\mathbf{v} = \pm \langle -\sqrt{2}/2, \sqrt{2}/2, 0 \rangle$ and $\mathbf{w} = \pm \langle 0, 0, 1 \rangle$.

13.3.57

a. $\text{proj}_{\mathbf{k}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{k}}{\mathbf{k} \cdot \mathbf{k}} \mathbf{k} = \frac{|\mathbf{u}| |\mathbf{k}| \cos \theta}{\mathbf{k} \cdot \mathbf{k}} \mathbf{k} = \frac{1}{2} \mathbf{k}$, which is independent of $\mathbf{u}$.

b. Yes, because the scalar projection is the length of the vector projection. In fact, using the above result, it is equal to $1/2$.

13.3.58 $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{3}{2} \langle 1, 1 \rangle$. Let $\mathbf{z} = \langle x, y \rangle$. Then $\text{proj}_{\mathbf{v}} \mathbf{z} = \frac{x+y}{2} \langle 1, 1 \rangle$, so we need $x + y = 3$. All vectors from the origin whose terminal point is on the line $x + y = 3$ will have the same projection onto $\mathbf{v}$. For example, $\langle 3, 0 \rangle$ will work.



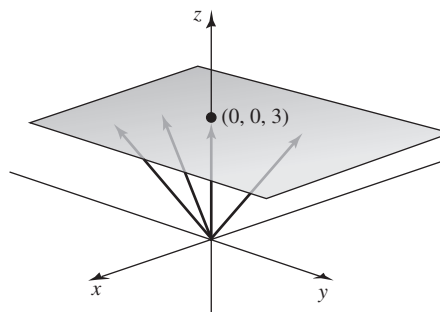
13.3.59 Using the idea from the last problem, any vector of the form $\langle x, y \rangle$ with $x + y = 3$ will work, so any vector of the form $\langle x, 3 - x \rangle$.

13.3.60 Note that

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle 1, 2, 3 \rangle \cdot \langle 1, 1, 1 \rangle}{3} \langle 1, 1, 1 \rangle = \langle 2, 2, 2 \rangle.$$

We are seeking $\langle x, y, z \rangle$ so that $\frac{x+y+z}{3} \langle 1, 1, 1 \rangle = \langle 2, 2, 2 \rangle$, so we require $x + y + z = 6$. For example, the vector $\langle 3, 2, 1 \rangle$ will work.

13.3.61 Note that $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle 1, 2, 3 \rangle \cdot \langle 0, 0, 1 \rangle}{1} \langle 0, 0, 1 \rangle = \langle 0, 0, 3 \rangle$. We are seeking $\langle x, y, z \rangle$ so that $\frac{z}{1} \langle 0, 0, 1 \rangle = \langle 0, 0, 3 \rangle$, so we require $z = 3$. Any vector of the form $\langle x, y, 3 \rangle$ will suffice.



13.3.62 Let $\mathbf{p} = \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{7}{2} \langle 1, 1 \rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle 4, 3 \rangle - \frac{7}{2} \langle 1, 1 \rangle = \langle 1/2, -1/2 \rangle$.

13.3.63 Let $\mathbf{p} = \text{proj}_{\mathbf{v}} \mathbf{u} = -\frac{2}{5} \langle 2, 1 \rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle -2, 2 \rangle - (-\frac{2}{5} \langle 2, 1 \rangle) = \langle -6/5, 12/5 \rangle$.

13.3.64 Let $\mathbf{p} = \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{7}{3} \langle 1, 1, 1 \rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle 4, 3, 0 \rangle - \frac{7}{3} \langle 1, 1, 1 \rangle = \langle 5/3, 2/3, -7/3 \rangle$.

13.3.65 Let $\mathbf{p} = \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{1}{2} \langle 2, 1, 1 \rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle -1, 2, 3 \rangle - \frac{1}{2} \langle 2, 1, 1 \rangle = \langle -2, 3/2, 5/2 \rangle$.

13.3.66 Let $P(x, y)$ be a point on ℓ . Then $\mathbf{n}$ is orthogonal to $\overrightarrow{P_0P} = \langle x - x_0, y - y_0 \rangle$, which implies that $\mathbf{n} \cdot \overrightarrow{P_0P} = \langle a, b \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0$, or $a(x - x_0) + b(y - y_0) = 0$.

13.3.67 We have $x_0 = 2$, $y_0 = 6$, and $\mathbf{n} = \langle a, b \rangle = \langle 3, -7 \rangle$. So the equation of the line is $3(x-2) - 7(y-6) = 0$, or $3x - 7y = -36$.

13.3.68 We have $x_0 = 1$, $y_0 = -3$, and $\mathbf{n} = \langle a, b \rangle = \langle 4, 0 \rangle$. So the equation of the line is $4(x-1) + 0(y+3) = 0$, or $x = 1$.

13.3.69 The vector $\mathbf{n}$ has a slope of $\frac{3}{5}$ and because this vector is orthogonal to the line, the line has a slope of $-\frac{5}{3}$.

13.3.70 $\mathbf{I} \cdot \mathbf{J} = -\frac{1}{2} + \frac{1}{2} = 0$. Also, $|\mathbf{I}| = \sqrt{1/2 + 1/2} = 1 = |\mathbf{J}|$.

13.3.71 $\mathbf{I} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{J} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{i} = \langle 1, 0 \rangle = \frac{\sqrt{2}}{2}(\mathbf{I} - \mathbf{J})$.
 $\mathbf{j} = \frac{\sqrt{2}}{2}(\mathbf{I} + \mathbf{J})$.

13.3.72 $\langle 2, -6 \rangle = 2\mathbf{i} - 6\mathbf{j} = 2 \cdot \frac{\sqrt{2}}{2}(\mathbf{I} - \mathbf{J}) - 6 \cdot \frac{\sqrt{2}}{2}(\mathbf{I} + \mathbf{J}) = -2\sqrt{2}\mathbf{I} - 4\sqrt{2}\mathbf{J}$.

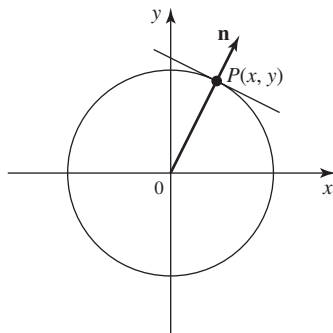
13.3.73

a. $|\mathbf{I}| = \sqrt{1/4 + 1/4 + 1/2} = 1$. $|\mathbf{J}| = \sqrt{1/2 + 1/2 + 0} = 1$. $|\mathbf{K}| = \sqrt{1/4 + 1/4 + 1/2} = 1$.

b. $\mathbf{I} \cdot \mathbf{J} = -1/2\sqrt{2} + 1/2\sqrt{2} = 0$. $\mathbf{I} \cdot \mathbf{K} = 1/4 + 1/4 - 1/2 = 0$, and $\mathbf{J} \cdot \mathbf{K} = -1/2\sqrt{2} + 1/2\sqrt{2} = 0$.

c. Let $\langle 1, 0, 0 \rangle = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. Then $\frac{1}{2}a - \frac{1}{\sqrt{2}}b + \frac{1}{2}c = 1$, $\frac{1}{2}a + \frac{1}{\sqrt{2}}b + \frac{1}{2}c = 0$, and $\frac{1}{\sqrt{2}}a - \frac{1}{\sqrt{2}}c = 0$. Solving this system of linear equations yields $a = \frac{1}{2}$, $b = -\frac{1}{\sqrt{2}}$, and $c = \frac{1}{2}$. Thus, $\langle 1, 0, 0 \rangle = \frac{1}{2}\mathbf{I} - \frac{1}{\sqrt{2}}\mathbf{J} + \frac{1}{2}\mathbf{K}$.

13.3.74



- a. The equation of the circle is $x^2 + y^2 = 1$, so $2x + 2y \frac{dy}{dx} = 0$, and we have $\frac{dy}{dx} = -\frac{x}{y}$. The slope of the line through the origin containing the vector $\langle x, y \rangle$ is $\frac{y}{x}$, and $\frac{y}{x} \cdot -\frac{x}{y} = -1$. Thus, the vector $\langle x, y \rangle$ is normal to the tangent vector.
- b. Let $x = \cos \theta$ and $y = \sin \theta$. Then the previous part shows that $\mathbf{n} = \langle x, y \rangle = \langle \cos \theta, \sin \theta \rangle$.
- c. The velocity is normal where $\langle x, y \rangle = k \langle 1, 2 \rangle$. Then $k^2 + 4k^2 = 1$, so $k = \frac{\pm 1}{\sqrt{5}}$, and the points are $(\pm 1/\sqrt{5}, \pm 2/\sqrt{5})$.
- d. We need $\langle 1, 2 \rangle \cdot \langle x, y \rangle = 0$, so $x + 2y = 0$. Because we also have $x^2 + y^2 = 1$, we see that $4y^2 + y^2 = 1$, so $y = \frac{\pm 1}{\sqrt{5}}$, and $x = \frac{\mp 2}{\sqrt{5}}$. The two points are $(2/\sqrt{5}, -1/\sqrt{5})$ and $(-2/\sqrt{5}, 1/\sqrt{5})$.
- e. The component of $\mathbf{v} = \langle 1, 2 \rangle$ normal to C is $\text{proj}_{\mathbf{n}} \mathbf{v} = \frac{x+2y}{x^2+y^2} \langle x, y \rangle = \langle x^2 + 2xy, xy + 2y^2 \rangle$. In terms of θ , this is $\langle \cos^2 \theta + 2 \cos \theta \sin \theta, \sin \theta \cos \theta + 2 \sin^2 \theta \rangle$.
- f. The net flow is 0, so there is no accumulation inside the circle.

13.3.75

- a. The faces on $y = 0$ and $z = 0$.
- b. The faces on $y = 1$ and $z = 1$.
- c. The faces on $x = 0$ and $x = 1$.
- d. Because $\mathbf{Q}$ is tangential on this face, the scalar component of $\mathbf{Q}$ normal to the face is 0.
- e. The scalar component of $\mathbf{Q}$ normal to $z = 1$ is 1. Note that a vector normal to $z = 1$ is $\langle 0, 0, 1 \rangle$.
- f. The scalar component of $\mathbf{Q}$ normal to $y = 0$ is 2. Note that a vector normal to $y = 0$ is $\langle 0, 1, 0 \rangle$.

13.3.76

- a. $\mathbf{r}_{0j} = \left\langle 2 \cos\left(\frac{(j-1)\pi}{3}\right), 2 \sin\left(\frac{(j-1)\pi}{3}\right) \right\rangle$ for $j = 1, 2, \dots, 6$.
- b. $\mathbf{r}_{12} = \langle 2 \cos(2\pi/3), 2 \sin(2\pi/3) \rangle = \langle -1, \sqrt{3} \rangle$.
 $\mathbf{r}_{34} = \langle 2 \cos(4\pi/3), 2 \sin(4\pi/3) \rangle = \langle -1, -\sqrt{3} \rangle$.
 $\mathbf{r}_{61} = \langle 2 \cos(\pi/3), 2 \sin(\pi/3) \rangle = \langle 1, \sqrt{3} \rangle$.
- c. $\mathbf{r}_{07} = \mathbf{r}_{01} + \mathbf{r}_{02} = \langle 3, \sqrt{3} \rangle$.
 $\mathbf{r}_{17} = \mathbf{r}_{02} = \langle 2 \cos(\pi/3), 2 \sin(\pi/3) \rangle = \langle 1, \sqrt{3} \rangle$.
 $\mathbf{r}_{47} = \langle 4, 0 \rangle + \mathbf{r}_{17} = \langle 5, \sqrt{3} \rangle$.
 $\mathbf{r}_{75} = \mathbf{r}_{70} + \mathbf{r}_{05} = -2 \langle 2, \sqrt{3} \rangle$.

13.3.77

- a. Let the coordinates of R be (x, y, z) . By symmetry, we have $y = 0$. We must have $x^2 + y^2 + z^2 = (x - \sqrt{3})^2 + (y+1)^2 + z^2 = (x - \sqrt{3})^2 + (y-1)^2 + z^2 = 4$. The first equality gives $x^2 + z^2 = x^2 - 2\sqrt{3}x + 3 + 1 + z^2$, so $2\sqrt{3}x = 4$, and $x = \frac{2}{\sqrt{3}}$. It then follows that $z = \frac{2\sqrt{2}}{\sqrt{3}}$.
- b. We have $\mathbf{r}_{OP} = \langle \sqrt{3}, -1, 0 \rangle$, $\mathbf{r}_{OQ} = \langle \sqrt{3}, 1, 0 \rangle$, $\mathbf{r}_{PQ} = \langle 0, 2, 0 \rangle$, $\mathbf{r}_{OR} = \langle 2/\sqrt{3}, 0, 2\sqrt{2}/\sqrt{3} \rangle$, and $\mathbf{r}_{PR} = \langle -\sqrt{3}/3, 1, 2\sqrt{2}/\sqrt{3} \rangle$.

13.3.78 $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$, so $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\cos \theta| \leq |\mathbf{u}| |\mathbf{v}|$, because $|\cos \theta| \leq 1$.

13.3.79 $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v} \cdot \mathbf{u}$.

13.3.80 $c(\mathbf{u} \cdot \mathbf{v}) = c(u_1v_1 + u_2v_2 + u_3v_3) = cu_1v_1 + cu_2v_2 + cu_3v_3 = (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 = (c\mathbf{u}) \cdot \mathbf{v}$. Using this result and the previous, we have $c(\mathbf{u} \cdot \mathbf{v}) = c(\mathbf{v} \cdot \mathbf{u}) = (c\mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot (c\mathbf{v})$.

13.3.81 $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) = u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3 + u_3w_3 = (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$.

13.3.82

- Using the previous results, we have $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$.
- If the two vectors are perpendicular, then $\mathbf{u} \cdot \mathbf{v} = 0$, so the above reduces to $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2$.
- Using the previous results, we have $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 - |\mathbf{v}|^2$.

13.3.83

- $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$, $\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$, and $\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$. Thus, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1$.
- We require $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{1}{2} + \frac{1}{2} + \cos^2 \gamma = 1$, so $\gamma = 90^\circ$. The vector could be $\langle 1, 1, 0 \rangle$; it makes a 90 degree angle with $\mathbf{k}$.
- We require $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{1}{4} + \frac{1}{4} + \cos^2 \gamma = 1$, so $\gamma = 45^\circ$. The vector could be $\langle 1, 1, \sqrt{2} \rangle$; it makes a 45 degree angle with $\mathbf{k}$.
- No. If so, we would have $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{3}{4} + \frac{3}{4} + \cos^2 \gamma = 1$, which would imply that $\cos^2 \gamma = -\frac{1}{2}$, which can't occur.
- If $\alpha = \beta = \gamma$, then $3\cos^2 \alpha = 1$, and $\alpha = \cos^{-1}(\sqrt{3}/3) \approx 54.7356$ degrees. The vector could be $\langle 1, 1, 1 \rangle$.

13.3.84 $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\cos \theta| = |\mathbf{u}| |\mathbf{v}|$ if and only if $\cos \theta = \pm 1$, which occurs only when $\mathbf{u}$ and $\mathbf{v}$ are parallel, or if either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

13.3.85 $\mathbf{u} \cdot \mathbf{v} = -24 - 15 + 6 = -33$. $|\mathbf{u}| = \sqrt{3^2 + (-5)^2 + 6^2} = \sqrt{70}$. $|\mathbf{v}| = \sqrt{(-8)^2 + 3^2 + 1^2} = \sqrt{74}$. Note that

$$33 < \sqrt{70}\sqrt{74},$$

so $|\mathbf{u} \cdot \mathbf{v}| < |\mathbf{u}| |\mathbf{v}|$.

13.3.86 We have $|\mathbf{u} \cdot \mathbf{v}| = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab} \leq |\mathbf{u}| |\mathbf{v}| = \sqrt{a+b}\sqrt{b+a} = a+b$. Thus,

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

13.3.87

- We have $|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$

b. Note that $2(\mathbf{u} \cdot \mathbf{v}) \leq 2|\mathbf{u} \cdot \mathbf{v}| \leq 2|\mathbf{u}||\mathbf{v}|$. Thus (using the previous part) we have,

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 \leq |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 \leq (|\mathbf{u}| + |\mathbf{v}|)^2.$$

c. Taking square roots of the previous result, and using the fact that the square root function is strictly increasing, we have $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$.

d. Because the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ form a triangle, we can interpret this as meaning that the sum of the lengths of any two sides of a triangle is greater than or equal to the length of the other side.

13.3.88 Let $\mathbf{v} = \langle u_2, u_3, u_1 \rangle$. Then $\mathbf{u} \cdot \mathbf{v} = u_1u_2 + u_2u_3 + u_1u_3$ and $|\mathbf{u}||\mathbf{v}| = u_1^2 + u_2^2 + u_3^2$. So the Cauchy-Schwarz inequality gives $u_1u_2 + u_2u_3 + u_3u_1 \leq |\mathbf{u}|^2$. Thus,

$$(u_1 + u_2 + u_3)^2 = u_1^2 + u_2^2 + u_3^2 + 2(u_1u_2 + u_2u_3 + u_3u_1) \leq u_1^2 + u_2^2 + u_3^2 + 2|\mathbf{u}|^2 = 3|\mathbf{u}|^2.$$

13.3.89

a. One diagonal consists of the sum of one side ($\mathbf{u}$) and the side opposite the side adjacent to $\mathbf{u}$, but because it is a parallelogram, the side opposite $\mathbf{v}$ is also $\mathbf{v}$. So the diagonal is $\mathbf{u} + \mathbf{v}$. The other diagonal is the difference of two adjacent sides, so it is $\mathbf{u} - \mathbf{v}$.

b. The two diagonals are equal when $|\mathbf{u} + \mathbf{v}| = |\mathbf{u} - \mathbf{v}|$. Squaring both sides, we see that this is equivalent to requiring $|\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 = |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$, which would imply that $2(\mathbf{u} \cdot \mathbf{v}) = -2(\mathbf{u} \cdot \mathbf{v})$, or $4(\mathbf{u} \cdot \mathbf{v}) = 0$. So if the diagonals are equal, the vectors are orthogonal. These steps are reversible, so the converse is also true.

c. $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$.

13.4 Cross Products

13.4.1 Two parallel vectors have $\sin \theta = 0$ where θ is the angle between them. Thus, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin \theta = 0$.

13.4.2 Two perpendicular vectors have $\sin \theta = 1$ where θ is the angle between them. Thus, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin \theta = |\mathbf{u}||\mathbf{v}|$.

13.4.3

a. This implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b. This implies that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

13.4.4 $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, so in particular, it is orthogonal to $\mathbf{u}$. So $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

13.4.5 $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\pi/4) = \sqrt{2}/2$.

13.4.6 $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(2\pi/3) = 12 \cdot (\sqrt{3}/2) = 6\sqrt{3}$.

13.4.7 Because $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$, we have $\mathbf{v} \times \mathbf{u} = -3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$.

13.4.8 Because the angle between $\mathbf{v}$ and itself is 0 radians, we have $|\mathbf{v} \times \mathbf{v}| = |\mathbf{v}||\mathbf{v}|\sin 0 = 0$. Thus $\mathbf{v} \times \mathbf{v}$ is the zero vector.

13.4.9 If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then $\mathbf{u} \times \mathbf{v}$ can be thought of as the determinant of the matrix

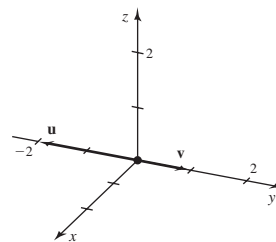
$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

13.4.10 The torque produced by the force $\mathbf{F}$ about the head of vector $\mathbf{r}$ is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

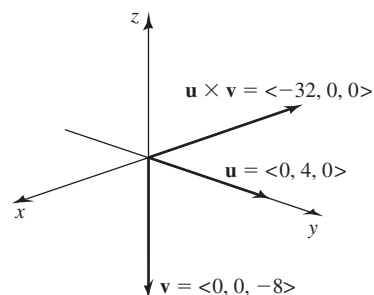
13.4.11 $\mathbf{u} \times \mathbf{v} = \langle 3, 0, 0 \rangle \times \langle 0, 5, 0 \rangle = \langle 0, 0, 15 \rangle$.

13.4.12 $\mathbf{u} \times \mathbf{v} = \langle -4, 0, 0 \rangle \times \langle 0, 0, 2 \rangle = \langle 0, 8, 0 \rangle$.

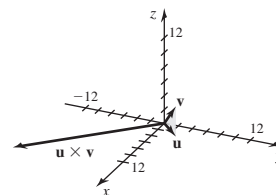
13.4.13 $\mathbf{u} \times \mathbf{v} = \langle 0, 0, 0 \rangle$, so $|\mathbf{u} \times \mathbf{v}| = 0$.



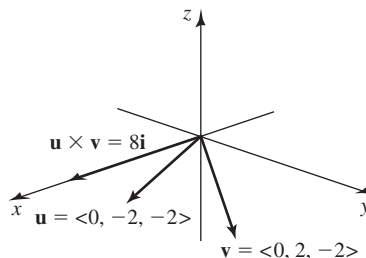
13.4.14 $\mathbf{u} \times \mathbf{v} = \langle -32, 0, 0 \rangle$, so $|\mathbf{u} \times \mathbf{v}| = 32$.



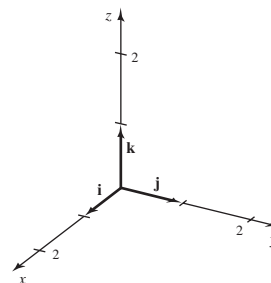
13.4.15 $\mathbf{u} \times \mathbf{v} = \langle 9\sqrt{2}, -9\sqrt{2}, 0 \rangle$, so $|\mathbf{u} \times \mathbf{v}| = 18$.



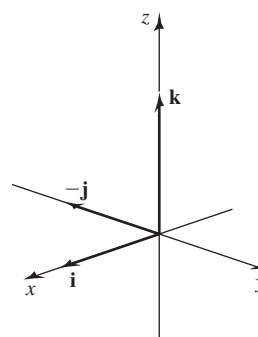
13.4.16 $\mathbf{u} \times \mathbf{v} = \langle 8, 0, 0 \rangle$, so $|\mathbf{u} \times \mathbf{v}| = 8$.



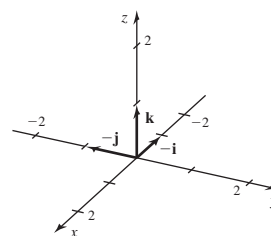
$$13.4.17 \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}.$$



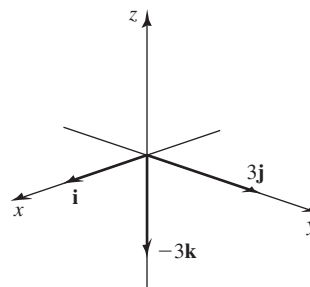
$$13.4.18 \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$



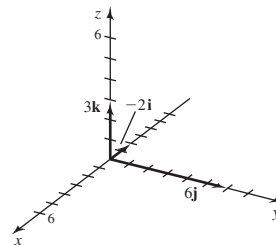
$$13.4.19 \quad -\mathbf{j} \times \mathbf{k} = -\mathbf{i}.$$



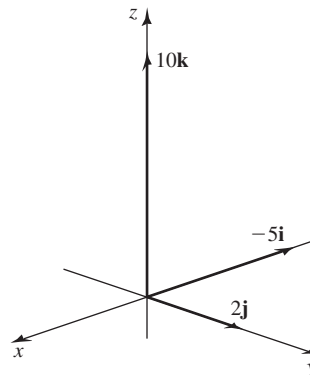
$$13.4.20 \quad 3\mathbf{j} \times \mathbf{i} = -3\mathbf{k}.$$



13.4.21 $-2\mathbf{i} \times 3\mathbf{k} = 6\mathbf{j}$.



13.4.22 $2\mathbf{j} \times -5\mathbf{i} = 10\mathbf{k}$.



13.4.23 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & 0 \\ 0 & 3 & -6 \end{vmatrix} = \langle -30, 18, 9 \rangle. \mathbf{v} \times \mathbf{u} = \langle 30, -18, -9 \rangle.$

13.4.24 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \langle -2, -4, -4 \rangle. \mathbf{v} \times \mathbf{u} = \langle 2, 4, 4 \rangle.$

13.4.25 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -9 \\ -1 & 1 & -1 \end{vmatrix} = \langle 6, 11, 5 \rangle. \mathbf{v} \times \mathbf{u} = \langle -6, -11, -5 \rangle.$

13.4.26 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 6 \\ 1 & 2 & -1 \end{vmatrix} = \langle -8, 9, 10 \rangle. \mathbf{v} \times \mathbf{u} = \langle 8, -9, -10 \rangle.$

13.4.27 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -2 \\ 1 & 3 & -2 \end{vmatrix} = \langle 8, 4, 10 \rangle. \mathbf{v} \times \mathbf{u} = \langle -8, -4, -10 \rangle.$

13.4.28 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -10 & 15 \\ .5 & 1 & -.6 \end{vmatrix} = \langle -9, 8.7, 7 \rangle. \mathbf{v} \times \mathbf{u} = \langle 9, -8.7, -7 \rangle.$

$$\mathbf{13.4.29} \quad |\mathbf{u} \times \mathbf{v}| = |\langle -2, -6, 9 \rangle| = \sqrt{4 + 36 + 81} = 11.$$

$$\mathbf{13.4.30} \quad |\mathbf{u} \times \mathbf{v}| = |\langle -2, 5, -3 \rangle| = \sqrt{4 + 9 + 25} = \sqrt{38}.$$

$$\mathbf{13.4.31} \quad |\mathbf{u} \times \mathbf{v}| = |\langle 5, -4, 7 \rangle| = \sqrt{25 + 16 + 49} = 3\sqrt{10}.$$

$$\mathbf{13.4.32} \quad |\mathbf{u} \times \mathbf{v}| = |\langle 4, 26, 28 \rangle| = \sqrt{16 + 676 + 784} = \sqrt{1476} = 6\sqrt{41}.$$

$$\mathbf{13.4.33} \quad \frac{1}{2} \cdot |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \cdot |\langle 3, 0, 1 \rangle \times \langle 1, 1, 0 \rangle| = \frac{1}{2} \cdot |\langle -1, 1, 3 \rangle| = \frac{\sqrt{11}}{2}.$$

$$\mathbf{13.4.34} \quad \text{Two of the sides are } \mathbf{u} = \langle 1, 2, 3 \rangle \text{ and } \mathbf{v} = \langle 6, 5, 4 \rangle.$$

$$\text{The area is } \frac{1}{2} |\mathbf{u} \times \mathbf{v}| = \frac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{vmatrix} \right\| = \frac{1}{2} |\langle -7, 14, -7 \rangle| = \frac{7\sqrt{6}}{2}.$$

$$\mathbf{13.4.35} \quad \frac{1}{2} \cdot |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \cdot |\langle 2, 10, 2 \rangle \times \langle 1, 1, 1 \rangle| = \frac{1}{2} \cdot |\langle 8, 0, -8 \rangle| = 4\sqrt{2}.$$

$$\mathbf{13.4.36} \quad \frac{1}{2} \cdot |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \cdot |\langle -2, 3, 2 \rangle \times \langle 1, 0, 2 \rangle| = \frac{1}{2} \cdot |\langle 6, 6, -3 \rangle| = \frac{9}{2}.$$

$$\mathbf{13.4.37} \quad \text{The area is } \frac{1}{2} |\mathbf{u} \times \mathbf{v}| = \frac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 3 \\ 6 & 0 & 6 \end{vmatrix} \right\| = \frac{1}{2} |\langle 18, 0, -18 \rangle| = 9\sqrt{2}.$$

$$\mathbf{13.4.38} \quad \text{The area is } \frac{1}{2} |\mathbf{u} \times \mathbf{v}| = \frac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ 4 & 4 & 4 \end{vmatrix} \right\| = \frac{1}{2} |\langle 24, 0, -24 \rangle| = 12\sqrt{2}.$$

13.4.39 If the points are collinear, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel, and $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{0}$. On the other hand, if the points are not collinear, then the area of area of triangle ABC is not 0. So $|\overrightarrow{AB} \times \overrightarrow{AC}| \neq 0$, and it follows that $\overrightarrow{AB} \times \overrightarrow{AC} \neq \mathbf{0}$.

$$\mathbf{13.4.40} \quad \overrightarrow{AB} \times \overrightarrow{AC} = \langle 2, 2, 6 \rangle \times \langle 6, 6, 18 \rangle = \langle 0, 0, 0 \rangle, \text{ so the points are collinear.}$$

$$\mathbf{13.4.41} \quad \overrightarrow{AB} \times \overrightarrow{AC} = \langle 4, 6, 6 \rangle \times \langle 7, 12, 13 \rangle \neq \langle 0, 0, 0 \rangle, \text{ so the points are not collinear.}$$

$$\mathbf{13.4.42} \quad \text{Let } \mathbf{u} = \langle 1, 2, 3 \rangle \text{ and } \mathbf{v} = \langle -2, 4, -1 \rangle. \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 4 & -1 \end{vmatrix} = \langle -14, -5, 8 \rangle \text{ is perpendicular to both } \mathbf{u} \text{ and } \mathbf{v}.$$

$$\mathbf{13.4.43} \quad \text{Let } \mathbf{u} = \langle 0, 1, 2 \rangle \text{ and } \mathbf{v} = \langle -2, 0, 3 \rangle. \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ -2 & 0 & 3 \end{vmatrix} = \langle 3, -4, 2 \rangle \text{ is perpendicular to both } \mathbf{u} \text{ and } \mathbf{v}.$$

$$\mathbf{13.4.44} \quad \text{Let } \mathbf{u} = \langle 8, 0, 4 \rangle \text{ and } \mathbf{v} = \langle -8, 2, 1 \rangle. \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 4 \\ -8 & 2 & 1 \end{vmatrix} = \langle -8, -40, 16 \rangle \text{ is perpendicular to both } \mathbf{u} \text{ and } \mathbf{v}.$$

$$13.4.45 \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 20 & 0 & 0 \end{vmatrix} = \langle 0, 20, -20 \rangle.$$

$$13.4.46 \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 10 & 10 & 0 \end{vmatrix} = \langle -20, 20, 20 \rangle.$$

$$13.4.47 \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 0 & 0 \\ 5 & 0 & -5 \end{vmatrix} = \langle 0, 50, 0 \rangle \text{ has magnitude } 50, \text{ while } \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 0 & 0 \\ 4 & -3 & 0 \end{vmatrix} = \langle 0, 0, -30 \rangle \text{ has magnitude } 30, \text{ so the first force has greater magnitude.}$$

$$13.4.48 \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & -5 \\ 1 & 0 & -10 \end{vmatrix} = \langle 0, 45, 0 \rangle. \text{ The magnitude is } 45 \text{ and the direction is in the positive } y \text{ direction.}$$

$$13.4.49 \quad |\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta = \frac{1}{4} \cdot 20 \cdot \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2} \text{ N} \cdot \text{m}.$$

$$13.4.50 \quad |\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta = \frac{5}{6} \cdot \frac{3}{2} \cdot \sin(\pi/2) = 1.25 \text{ ft} \cdot \text{lb}.$$

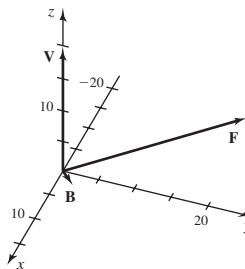
$$13.4.51 \quad \text{Note that } \mathbf{r} = 0.66\mathbf{k}, \text{ and } \mathbf{F} = 40\mathbf{j}. \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.66 \\ 0 & 40 & 0 \end{vmatrix} = \langle -26.4, 0, 0 \rangle. \text{ The magnitude of the torque is } 26.4 \text{ Newton-meters and the direction is on the negative } x \text{ axis.}$$

13.4.52

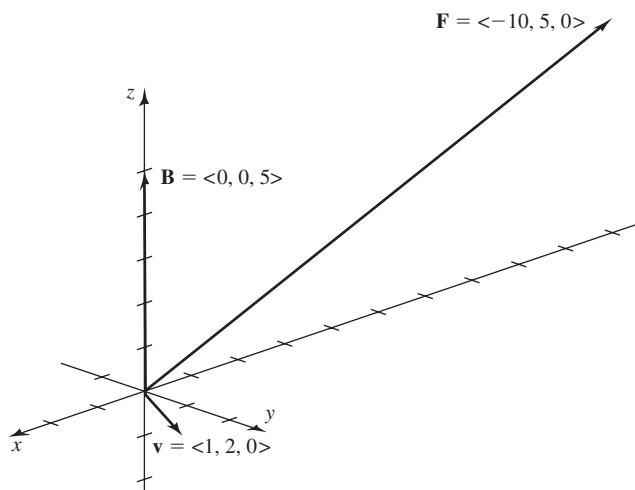
- The torque of the shoulder has magnitude $2 \cdot 20 = 40$ ft lbs and the direction is perpendicular to $\mathbf{r}$ and $\mathbf{F}$, into the page.
- The torque of the elbow has magnitude $1 \cdot 20 = 20$ foot pounds, and the direction is the same as the torque of the shoulder, into the page.

$$13.4.53 \quad \mathbf{F} = 1 \cdot (\mathbf{v} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 20 \\ 1 & 1 & 0 \end{vmatrix} = \langle -20, 20, 0 \rangle.$$

The magnitude of $\mathbf{F}$ is $20\sqrt{2}$ and the angle of the force is 135 degrees with the positive x axis in the xy -plane.



$$13.4.54 \quad \mathbf{F} = -1 \cdot (\mathbf{v} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = \langle -10, 5, 0 \rangle. \text{ The magnitude of } \mathbf{F} \text{ is } 5\sqrt{5} \text{ and the angle of the force is about } 153.435 \text{ degrees with the positive } x \text{ axis in the } xy\text{-plane.}$$



13.4.55 $|\mathbf{F}| = |q(\mathbf{v} \times \mathbf{B})| = |-1.6 \cdot 10^{-19}| \text{ C} \cdot 2 \cdot 10^5 \cdot 2 \cdot \sin 45^\circ = 4.53 \cdot 10^{-14} \text{ kg} \cdot \text{m/s}^2.$

13.4.56 $\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) = 5 \cdot 10^{-12} \mathbf{k}$, $\mathbf{v} = 2 \cdot 10^6 \mathbf{j}$. We must have $\mathbf{B} = b\mathbf{i}$ for some scalar b . We have

$$1.6 \cdot 10^{-19} (2 \cdot 10^6 \mathbf{j} \times b\mathbf{i}) = 5 \cdot 10^{-12} \mathbf{k},$$

so $b = -\frac{5 \cdot 10^{-12}}{1.6 \cdot 10^{-19} \cdot 2 \cdot 10^6} = -\frac{125}{8}$. So $\mathbf{B} = -15.625\mathbf{i}$.

13.4.57

- False. For example $\mathbf{i} \times \mathbf{i} = \langle 0, 0, 0 \rangle$, even though $\mathbf{i} \neq \langle 0, 0, 0 \rangle$.
- False. For example, $2\mathbf{i} \times 4\mathbf{j} = 8\mathbf{k}$ has magnitude 8, while $2\mathbf{i}$ has magnitude 2 and $4\mathbf{j}$ has magnitude 4.
- False. If the compass directions are thought to lie in a plane, $\mathbf{u} \times \mathbf{v}$ doesn't lie in that plane, so it can't be a compass direction.
- True. If both were nonzero, the first statement implies that the vectors are parallel, and the second that they are perpendicular, which can't both occur. So at least one of the vectors must be the zero vector.
- False. $\mathbf{i} \times 2\mathbf{i} = \langle 0, 0, 0 \rangle = \mathbf{i} \times 3\mathbf{i}$, but $2\mathbf{i} \neq 3\mathbf{i}$.

13.4.58 Note that $\langle a, a, 2 \rangle \times \langle 1, a, 3 \rangle = \langle a, 2 - 3a, -a + a^2 \rangle$. So $a = 2$.

13.4.59 Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$. Then we have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ u_1 & u_2 & u_3 \end{vmatrix} = \langle -1, -1, 2 \rangle,$$

so $u_3 - u_2 = -1$, $u_1 - u_3 = -1$, and $u_2 - u_1 = 2$. The solutions to this system of linear equations can be characterized by letting u_1 be arbitrary, and by letting $u_2 = u_1 + 2$ and $u_3 = u_1 + 1$. Thus, $\mathbf{u} = \langle u_1, u_1 + 2, u_1 + 1 \rangle$ for any real number u_1 .

13.4.60 Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$. Then we have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ u_1 & u_2 & u_3 \end{vmatrix} = \langle 0, 0, 1 \rangle,$$

so $u_3 - u_2 = 0$, $u_1 - u_3 = 0$, and $u_2 - u_1 = 1$. This system of linear equations has no solutions, so there are no vectors $\mathbf{u}$ which satisfy the given equation.

13.4.61 Two of the sides of the triangle are $\mathbf{u} = \langle -a, b, 0 \rangle$ and $\mathbf{v} = \langle -a, 0, c \rangle$.

$$\text{The area is } \frac{1}{2} |\mathbf{u} \times \mathbf{v}| = \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \frac{1}{2} |\langle bc, ac, ab \rangle| = \frac{1}{2} \sqrt{b^2c^2 + a^2c^2 + a^2b^2}.$$

13.4.62 $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_2w_3 - v_3w_2, v_1w_3 - v_1w_3, v_1w_2 - w_1v_2 \rangle = u_1(v_2w_3 - v_3w_2) + u_2(w_1v_3 - v_1w_3) + u_3(v_1w_2 - w_1v_2)$. Note that this is exactly the expression for the determinant of the given matrix, as can be seen by expanding by cofactors across the top row.

13.4.63 $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}| |\mathbf{v} \times \mathbf{w}| |\cos \theta|$. Because $|\mathbf{v} \times \mathbf{w}|$ represents the area of the base, we just need to see that the height of the parallelepiped is $|\mathbf{u}| |\cos \theta|$. Note that the height is given by the scalar projection of $\mathbf{u}$ on $\mathbf{v} \times \mathbf{w}$, which has value $|\cos \theta| |\mathbf{u}|$. Thus the given expression represents the volume of the parallelepiped.

13.4.64 The volume is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\langle 3, 1, 0 \rangle \cdot \langle 19, -9, -2 \rangle| = 57 - 9 + 0 = 48$.

13.4.65 If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are coplanar, then the volume of the corresponding parallelepiped is 0 which implies that $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 0$. If vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are not coplanar, then the volume of the corresponding parallelepiped is not 0 which implies that $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \neq 0$.

13.4.66 Using one of the results above, we see that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ and similarly we have

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$. These two determinants are equal, as can be seen by fact that the underlying matrices differ by an even number of row transpositions.

13.4.67 Because $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, we have $|\mathbf{F}| = |q| |\mathbf{v}| |\mathbf{B}| \sin \theta$. Thus, $\frac{m |\mathbf{v}|^2}{R} = |q| |\mathbf{v}| |\mathbf{B}| \sin \pi/2$. Therefore,

$$|\mathbf{v}| = \frac{R |q| |\mathbf{B}|}{m} = \frac{0.002 \cdot 1.6 \cdot 10^{-19} \cdot .05}{9 \cdot 10^{-31}} \approx 1.758 \cdot 10^7 \text{ m/s}.$$

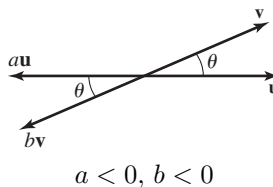
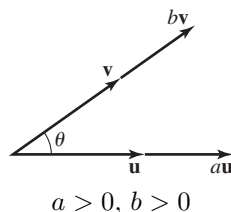
13.4.68

a. $|\mathbf{u} \times \mathbf{u}| = |\mathbf{u}|^2 \sin 0 = 0$.

b. $\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \langle 0, 0, 0 \rangle$.

c. Because $\mathbf{u} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{u})$, we must have $2(\mathbf{u} \times \mathbf{u}) = 0$, so $\mathbf{u} \times \mathbf{u} = 0$.

13.4.69 The result is trivial if either $a = 0$ or $b = 0$, so assume $ab \neq 0$. Note that the sine of the angle between $a\mathbf{u}$ and $b\mathbf{v}$ is the same as the sine of the angle between $\mathbf{u}$ and $\mathbf{v}$, as is demonstrated in the following diagrams.





By the definition: $|(a\mathbf{u}) \times (b\mathbf{v})| = |a\mathbf{u}||b\mathbf{v}|\sin\theta$, where θ is the angle between $a\mathbf{u}$ and $b\mathbf{v}$. But this is equal to $|a||\mathbf{u}||b||\mathbf{v}|\sin\theta = |ab|(|\mathbf{u}||\mathbf{v}|\sin\theta) = |ab|(|\mathbf{u} \times \mathbf{v}|)$. When a and b have the same sign, the directions are also the same, because they are determined by the right-hand rule (see diagrams above.) When a and b have opposite signs, the directions are opposite, but then $ab < 0$.

Using the determinant formula:

$$(a\mathbf{u}) \times (b\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ au_1 & au_2 & au_3 \\ bv_1 & bv_2 & bv_3 \end{vmatrix} = ab \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = ab(\mathbf{u} \times \mathbf{v}).$$

13.4.70 False. For example, $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq \mathbf{0}$.

13.4.71 True. $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{v} - (\mathbf{v} \times \mathbf{u}) - (\mathbf{v} \times \mathbf{v}) = 2(\mathbf{u} \times \mathbf{v}) = (2\mathbf{u} \times \mathbf{v})$.

13.4.72 This follows from number 66 above, together with the commutativity of the dot product.

13.4.73

$$\begin{aligned}
 \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} \\
 &= \langle u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3), u_3(v_2w_3 - v_3w_2) \\
 &\quad - u_1(v_1w_2 - v_2w_1), u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2) \rangle \\
 &= \langle v_1(u_2w_2 + u_3w_3) - w_1(u_2v_2 + u_3v_3), v_2(u_1w_1 + u_3w_3) \\
 &\quad - w_2(u_1v_1 + u_3v_3), v_3(u_1w_1 + u_2w_2) - w_3(u_1v_1 + u_2v_2) \rangle \\
 &= \langle v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v}), v_2(\mathbf{u} \cdot \mathbf{w}) - w_2(\mathbf{u} \cdot \mathbf{v}), v_3(\mathbf{u} \cdot \mathbf{w}) - w_3(\mathbf{u} \cdot \mathbf{v}) \rangle \\
 &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.
 \end{aligned}$$

13.4.74 Consider the quantity $(\mathbf{w} \times \mathbf{x})$ as a single vector. Using the result of exercise 72, we have $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = \mathbf{u} \cdot (\mathbf{v} \times (\mathbf{w} \times \mathbf{x}))$. Now applying the result of exercise 73, this is equal to $\mathbf{u} \cdot ((\mathbf{v} \cdot \mathbf{x})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$ as desired.

13.4.75

- a. Suppose $\mathbf{u} \times \mathbf{z} = \mathbf{v}$. Then $\mathbf{v} \times (\mathbf{u} \times \mathbf{z}) = \mathbf{v} \times \mathbf{v} = \langle 0, 0, 0 \rangle$. Now $\mathbf{v} \times (\mathbf{u} \times \mathbf{z}) = \mathbf{u}(\mathbf{v} \cdot \mathbf{z}) - \mathbf{z}(\mathbf{v} \cdot \mathbf{u})$ by exercise 73. If $\mathbf{u} \cdot \mathbf{v} = 0$, then we have $\mathbf{u}(\mathbf{v} \cdot \mathbf{z}) = \langle 0, 0, 0 \rangle$. Any vector $\mathbf{z}$ which is perpendicular to $\mathbf{v}$ is a solution to this equation.

Now suppose that the equation $\mathbf{u} \times \mathbf{z} = \mathbf{v}$ has a nonzero solution. Because the cross product of any two vectors is perpendicular to both of the vectors, we must have that $\mathbf{u} \times \mathbf{z} \cdot \mathbf{u} = 0$. But this means that $\mathbf{v} \cdot \mathbf{u} = 0$, as desired.

- b. If there exists a vector $\mathbf{z}$ so that $\mathbf{u} \times \mathbf{z} = \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ must be perpendicular. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular nonzero vectors, then there must be a plane which contains $\mathbf{u}$ and a nonzero vector $\mathbf{z}$ so that $\mathbf{u} \times \mathbf{z} = \mathbf{v}$.

13.5 Lines and Planes in Space

13.5.1 $\langle 4, -8, 9 \rangle$.

13.5.2 $x = 1 + 4t$, $y = 2$, $z = 3 - 6t$.

13.5.3 Subtract componentwise to obtain the vector $\mathbf{d} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

13.5.4 Let $\mathbf{d}$ be the direction vector as in the previous problem. Then $\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\mathbf{d}$.

13.5.5 The direction vector of the line is $\langle 1, 1, 1 \rangle$, which is the direction normal to the plane. Thus, the line is perpendicular to the plane.

13.5.6 The direction vector of the line is $\langle 1, -2, 1 \rangle$ which is perpendicular to the normal to the plane (because $\langle 1, -2, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 0$), so the line is parallel to the plane.

13.5.7 One point and a normal vector determine a plane

13.5.8 The vector $\mathbf{n} = \langle -2, -3, 4 \rangle$ is normal to this plane.

13.5.9 The point (x, y, z) where this plane intersects the x -axis has $y = z = 0$; substituting in the equation of the plane gives $x = -6$. Similarly, we see that the plane meets the y -axis at $y = -4$ and the z -axis at $z = 3$.

13.5.10 Substituting in the general equation for a plane gives $(x - 1) + (y - 0) + (z - 0) = 0$, which simplifies to $x + y + z = 1$.

13.5.11 The line is $\mathbf{r}(t) = \langle 0, 0, 1 \rangle + t \langle 4, 7, 0 \rangle$. The parametric equations are $x = 4t$, $y = 7t$, $z = 1$.

13.5.12 The line is $\mathbf{r}(t) = \langle -3, 2, -1 \rangle + t \langle 1, -2, 0 \rangle$. The parametric equations are $x = -3 + t$, $y = 2 - 2t$, $z = -1$.

13.5.13 The direction is $\langle 0, 1, 0 \rangle$, so the line is $\mathbf{r}(t) = \langle 0, 0, 1 \rangle + t \langle 0, 1, 0 \rangle$. The parametric equations are $x = 0$, $y = t$, $z = 1$.

13.5.14 The direction is $\langle 1, 0, 0 \rangle$ so the line is $\langle -2, 4, 3 \rangle + t \langle 1, 0, 0 \rangle$. The parametric equations are $x = -2 + t$, $y = 4$, $z = 3$.

13.5.15 The direction is $\langle 1, 2, 3 \rangle$, so the line is $\mathbf{r}(t) = t \langle 1, 2, 3 \rangle$. The parametric equations are $x = t$, $y = 2t$, $z = 3t$.

13.5.16 The direction is $\langle 8, -5, -6 \rangle$, so the line is $\mathbf{r}(t) = \langle -3, 4, 6 \rangle + t \langle 8, -5, -6 \rangle$. The parametric equations are $x = -3 + 8t$, $y = 4 - 5t$, $z = 6 - 6t$.

13.5.17 The direction is $\langle -2, 8, -4 \rangle$, so the line is $\mathbf{r}(t) = t \langle -2, 8, -4 \rangle$. The parametric equations are $x = -2t$, $y = 8t$, $z = -4t$.

13.5.18 The direction is $\langle 4, -1, 0 \rangle$, so the line is $\mathbf{r}(t) = \langle 1, -3, 4 \rangle + t \langle 4, -1, 0 \rangle$. The parametric equations are $x = 1 + 4t$, $y = -3 - t$, $z = 4$.

13.5.19 The direction is $\langle 1, 0, 2 \rangle \times \langle 0, 1, 1 \rangle = \langle -2, -1, 1 \rangle$, so the line is $\mathbf{r}(t) = t \langle -2, -1, 1 \rangle$. The parametric equations are $x = -2t$, $y = -t$, $z = t$.

13.5.20 The direction is $\langle 1, 1, -5 \rangle \times \langle 0, 4, 0 \rangle = \langle 20, 0, 4 \rangle$, so the line is $\mathbf{r}(t) = \langle -3, 4, 2 \rangle + t \langle 20, 0, 4 \rangle$. The parametric equations are $x = -3 + 20t$, $y = 4$, $z = 2 + 4t$.

13.5.21 The direction is $\langle 1, 1, 2 \rangle \times \langle 1, 0, 0 \rangle = \langle 0, 2, -1 \rangle$, so the line is $\mathbf{r}(t) = \langle -2, 5, 3 \rangle + t \langle 0, 2, -1 \rangle$. The parametric equations are $x = -2$, $y = 5 + 2t$, $z = 3 - t$.

13.5.22 The direction is $\langle 4, 3, -5 \rangle \times \langle 0, 0, 1 \rangle = \langle 3, -4, 0 \rangle$, so the line is $\mathbf{r}(t) = \langle 0, 2, 1 \rangle + t \langle 3, -4, 0 \rangle$. The parametric equations are $x = 3t$, $y = 2 - 4t$, $z = 1$.

13.5.23 The direction is $\langle -2, 8, -4 \rangle \times \langle -2, 1, -1 \rangle = \langle -4, 6, 14 \rangle$, so the line is $\mathbf{r}(t) = \langle 1, 2, 3 \rangle + t \langle -4, 6, 14 \rangle$. The parametric equations are $x = 1 - 4t$, $y = 2 + 6t$, $z = 3 + 14t$.

13.5.24 The direction is $\langle 2, 3, -4 \rangle \times \langle 1, 1, -1 \rangle = \langle 1, -2, -1 \rangle$, so the line is $\langle 1, 0, -1 \rangle + t \langle 1, -2, -1 \rangle$. The parametric equations are $x = 1 + t$, $y = -2t$, $z = -1 - t$.

13.5.25 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 1$ and $s = 5$, so the point of intersection of the lines occurs at $\mathbf{r}(1) = \mathbf{R}(5) = \langle 4, 3, 3 \rangle$. The direction of the line perpendicular to both of these is $\langle 4, 2, 3 \rangle \times \langle 1, 2, 3 \rangle = \langle 0, -9, 6 \rangle$. The line we are seeking is therefore $\langle 4, 3, 3 \rangle + t \langle 0, -9, 6 \rangle$. The parametric equations are $x = 4$, $y = 3 - 9t$, $z = 3 + 6t$.

13.5.26 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 0$ and $s = 4$, so the point of intersection of the lines occurs at $\mathbf{r}(0) = \mathbf{R}(4) = \langle -2, 0, 0 \rangle$. The direction of the line perpendicular to both of these is $\langle 3, 2, 3 \rangle \times \langle 1, 2, 3 \rangle = \langle 0, -6, 4 \rangle$. The line we are seeking is therefore $\langle -2, 0, 0 \rangle + t \langle 0, -6, 4 \rangle$. The parametric equations are $x = -2$, $y = -6t$, $z = 4t$.

13.5.27 The line segment is $\mathbf{r}(t) = t \langle 1, 2, 3 \rangle$, where $0 \leq t \leq 1$. The parametric equations are $x = t$, $y = 2t$, $z = 3t$, $0 \leq t \leq 1$.

13.5.28 The line segment is $\mathbf{r}(t) = \langle 1, 0, 1 \rangle + t \langle -1, -2, 0 \rangle$, where $0 \leq t \leq 1$. The parametric equations are $x = 1 - t$, $y = -2t$, $z = 1$, $0 \leq t \leq 1$.

13.5.29 The line segment is $\mathbf{r}(t) = \langle 2, 4, 8 \rangle + t \langle 5, 1, -5 \rangle$, where $0 \leq t \leq 1$. The parametric equations are $x = 2 + 5t$, $y = 4 + t$, $z = 8 - 5t$, $0 \leq t \leq 1$.

13.5.30 The line segment is $\mathbf{r}(t) = \langle -1, -8, 4 \rangle + t \langle -8, 13, -7 \rangle$, where $0 \leq t \leq 1$. The parametric equations are $x = -1 - 8t$, $y = -8 + 13t$, $z = 4 - 7t$, $0 \leq t \leq 1$.

13.5.31 The direction vectors are not parallel, so the lines either intersect in a single point or are skew. Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 0$ and $s = -3$, so the point of intersection of the lines occurs at $\mathbf{r}(0) = \mathbf{R}(-3) = \langle 1, 3, 2 \rangle$.

13.5.32 The direction vectors for the two lines are $\langle 2, 1, 3 \rangle$ and $\langle 5, 1, 5 \rangle$, which are not parallel, so the lines either intersect in a point or are skew. Setting $2t = 5s - 2$, $t + 2 = s + 4$, and $3t - 1 = 5s + 1$ yields $t = 4$ and $s = 2$. Thus the point of intersection is $(8, 6, 11)$.

13.5.33 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and attempting to solve the corresponding system of linear equations yields no solution. The lines aren't parallel since $\langle 0, -1, 1 \rangle$ is not a multiple of $\langle -7, 4, -1 \rangle$. Therefore the lines are skew.

13.5.34 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and attempting to solve the corresponding system of linear equations yields no solution. The lines aren't parallel since $\langle 5, -2, 3 \rangle$ is not a multiple of $\langle 10, 4, 6 \rangle$. Therefore the lines are skew.

13.5.35 These equations represent the same line. (So they are parallel and intersecting.) Note that $\mathbf{r}(3t - 5) = \langle 1 + 2(3t - 5), 7 - 3(3t - 5), 6 + (3t - 5) \rangle = \langle -9 + 6t, 22 - 9t, 1 + 3t \rangle = \mathbf{R}(t)$.

13.5.36 Because the direction vectors are multiples of each other ($\langle 4, -6, 4 \rangle = -2 \langle -2, 3, -2 \rangle$), the lines are parallel. (Note also that the lines don't coincide.)

13.5.37 Because the direction vectors are multiples of each other ($\langle 1, -2, 3 \rangle = -\frac{1}{7} \langle -7, 14, -21 \rangle$), the lines are parallel. (Note also that the lines don't coincide.)

13.5.38

- If they are to intersect, then we must have $2 + 2t = 6 + s$, $8 + t = 10 - 2s$, and $10 + 3t = 16 - s$. This system of linear equations has the solution $t = 2$ and $s = 0$. So the lines intersect at $(6, 10, 16)$.

b. They do not collide, because they arrive at the intersection point at different times.

13.5.39 Let the point on the line be $P(7, 2, 4)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle -12, 0, 5 \rangle$. Then

$$d = \frac{|\langle -12, 0, 5 \rangle \times \langle 5, -1, 12 \rangle|}{|\langle 5, -1, 12 \rangle|} = \frac{|\langle 5, 169, 12 \rangle|}{|\langle 5, -1, 12 \rangle|} = \frac{13\sqrt{170}}{\sqrt{170}} = 13.$$

13.5.40 Let the point on the line be $P(1, 3, 1)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle 4, 3, 0 \rangle$. Then

$$d = \frac{|\langle 4, 3, 0 \rangle \times \langle 3, -4, 1 \rangle|}{|\langle 3, -4, 1 \rangle|} = \frac{|\langle 3, -4, -25 \rangle|}{|\langle 3, -4, 1 \rangle|} = \frac{5\sqrt{26}}{\sqrt{26}} = 5.$$

13.5.41

a. Yes. The distance between $A(5, 45)$ and the line through $(25, 16)$ parallel to $\mathbf{v} = \langle \cos 122^\circ, \sin 122^\circ \rangle$ is $\approx 1.3 < 2.25$.

b. No. The distance between $B(76, 40)$ and the line through $P(25, 16)$ and $(50, 25)$ (i.e. $x = 25 + 25t$, $y = 16 + 9t$) is $\approx 5.3 > 2.25$.

c. $13.16^\circ < \theta < 18.12^\circ$. A vector parallel to the line of the shot is $\langle \cos \theta, \sin \theta \rangle$ and $\overrightarrow{PC} = \langle 50, 14 \rangle$. We need $d = |14 \cos \theta - 50 \sin \theta| < 2.25$. Using a root finder to solve $14 \cos \theta - 50 \sin \theta - 2.25 = 0$ and $14 \cos \theta - 50 \sin \theta + 2.25 = 0$, results in $\theta \approx 0.2297$ and $\theta \approx 0.3163$, respectively. Converting to degrees results in the interval given.

13.5.42 No, the balls will not collide. The cue ball bounces when its center reaches $(40, 1)$ and it follows the line $x = 40 + \frac{\sqrt{2}}{2}t$, $y = 1 + \frac{\sqrt{2}}{2}t$, $t \geq 0$ after the bounce. The distance from $D(59, 17)$ to this line is $d \approx 2.12$; d needs to be less than 2 for a collision to occur.

13.5.43 Substituting in the general equation for a plane gives $(x - 0) + (y - 2) - (z - (-2)) = 0$, which simplifies to $x + y - z = 4$.

13.5.44 Substituting in the general equation for a plane gives $-1(x - 2) + 2(y - 3) - 3(z - 0) = 0$, which simplifies to $-x + 2y - 3z = 4$.

13.5.45 A vector normal to the plane is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle -2, -1, 2 \rangle$. Substituting in the general equation for a plane gives $-2(x - 1) - 1(y - 2) + 2(z - 3) = 0$, which simplifies to $2x + y - 2z = -2$.

13.5.46 A vector normal to the plane is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 4 & 2 & 0 \end{vmatrix} = \langle -2, 4, 14 \rangle$. Substituting in the general equation for a plane gives $-2(x - 3) + 4(y - 0) + 2(z + 2) = 0$, which simplifies to $x - 2y - 7z = 17$.

13.5.47 A normal to the plane is the direction vector of the line, which is $\langle 1, 4, 7 \rangle$. Substituting in the general equation for a plane gives $1(x - 0) + 4(y - 0) + 7(z - 0) = 0$, or $x + 4y + 7z = 0$.

13.5.48 A normal to the plane is the direction vector of the line, which is $\langle 2, 3, 4 \rangle$. Substituting in the general equation for a plane gives $2(x - 2) + 3(y - (-3)) + 4(z - 5) = 0$, or $2x + 3y + 4z = 15$.

13.5.49 Let $P = (1, 0, 3)$, $Q = (0, 4, 2)$ and $R = (1, 1, 1)$. Then the vectors $\overrightarrow{PQ} = \langle -1, 4, -1 \rangle$ and $\overrightarrow{PR} = \langle 0, 1, -2 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & -1 \\ 0 & 1 & -2 \end{vmatrix} = -7\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ is normal to the plane. The plane has equation $7(x - 1) + 2(y - 0) + 1(z - 3) = 0$, which simplifies to $7x + 2y + z = 10$.

13.5.50 Let $P = (2, -1, 4)$, $Q = (1, 1, -1)$ and $R = (-4, 1, 1)$. Then the vectors $\overrightarrow{PQ} = \langle -1, 2, -5 \rangle$ and $\overrightarrow{PR} = \langle -6, 2, -3 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -5 \\ -6 & 2 & -3 \end{vmatrix} = 4\mathbf{i} + 27\mathbf{j} + 10\mathbf{k}$ is normal to the plane.

The plane has equation $4(x - 2) + 27(y - (-1)) + 10(z - 4) = 0$, which simplifies to $4x + 27y + 10z = 21$.

13.5.51 The plane Q has normal vector $\langle -1, 2, -4 \rangle$; therefore the parallel plane passing through the point $P_0(1, 0, 4)$ has equation $-1(x - 1) + 2(y - 0) - 4(z - 4) = 0$, which simplifies to $-x + 2y - 4z = -17$.

13.5.52 The plane Q has normal vector $\langle 2, 1, -1 \rangle$; therefore the parallel plane passing through the point $P_0(0, 2, -2)$ has equation $2(x - 0) + 1(y - 2) - 1(z - (-2)) = 0$, which simplifies to $2x + y - z = 4$.

13.5.53 Observe that the vectors $\langle 1, 0, 0 \rangle$ and $\langle 1, 2, 3 \rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = -3\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has equation $0(x - 0) - 3(y - 0) + 2(z - 0) = 0$, which simplifies to $3y - 2z = 0$.

13.5.54 Observe that the vectors $\langle 0, 0, 1 \rangle$ and $\langle 3, -1, 2 \rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j}$ is normal to the plane. The plane has equation $1(x - 0) + 3(y - 0) + 0(z - 0) = 0$, which simplifies to $x + 3y = 0$.

13.5.55 Observe that the vectors $\langle 1, 2, 3 \rangle$ and $\langle -1, 0, 4 \rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix} = 8\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has equation $8(x - 0) - 7(y - 0) + 2(z - 0) = 0$, which simplifies to $8x - 7y + 2z = 0$.

13.5.56 Observe that the points $P = (0, 0, 0)$ and $Q = (1, -1, 2)$ lie on the line ℓ . Therefore the vectors $\overrightarrow{PP_0} = \langle 1, -2, 3 \rangle$ and $\overrightarrow{PQ} = \langle 1, -1, 2 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PP_0} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is normal to the plane. The plane has equation $-1(x - 0) + 1(y - 0) + 1(z - 0) = 0$, which simplifies to $x - y - z = 0$.

13.5.57 Observe that the points $P = (-2, 0, 1)$ and $Q = (0, -2, -3)$ lie on the line ℓ . Therefore the vectors $\overrightarrow{PP_0} = \langle -2, 1, 1 \rangle$ and $\overrightarrow{PQ} = \langle 2, -2, -4 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PP_0} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -2 & -4 \end{vmatrix} = \langle -2, -6, 2 \rangle = -2\langle 1, 3, -1 \rangle$ is normal to the plane. The plane has equation $1(x - (-2)) + 3(y - 0) - 1(z - 1) = 0$, which simplifies to $x + 3y - z = -3$.

13.5.58 Observe that two points which lie in the intersection of the planes are $(4, 0, -2)$ and $(2, -2, 0)$. So the vectors $\langle 4, 0, -2 \rangle$ and $\langle 2, -2, 0 \rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -2 \\ 2 & -2 & 0 \end{vmatrix} = \langle -4, -4, -8 \rangle$ is normal to the plane. Scaling gives the normal vector $\langle 1, 1, 2 \rangle$. The plane has equation $x + y + 2z = 0$.

13.5.59 Yes. A direction vector for the line is $\langle 1, 2, 4 \rangle$ which is perpendicular to the normal to the plane $\mathbf{n} = \langle 2, -1, 0 \rangle$, so the line is parallel to the plane. A point on the line is $(1, 3, 5)$, so the plane $2x - y = -1$ contains the line and is parallel to the given plane.

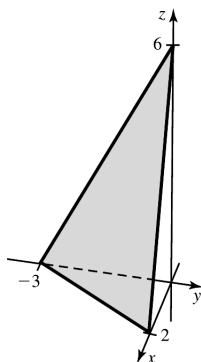
13.5.60 Yes. Solving the system of linear equations that arises from setting the coordinates equal to each other gives $t = 0$ and $s = 1$, which means that the point of intersection is $(0, 1, 4)$. To find $\mathbf{n}$ for the plane,

we calculate $v \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} = \langle 0, 3, -2 \rangle$. The equation of the plane is therefore

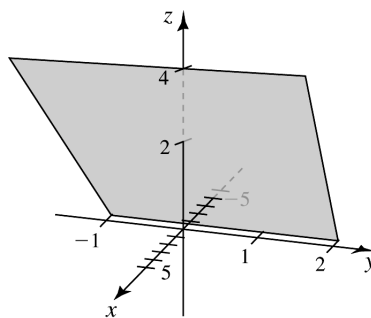
$$0(x - 0) + 3(y - 1) - 2(z - 4) = 0$$

or $3y - 2z = -5$.

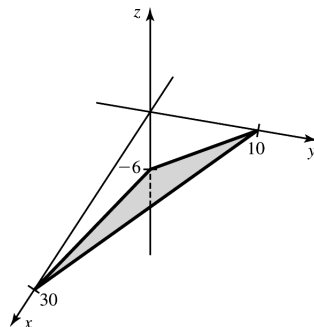
13.5.61 The x -intercept is found by setting $y = z = 0$ and solving $3x = 6$ to get $x = 2$. Similarly, we see that the y -intercept is -3 and the z -intercept is 6 . The xy -trace is found by setting $z = 0$, which gives $3x - 2y = 6$. Similarly, the xz -trace is $3x + z = 6$ and the yz -trace is $-2y + z = 6$.



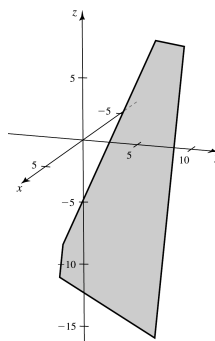
13.5.62 The x -intercept is found by setting $y = z = 0$ and solving $-4x = 16$ to get $x = -4$. Similarly, we see that the z -intercept is 2 . Setting $x = z = 0$ gives $0 = 16$, so this plane does not intersect the y -axis. The xy -trace is found by setting $z = 0$, which gives $-4x = 16$ or $x = -4$. Similarly, the xz -trace is $-4x + 8z = 16$ or $x - 2z = -4$ and the yz -trace is $z = 2$.



13.5.63 The x -intercept is found by setting $y = z = 0$ which gives $x = 30$. Similarly, we see that the y -intercept is 10 and the z -intercept is -6 . The xy -trace is found by setting $z = 0$, which gives $x + 3y = 30$. Similarly, the xz -trace is $x - 5z = 30$ and the yz -trace is $3y - 5z = 30$.



13.5.64 The x -intercept is found by setting $y = z = 0$ and solving $12x + 72 = 0$ to get $x = -6$. Similarly, we see that the y -intercept is 8 and the z -intercept is -18 . The xy -trace is found by setting $z = 0$, which gives $12x - 9y = -72$ or $4x - 3y = -24$. Similarly, the xz -trace is $12x + 4z = -72$ or $3x + z = -18$ and the yz -trace is $9y - 4z = 72$.



13.5.65 The normal vectors to the planes are $\langle 1, 1, 4 \rangle$ and $\langle -1, -3, 1 \rangle$, and the dot product of these vectors is $-1 - 3 + 4 = 0$, so the planes are orthogonal.

13.5.66 The normal vectors to the planes are $\langle 2, 2, -3 \rangle$ and $\langle -10, -10, 15 \rangle$, and these are parallel because $\langle -10, -10, 15 \rangle = -5 \langle 2, 2, -3 \rangle$. Thus, the planes are parallel.

13.5.67 The normal vectors to the planes are $\langle 3, 2, -3 \rangle$ and $\langle -6, -10, 1 \rangle$. These are neither parallel nor perpendicular, because one is not a multiple of the other and because their dot product is not 0. Thus, the planes are neither parallel nor perpendicular.

13.5.68 The normal vectors to the planes are $\langle 3, 2, 2 \rangle$ and $\langle -6, -10, 19 \rangle$, and the dot product of these vectors is $-18 - 20 + 38 = 0$, so the planes are orthogonal.

13.5.69 Rewrite R and T so we have $Q : 3x - 2y + z = 12$, $R : 3x - 2y + z = 0$, $T : 3x - 2y + z = 12$. This shows that Q and T are identical, and Q , R and T are parallel. Note that $\langle 3, -2, 1 \rangle \cdot \langle -1, 2, 7 \rangle = 0$ so S is orthogonal to Q , R and T .

13.5.70 The planes Q , R , S and T have normal vectors $\langle 1, 1, -1 \rangle$, $\langle 0, 1, 1 \rangle$, $\langle 1, -1, 0 \rangle$ and $\langle 1, 1, 1 \rangle$ respectively. None of these vectors are scalar multiples of each other, so no two of these planes are parallel (or identical). Observe that $\langle 1, 1, -1 \rangle \cdot \langle 0, 1, 1 \rangle = \langle 1, 1, -1 \rangle \cdot \langle 1, -1, 0 \rangle = 0$; therefore Q is orthogonal to R and S . We also have $\langle 1, -1, 0 \rangle \cdot \langle 1, 1, 1 \rangle = 0$, so S and T are orthogonal. All other pairs have non-zero dot products, so no other pairs are orthogonal.

13.5.71 The direction of the line is $\langle 2, -4, 1 \rangle$, so the line is given by

$$\langle 2, 1, 3 \rangle + t \langle 2, -4, 1 \rangle = \langle 2 + 2t, 1 - 4t, 3 + t \rangle.$$

13.5.72 The direction of the line is $\langle 1, 0, 4 \rangle$, so the line is given by $\langle 0, -10, -3 \rangle + t \langle 1, 0, 4 \rangle = \langle t, -10, -3 + 4t \rangle$.

13.5.73 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle -1, 2, 1 \rangle$ and $\mathbf{n}_R = \langle 1, 1, 1 \rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ . We need to find a point on ℓ and a vector in the direction of ℓ . Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz -plane: $2y + z = 1$, $y + z = 0$. Solving these equations simultaneously gives $y = 1$ and $z = -1$, so $(0, 1, -1)$ is a point on ℓ . A vector in the direction of ℓ

is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} = \langle 1, 2, -3 \rangle$. Therefore ℓ has equation $\mathbf{r}(t) = \langle 0, 1, -1 \rangle + t \langle 1, 2, -3 \rangle = \langle t, 1 + 2t, -1 - 3t \rangle$, or $x = t$, $y = 1 + 2t$, $z = -1 - 3t$.

13.5.74 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 1, 2, -1 \rangle$ and $\mathbf{n}_R = \langle 1, 1, 1 \rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ . We need to find a point on ℓ and a vector in the direction of ℓ . Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy -plane: $x + 2y = 1$, $x + y = 1$. Solving these equations simultaneously gives $x = 1$ and $y = 0$, so $(1, 0, 0)$ is a point on ℓ . A vector in the direction of ℓ is

$$\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k} = \langle 3, -2, -1 \rangle. \text{ Therefore } \ell \text{ has equation } \mathbf{r}(t) = \langle 1, 0, 0 \rangle + t\langle 3, -2, -1 \rangle = \langle 1 + 3t, -2t, -t \rangle, \text{ or } x = 1 + 3t, y = -2t, z = -t.$$

13.5.75 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 2, -1, 3 \rangle$ and $\mathbf{n}_R = \langle -1, 3, 1 \rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ . We need to find a point on ℓ and a vector in the direction of ℓ . Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy -plane: $2x - y = 1$, $-x + 3y = 4$. Solving these equations simultaneously gives $x = \frac{7}{5}$ and $y = \frac{9}{5}$, so $(\frac{7}{5}, \frac{9}{5}, 0)$ is a point on ℓ . A vector in the

$$\text{direction of } \ell \text{ is } \mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = -10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k} = -5\langle 2, 1, -1 \rangle. \text{ Therefore } \ell \text{ has equation } \mathbf{r}(t) = \langle \frac{7}{5}, \frac{9}{5}, 0 \rangle + t\langle 2, 1, -1 \rangle = \langle \frac{7}{5} + 2t, \frac{9}{5} + t, -t \rangle, \text{ or } x = \frac{7}{5} + 2t, y = \frac{9}{5} + t, z = -t.$$

13.5.76 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 1, -1, -2 \rangle$ and $\mathbf{n}_R = \langle 1, 1, 1 \rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ . We need to find a point on ℓ and a vector in the direction of ℓ . Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy -plane: $x - y = 1$, $x + y = -1$. Solving these equations simultaneously gives $x = 0$ and $y = -1$, so $(0, -1, 0)$ is a point on ℓ . A vector in the direction of ℓ is

$$\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} = \langle 1, -3, 2 \rangle. \text{ Therefore } \ell \text{ has equation } \mathbf{r}(t) = \langle 0, -1, 0 \rangle + t\langle 1, -3, 2 \rangle = \langle t, -1 - 3t, 2t \rangle, \text{ or } x = t, y = -1 - 3t, z = 2t.$$

13.5.77 The line and plane intersect for $x = t = 3$, so the point of intersection is $(3, 3, 3)$.

13.5.78 The intersection occurs for $y = -2 = -t + 4$, so for $t = 6$. The point of intersection is $(13, -2, 0)$.

13.5.79 The intersection occurs for $3(-2t + 5) + 2(3t - 5) - 4(4t - 6) = -3$, which occurs for $t = 2$. So the point is $(1, 1, 2)$.

13.5.80 The plane and line don't intersect. Notice that the direction of the line is $\langle 3, 1, -1 \rangle$ which is perpendicular to the normal to the plane $\langle 2, -3, 3 \rangle$. So the line is parallel to the plane and doesn't intersect it.

13.5.81

- True. This curve passes through the origin at $t = -1/2$.
- False. For example, the x -axis is not parallel to the line $\langle 0, 0, 1 \rangle + t\langle 0, 1, 0 \rangle$, but neither do they intersect.
- False. The direction vector for the line is parallel to the normal to the plane, so the line is perpendicular to the plane, not parallel.
- True. Because the direction vectors are multiples of each other, these represent the same line.
- False. For example, the point $(0, 0, -1)$ is on the first plane but isn't on the second.

f. False. Two distinct lines determine a plane only if the lines are parallel or if they intersect.

g. True. Note that $\langle 5, -1, 0 \rangle \cdot \langle -1, -5, 7 \rangle = 0$ and $\langle 7, 0, 1 \rangle \cdot \langle -1, -5, 7 \rangle = 0$.

13.5.82 The length of the orthogonal projection of $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$ is the magnitude of the scalar component of $\overrightarrow{PQ}$ in the direction of $\mathbf{n}$ which is $\frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|}$.

13.5.83 Let $P(2, 0, 0)$ be a point on the plane. Then $\overrightarrow{PQ} = \langle 4, -2, 4 \rangle$, and the distance from the point $Q(6, -2, 4)$ to the plane is

$$\frac{|\langle 4, -2, 4 \rangle \cdot \langle 2, -1, 2 \rangle|}{\sqrt{4 + 1 + 4}} = \frac{18}{3} = 6.$$

13.5.84 Let $P(0, 0, -1)$ be a point on the plane. Then $\overrightarrow{PQ} = \langle 1, 2, -3 \rangle$, and the distance from the point $Q(1, 2, -4)$ to the plane is

$$\frac{|\langle 1, 2, -3 \rangle \cdot \langle 2, 0, -5 \rangle|}{\sqrt{4 + 0 + 25}} = \frac{17}{\sqrt{29}}.$$

13.5.85 We have $x = 1 + 4t$, $y = 2 + 7t$, and $z = 2t$. Solving for t and setting equal gives

$$\frac{x-1}{4} = \frac{y-2}{7} = \frac{z}{2}.$$

13.5.86 The line is given by $\langle 1, -2, 3 \rangle + t \langle 1, 5, -4 \rangle$, so $x = 1 + t$, $y = -2 + 5t$, and $z = 3 - 4t$. Solving for t and setting equal gives

$$\frac{x-1}{1} = \frac{y+2}{5} = \frac{z-3}{-4}.$$

13.5.87 The angle θ between the vectors $\mathbf{n}_1 = \langle 5, 2, -1 \rangle$ and $\mathbf{n}_2 = \langle -3, 1, 2 \rangle$ satisfies $\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\frac{15}{\sqrt{30}\sqrt{14}} = -\frac{\sqrt{105}}{14}$, so $\theta = \cos^{-1}\left(-\frac{\sqrt{105}}{14}\right) \approx 2.392 \text{ rad} \approx 137^\circ$.

13.5.88 The x -intercept is found by setting $y = z = 0$ and solving for x , which gives the point $(\frac{d}{a}, 0, 0)$, assuming $a \neq 0$. If $a = 0$, then all points on the x -axis lie on this plane when $d = 0$, and no points on the x -axis lie on the plane when $d \neq 0$. Similarly, the intersection of the plane with the y -axis is the point $(0, \frac{d}{b}, 0)$ when $b \neq 0$, the entire y -axis when $b = d = 0$ and empty when $b = 0$, $d \neq 0$; and the intersection of the plane with the z -axis is the point $(0, 0, \frac{d}{c})$ when $c \neq 0$, the entire z -axis when $c = d = 0$ and empty when $c = 0$, $d \neq 0$.

13.5.89 These planes have normal $\langle 2, 3, 0 \rangle \times \langle -1, -1, 2 \rangle = \langle 6, -4, 1 \rangle$, so the planes all have an equation of the form $6x - 4y + z = d$ for some real number d .

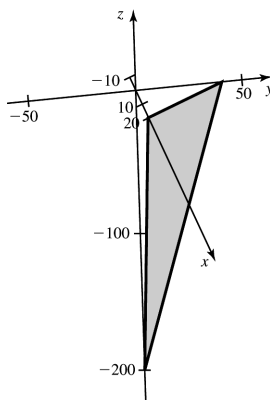
13.5.90 The normal to the plane we are seeking is $\langle 2, 5, -3 \rangle \times \langle -1, 5, 2 \rangle = \langle 25, -1, 15 \rangle$. The equation of the plane has the form $25x - y + 15z = d$ for some d . Because the point $(0, -2, 4)$ is on the plane, we have $0 - (-2) + 60 = d$, so the plane is given by $25x - y + 15z = 62$.

13.5.91 First we find the line of intersection of the first two planes. The direction of the line of intersection is $\langle 1, 0, 3 \rangle \times \langle 0, 1, 4 \rangle = \langle -3, -4, 1 \rangle$, and by inspection, a point on both planes is $(0, 2, 1)$. Thus the line of intersection is given by $\langle -3t, 2 - 4t, 1 + t \rangle$. This intersects the plane $x + y + 6z = 9$ when $-3t + (2 - 4t) + 6(1 + t) = 9$, or $8 - t = 9$, so $t = -1$. Thus, the intersection is the single point $(3, 6, 0)$.

13.5.92 First we find the line of intersection of the first two planes. The direction of the line of intersection is $\langle 1, 2, 2 \rangle \times \langle 0, 1, 4 \rangle = \langle 6, -4, 1 \rangle$, and by inspection, a point on both planes is $(3, -2, 2)$. Thus the line of intersection is given by $\langle 3 + 6t, -2 - 4t, 2 + t \rangle$. This intersects the plane $x + 2y + 8z = 9$ when $3 + 6t + 2(-2 - 4t) + 8(2 + t) = 9$, or $6t + 15 = 9$, so $t = -1$. Thus, the intersection is the single point $(-3, 2, 1)$.

13.5.93

a.

b. The profit is $z = 10 \cdot 20 + 5 \cdot 10 - 200 = \50 which is positive.c. The profit is 0 when x and y lie on the line $2x + y = 40$.**13.6 Cylinders and Quadric Surfaces**

13.6.1 Because z is absent from the equation $x^2 + 2y^2 = 8$, this cylinder is parallel to the z -axis. Similarly, $z^2 + 2y^2 = 8$ is parallel to the x -axis and $x^2 + 2z^2 = 8$ is parallel to the y -axis.

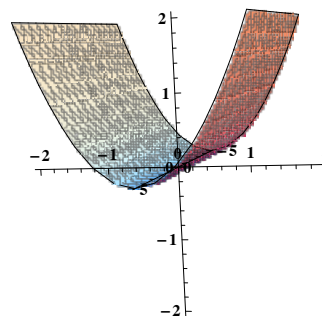
13.6.2 This is a cylinder consisting of all lines parallel to the y -axis that pass through the parabola $x = z^2$ in the xz -plane.

13.6.3 The traces of a surface are the sets of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

13.6.4 This is an elliptic paraboloid.

13.6.5 This is an ellipsoid.

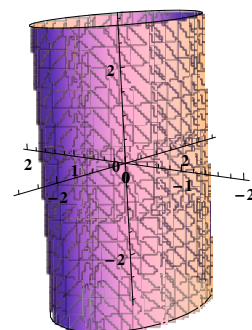
13.6.6 This is a hyperboloid of two sheets.

13.6.7a. The cylinder is parallel to the x -axis.

b.

13.6.8

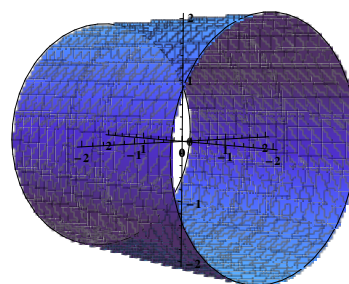
- a. The cylinder is parallel to the z -axis.



b.

13.6.9

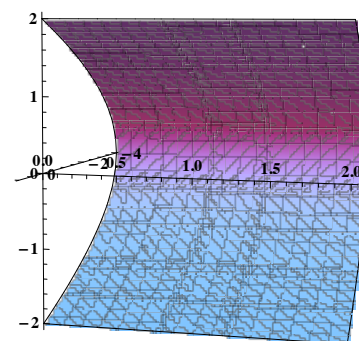
- a. The cylinder is parallel to the y -axis.



b.

13.6.10

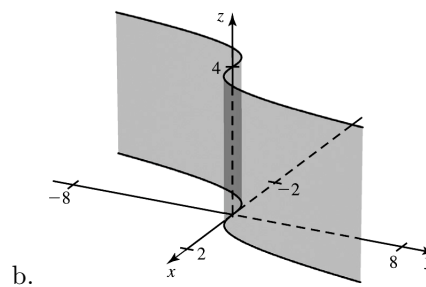
- a. The cylinder is parallel to the y -axis.



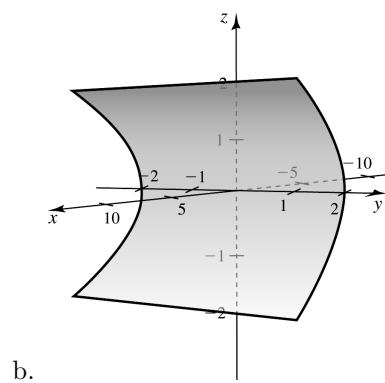
b.

13.6.11

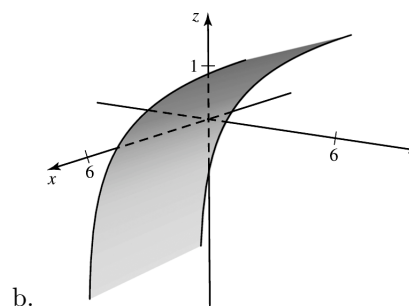
- a. The cylinder is parallel to the z -axis.

**13.6.12**

- a. The cylinder is parallel to the y -axis.

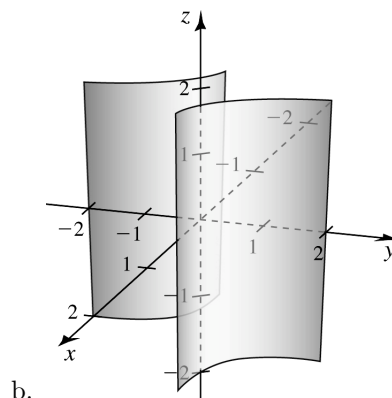
**13.6.13**

- a. The cylinder is parallel to the x -axis.



13.6.14

a. The cylinder is parallel to the z -axis.



13.6.15 This is an ellipsoid. The xy -trace is $x^2 + y^2 = 1$, which is a circle. The xz -trace is $x^2 + \frac{z^2}{25} = 1$ which is an ellipse. The yz -trace is $y^2 + \frac{z^2}{25} = 1$ which is also an ellipse.

13.6.16 This is a hyperboloid of one sheet. The xy -trace is $x^2 + y^2 = 1$, which is a circle. The xz -trace is $x^2 - \frac{z^2}{25} = 1$ which is a hyperbola. The yz -trace is $y^2 - \frac{z^2}{25} = 1$ which is also a hyperbola.

13.6.17 This is a paraboloid. The xy -trace is the single point $(0, 0, 0)$. The xz -trace is $z = 25x^2$ which is a parabola. The yz -trace is $z = -25y^2$ which is also a parabola.

13.6.18 This is a hyperbolic paraboloid. The xy -trace is $y = \pm x$ which are lines. The xz -trace is $z = 25x^2$ which is a parabola. The yz -trace is $z = -25y^2$ which is a parabola.

13.6.19 This is a hyperboloid of two sheets. The xz -trace is $z^2 - 25x^2 = 25$ which is a hyperbola. The yz -trace is $z^2 - 25y^2 = 25$ which is also a hyperbola.

13.6.20 This is an elliptic cone. The xy -trace is the single point $(0, 0, 0)$. The xz -trace is $z = \pm 5x$ which are lines. The yz -trace is $z = \pm 5y$ which are also lines.

13.6.21 This a hyperbolic paraboloid.

13.6.22 This is a hyperboloid of two sheets.

13.6.23 This is an elliptic paraboloid.

13.6.24 This is a hyperboloid of one sheet.

13.6.25 This is a hyperbolic cylinder.

13.6.26 This is an elliptic cylinder.

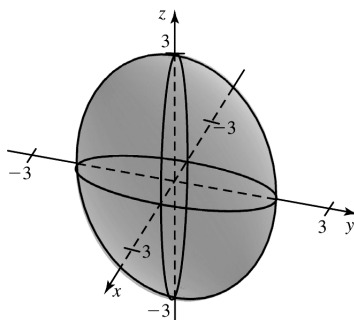
13.6.27 This is an elliptic paraboloid.

13.6.28 This is an elliptic cone.

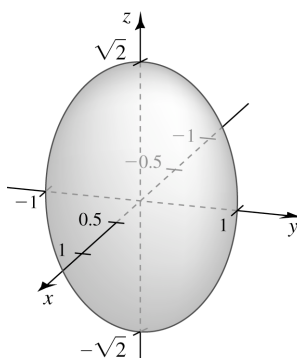
13.6.29

- a. The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 1$, so the x -intercepts are $x = \pm 1$. Similarly we see that the y -intercepts are $y = \pm 2$ and the z -intercepts are $z = \pm 3$.

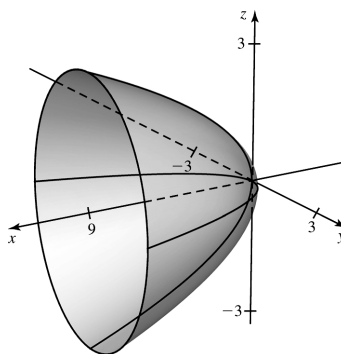
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 + \frac{y^2}{4} = 1$, $x^2 + \frac{z^2}{9} = 1$, $\frac{y^2}{4} + \frac{z^2}{9} = 1$.
- c. This is an ellipsoid.

**13.6.30**

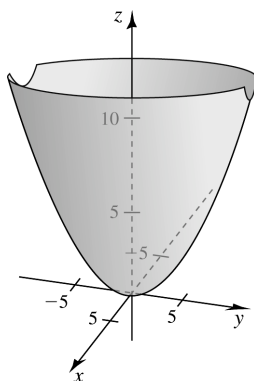
- a. The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $4x^2 = 1$, so the x -intercepts are $x = \pm\frac{1}{2}$. Similarly we see that the y -intercepts are $y = \pm 1$ and the z -intercepts are $z = \pm\sqrt{2}$.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $4x^2 + y^2 = 1$, $4x^2 + \frac{z^2}{2} = 1$, $y^2 + \frac{z^2}{2} = 1$.
- c. This is an ellipsoid.

**13.6.31**

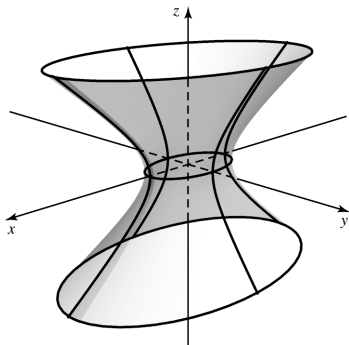
- a. The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x = y^2$, $x = z^2$, and $y^2 + z^2 = 0$ (which implies that $x = y = z = 0$).
- c. This is an elliptic paraboloid.

**13.6.32**

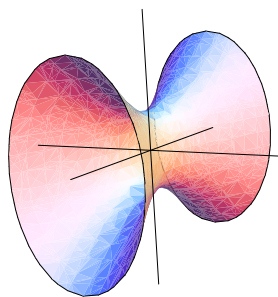
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $0 = \frac{x^2}{4} + \frac{y^2}{9}$ (which implies that $x = y = z = 0$), $z = \frac{x^2}{4}$, $z = \frac{y^2}{9}$.
- This is an elliptic paraboloid.

**13.6.33**

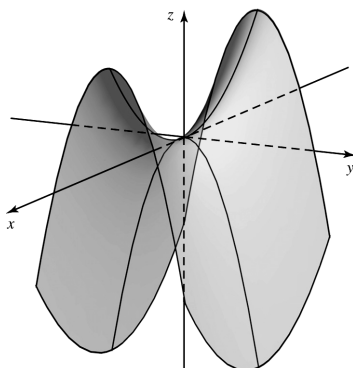
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 25$, so the x -intercepts are $x = \pm 5$. Similarly we see that the y -intercepts are $y = \pm 3$ and there are no z -intercepts.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{25} + \frac{y^2}{9} = 1$, $\frac{x^2}{25} - z^2 = 1$, $\frac{y^2}{9} - z^2 = 1$.
- This is a hyperboloid of one sheet.

**13.6.34**

- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which results in no intercepts. Similarly we see that the y -intercepts are $y = \pm 2$ and the z -intercepts are $z = \pm 3$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{y^2}{4} - \frac{x^2}{16} = 1$, $\frac{z^2}{9} - \frac{x^2}{16} = 1$, $\frac{y^2}{4} + \frac{z^2}{9} = 1$.
- This is a hyperboloid of one sheet.

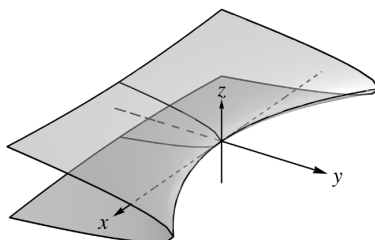
**13.6.35**

- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{9} - y^2 = 0$, $z = \frac{x^2}{9}$, $z = -y^2$.
- This is a hyperbolic paraboloid.

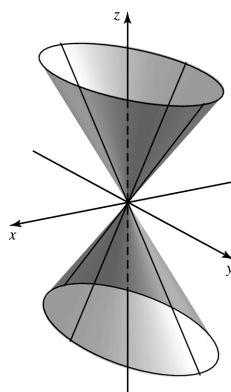


13.6.36

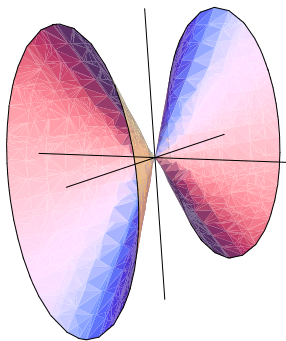
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $y = \frac{x^2}{16}$, $\frac{x^2}{16} - 4z^2 = 0$, $y = -4z^2$.
- This is a hyperbolic paraboloid.

**13.6.37**

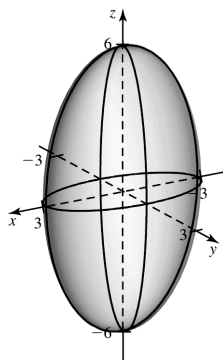
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 + \frac{y^2}{4} = 0$ (which implies that $x = y = z = 0$), $x^2 = z^2$, $\frac{y^2}{4} = z^2$.
- This is an elliptic cone.

**13.6.38**

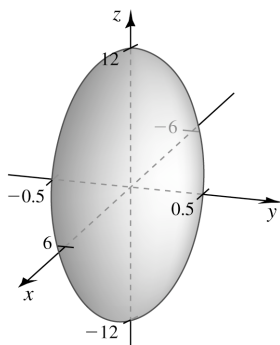
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $4y^2 = x^2$, $x^2 = z^2$, $4y^2 + z^2 = 0$ (which implies that $x = y = z = 0$).
- This is an elliptic cone.

**13.6.39**

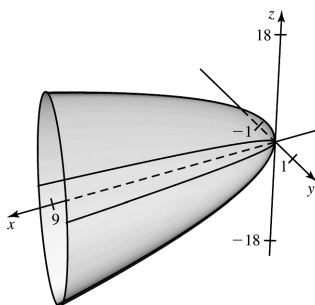
- a. The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 9$, so the x -intercepts are $x = \pm 3$. Similarly we see that the y -intercepts are $y = \pm 1$ and the z -intercepts are $z = \pm 6$.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{3} + 3y^2 = 3$, $\frac{x^2}{3} + \frac{z^2}{12} = 3$, $3y^2 + \frac{z^2}{12} = 3$.
- c. This is an ellipsoid.

**13.6.40**

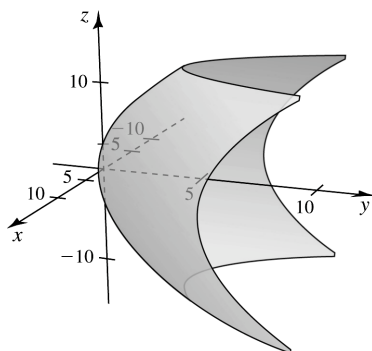
- a. The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 36$, so the x -intercepts are $x = \pm 6$. Similarly we see that the y -intercepts are $y = \pm \frac{1}{2}$ and the z -intercepts are $z = \pm 12$.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{6} + 24y^2 = 6$, $\frac{x^2}{6} + \frac{z^2}{24} = 6$, $24y^2 + \frac{z^2}{24} = 6$.
- c. This is an ellipsoid.

**13.6.41**

- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x - 9y^2 = 0$, $9x - \frac{z^2}{4} = 0$, $81y^2 + \frac{z^2}{4} = 0$ (which implies that $x = y = z = 0$).
- This is an elliptic paraboloid.

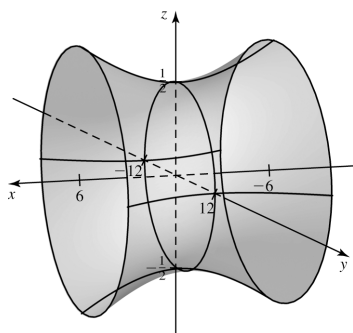
**13.6.42**

- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $y - \frac{x^2}{16} = 0$, $\frac{x^2}{8} + \frac{z^2}{18} = 0$ (which implies that $x = y = z = 0$), $y - \frac{z^2}{36} = 0$.
- This is an elliptic paraboloid.

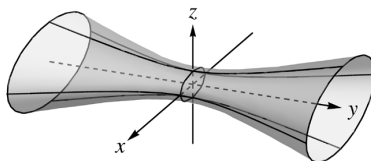


13.6.43

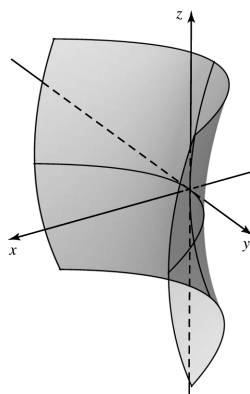
- The y -intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $y^2 = 144$, so the y -intercepts are $y = \pm 12$. Similarly we see that the z -intercepts are $z = \pm \frac{1}{2}$ and there are no x -intercepts.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\frac{x^2}{4} + \frac{y^2}{16} - 9 = 0$, $-\frac{x^2}{4} + 36z^2 - 9 = 0$, $\frac{y^2}{16} + 36z^2 - 9 = 0$.
- This is an hyperboloid of one sheet.

**13.6.44**

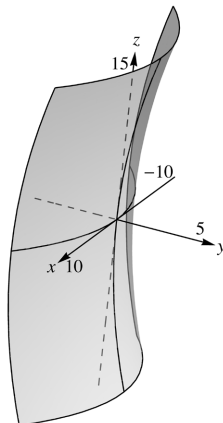
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 1$, so the x -intercepts are $x = \pm 1$. Similarly we see that the z -intercepts are $z = \pm \frac{1}{3}$ and there are no y -intercepts.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 - \frac{y^2}{3} - 1 = 0$, $9z^2 + x^2 - 1 = 0$, $9z^2 - \frac{y^2}{3} - 1 = 0$.
- This is a hyperboloid of one sheet.

**13.6.45**

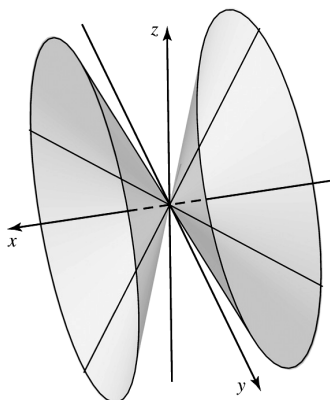
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $5x - \frac{y^2}{5} = 0$, $5x + \frac{z^2}{20} = 0$, $-\frac{y^2}{25} + \frac{z^2}{20} = 0$.
- This is a hyperbolic paraboloid.

**13.6.46**

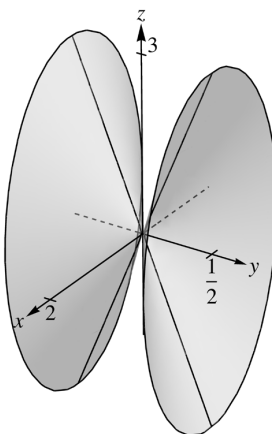
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $6y + \frac{x^2}{6} = 0$, $\frac{x^2}{6} - \frac{z^2}{24} = 0$, $6y - \frac{z^2}{24} = 0$.
- This is a hyperbolic paraboloid.

**13.6.47**

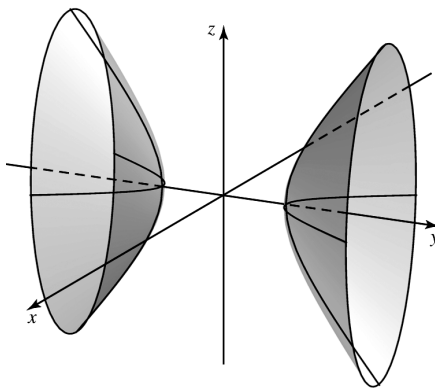
- The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{y^2}{18} = 2x^2$, $\frac{z^2}{32} = 2x^2$, $\frac{z^2}{32} + \frac{y^2}{18} = 0$ (which implies that $x = y = z = 0$).
- This is an elliptic cone.

**13.6.48**

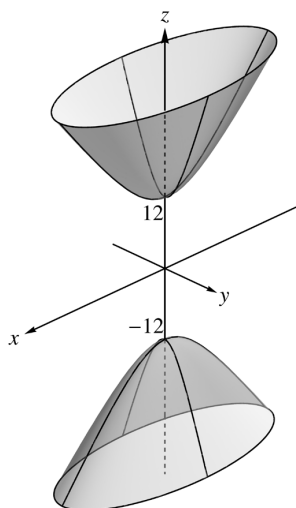
- a. The x -intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y -intercept is $y = 0$ and the z -intercept is $z = 0$.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{3} = 3y^2$, $\frac{x^2}{3} + \frac{z^2}{12} = 0$, (which implies that $x = y = z = 0$), $\frac{z^2}{12} = 3y^2$.
- c. This is an elliptic cone.

**13.6.49**

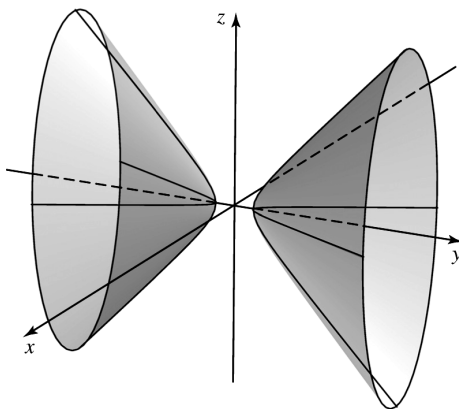
- a. The y -intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $y^2 = 4$, so the y -intercepts are $y = \pm 2$. There are no x or z -intercepts.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-x^2 + \frac{y^2}{4} = 1$, $-x^2 - \frac{z^2}{9} = 1$ (no xz -trace), $\frac{y^2}{4} - \frac{z^2}{9} = 1$.
- c. This is a hyperboloid of two sheets.

**13.6.50**

- a. The z -intercept is found by setting $x = y = 0$ in the equation of this surface, which gives $z^2 = 24 \cdot 6$, so the z -intercepts are $z = \pm 12$. There are no x or y -intercepts.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\frac{x^2}{6} - 24y^2 - 6 = 0$ (no xy -trace), $-\frac{x^2}{6} + \frac{z^2}{24} - 6 = 0$, $-24y^2 + \frac{z^2}{24} - 6 = 0$.
- c. This is a hyperboloid of two sheets.

**13.6.51**

- a. The y -intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $3y^2 = 1$, so the y -intercepts are $y = \pm \frac{\sqrt{3}}{3}$. There are no x or z -intercepts.
- b. The equations for the xy -, xz - and yz -traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\frac{x^2}{3} + 3y^2 = 1$, $-\frac{x^2}{3} - \frac{z^2}{12} = 1$ (no xz -trace), $3y^2 - \frac{z^2}{12} = 1$.
- c. This is a hyperboloid of two sheets.



13.6.52 The equation $x^2 + y^2 - z^2 + 2z = 1$ can be written as $x^2 + y^2 - (z - 1)^2 = 0$. This describes a cone shifted up one unit from the graph of $x^2 + y^2 - z^2 = 0$.

13.6.53 The equation $x^2 + 4y^2 + 9z^2 + 54z = 19$ can be written $x^2 + 4y^2 + 9(z^2 + 6z + 9) = 100$, or $x^2 + 4y^2 + 9(z + 3)^2 = 100$. The graph of the corresponding ellipsoid is obtained by shifting the graph of the ellipsoid $x^2 + 4y^2 + 9z^2 = 100$ down 3 units.

13.6.54 Completing the square and rewriting the equation of the surface as $(x + 1)^2 + y^2 + 4z^2 = 1$ shows that this surface is an ellipsoid centered at the point $(-1, 0, 0)$.

13.6.55 Completing the square and rewriting the equation of the surface as $9x^2 + (y + 1)^2 - 4z^2 = 1$ shows that this surface is a hyperboloid of one sheet with axis the line $\ell: \mathbf{r} = \langle 0, -1, t \rangle$.

13.6.56 Completing the square and rewriting the equation of the surface as $-(x - 3)^2 - (y + 4)^2 + \frac{z^2}{9} = 1$ shows that this surface is a hyperboloid of two sheets with axis the line $\ell: \mathbf{r} = \langle 3, -4, t \rangle$.

13.6.57 Completing the square and rewriting the equation of the surface as $z^2 = \frac{(x - 4)^2}{4} + (y - 5)^2 + 12$ shows that this surface is a hyperboloid of two sheets.

13.6.58 Writing this as $z = -(x^2 + y^2)$, we see that it is a paraboloid opening downward

13.6.59

- True.
- True. Letting $z = 0$ in each equation gives the same result.
- True. Letting z be a constant gives $y = 3x^2 - k$ for some constant k .
- False. For example, letting $y = 0$ gives $3x^2 = z^2$ which does not represent a parabola.
- False. Rather, up 4 units.

13.6.60

- D. This surface is a cylinder parallel to the parabola $y = z^2$ in the yz -plane.
- A. This surface is a plane.
- E. This surface is an ellipsoid.
- F. This surface is a hyperboloid of one sheet.
- B. This surface is an elliptic cone.

f. C. This surface is a cylinder parallel to the graph $y = |x|$ in the xy -plane.

13.6.61 All of the quadric surfaces in Table 13.1 except the hyperbolic paraboloid can have circular cross-sections around a coordinate axis, and so can be generated by revolving a curve in one of the coordinate planes about a coordinate axis.

13.6.62

a. The ellipsoid has equal y - and z -intercepts, so the equation is $x^2 + 4y^2 + 4z^2 = 1$.

b. The ellipsoid has equal x - and z -intercepts, so the equation is $x^2 + 4y^2 + z^2 = 1$.

13.6.63 The volume is given by $3 \int_{-1}^1 (2 - x^2 - x^2) dx = 3 \left(2x - \frac{2x^3}{3} \right) \Big|_{-1}^1 = 3 \left(2 - \frac{2}{3} - \left(-2 + \frac{2}{3} \right) \right) = 8$.

13.6.64

a. The light cone consists of all points (x, y, t) such that the distance from (x, y) to the origin is $c|t|$, where c is the speed of light; hence in these units an equation of the light cone is $x^2 + y^2 = t^2$.

b. The light cone consists of all points (x, y, t) such that the distance from (x, y) to the origin is $c|t|$, where c is the speed of light; hence in these units the equation of the light cone is $x^2 + y^2 = 9 \times 10^{16} t^2$.

13.6.65

a. Suppose that the y -axis is the major axis of an ellipse. When we rotate this ellipse about its major axis, we obtain an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The long axis therefore has a length of $2b$ and the other two axes have the same length ($c = a$). So we obtain the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$, or $\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b > a$.

b. The length of the long axis is $2b = 11.1$, which implies that $b = 5.55$. The xz -trace is a circle of radius a with a circumference of 21.1 inches. So $2\pi a = 21.1$, which implies that $a = \frac{10.55}{\pi}$. So an equation for the football is $\frac{x^2 + z^2}{(10.55/\pi)^2} + \frac{y^2}{(5.55)^2} = 1$.

13.6.66 We are designing the upper half of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-3)^2}{c^2} = 1$ that equals $\frac{4x^2}{9} + 4y^2 = 1$ when $z = 3$. This implies that $a^2 = \frac{9}{4}$ and $b^2 = \frac{1}{4}$. Because the distance from the center of the base of the half-ellipsoid to the top of the ellipsoid is $\frac{1}{2}$, $c^2 = \frac{1}{4}$. So the desired equation is $\frac{4x^2}{9} + 4y^2 + 4(z-3)^2 = 1$.

13.6.67 The desired ellipsoid is obtained by shifting the standard equation of an ellipsoid up three units to obtain $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-3)^2}{c^2} = 1$. Now we need to find the constants a , b , and c . For $z = 3$, the equation of the cone reduces to $\frac{x^2}{2} + y^2 = \frac{9}{8}$, or $\frac{x^2}{(9/4)} + \frac{y^2}{(9/8)} = 1$ and the equation of the ellipsoid reduces to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Because the cone and ellipsoid coincide at $z = 3$, we let $a^2 = 9/4$ and $b^2 = 9/8$. Updating the equation of the ellipsoid, we have $\frac{4x^2}{9} + \frac{8y^2}{9} + \frac{(z-3)^2}{c^2} = 1$. Because the height of the truncated ellipsoid is $3/2$, we require the value of z to equal $9/2$ when $x = y = 0$. Substituting these values into the updated equation, we obtain $\frac{(9/2-3)^2}{c^2} = 1$, which implies that $c^2 = 9/4$. So the equation of the upper half of the ellipsoid is $\frac{4x^2}{9} + \frac{8y^2}{9} + \frac{4(z-3)^2}{9} = 1$, or $4x^2 + 8y^2 + 4(z-3)^2 = 9$, for $3 \leq z \leq 4.5$.

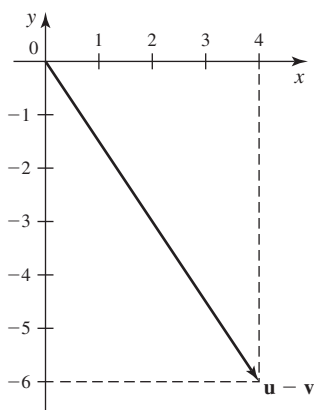
13.6.68 Because horizontal traces of the cup are circular, $a = b$. For $z = 0$, we have $x^2 + y^2 = 1$, which implies that $a^2 = b^2 = 1$. For $z = 6$, $x^2 + y^2 = 1 + \frac{36}{c^2}$, and also $x^2 + y^2 = 4$, which implies that $c^2 = 12$.

Chapter Thirteen Review

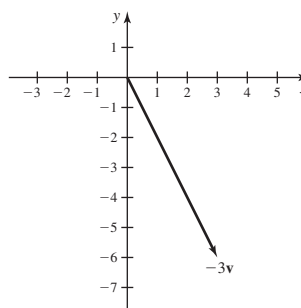
1

- a. True. Addition of vectors is commutative.
- b. False. For example, the vector in the direction of $\mathbf{i}$ with the length of $\mathbf{j}$ is $\mathbf{i}$, but the vector in the direction of $\mathbf{j}$ with the length of $\mathbf{i}$ is $\mathbf{j}$, and $\mathbf{i} \neq \mathbf{j}$.
- c. True, because it then follows that $\mathbf{u} = -\mathbf{v}$, so the two are parallel.
- d. False. The direction vectors for the lines are not parallel, so the lines are not parallel.
- e. True. The direction vector for the line is $\langle 1, 2, -1 \rangle$ and the normal to the plane is $\langle 1, 2, 5 \rangle$ and $\langle 1, 2, -1 \rangle \cdot \langle 1, 2, 5 \rangle = 1 + 4 - 5 = 0$, so the line and the plane are parallel.
- f. True. The cross product of the normals to the two intersecting planes is the normal to the plane in question.

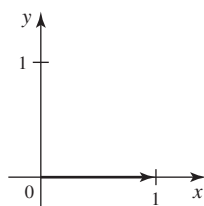
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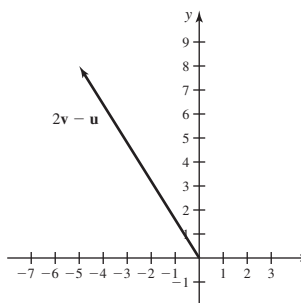
3



4



5



$$6 \quad \mathbf{u} - 3\mathbf{v} = \langle 2, 4, -5 \rangle - \langle -18, 30, 6 \rangle = \langle 20, -26, -11 \rangle.$$

$$7 \quad |\mathbf{u} + \mathbf{v}| = |\langle -4, 14, -3 \rangle| = \sqrt{16 + 196 + 9} = \sqrt{221}.$$

$$8 \quad \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{4 + 16 + 25}} \langle 2, 4, -5 \rangle = \frac{1}{\sqrt{45}} \langle 2, 4, -5 \rangle.$$

$$9 \quad 12 \frac{\mathbf{w}}{|\mathbf{w}|} = 12 \frac{\langle 4, -8, 8 \rangle}{\sqrt{16 + 64 + 64}} = 12 \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle.$$

$$10 \quad 10\mathbf{w} = 10 \langle 4, -8, 8 \rangle = \langle 40, -80, 80 \rangle.$$

$$11 \quad 10 \frac{\mathbf{w}}{|\mathbf{w}|} = \left\langle \frac{10}{3}, -\frac{20}{3}, \frac{20}{3} \right\rangle.$$

$$12 \quad \mathbf{u} \cdot \mathbf{v} = 2(-6) + 4(10) + -5(2) = -12 + 40 - 10 = 18.$$

$$13 \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -6 & 10 & 2 \end{vmatrix} = \langle 58, 26, 44 \rangle$$

$$14 \quad \text{We need } \langle -6, 10, 2 \rangle \cdot \langle a, 1, -3 \rangle = 0, \text{ so } -6a + 10 - 6 = 0, \text{ so } a = \frac{2}{3}.$$

$$15 \quad \text{We need } \langle 4, -8, 8 \rangle = k \langle a, 6, -6 \rangle \text{ for some } k. \text{ So } k = -\frac{4}{3} \text{ and } a = -3.$$

$$16 \quad \text{We must have } a + c = 2, a + b = 2, \text{ and } b + c = 2. \text{ Thus } a = b = c = 1.$$

17

$$\text{a. } \mathbf{v} = 550 \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle = \langle -275\sqrt{2}, 275\sqrt{2} \rangle.$$

$$\text{b. } \mathbf{v} = 550 \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle + \langle 0, 40 \rangle = \langle -275\sqrt{2}, 275\sqrt{2} + 40 \rangle.$$

18

$$\text{a. This is equal to } \langle 2, -8, 5 \rangle - \langle 2, 0, 6 \rangle = \langle 0, -8, -1 \rangle.$$

$$\text{b. The midpoint is } \left(\frac{2+2}{2}, \frac{-8+0}{2}, \frac{5+6}{2} \right) = (2, -4, 5.5). \text{ The magnitude of } \overrightarrow{PM} \text{ is } \frac{1}{2} |\langle 0, -8, -1 \rangle| = \frac{1}{2} \sqrt{64+1} = \frac{1}{2} \sqrt{65}.$$

$$\text{c. } -\frac{8}{\sqrt{65}} \langle 0, -8, -1 \rangle = \frac{8}{\sqrt{65}} \langle 0, 8, 1 \rangle.$$

$$19 \quad \{(x, y, z) \mid (x-1)^2 + y^2 + (z+1)^2 = 16\}.$$

$$20 \quad \{(x, y, z) \mid (x-2)^2 + (y-4)^2 + (z+3)^2 < 100\}.$$

$$21 \quad \{(x, y, z) \mid x^2 + (y-1)^2 + z^2 > 4\}.$$

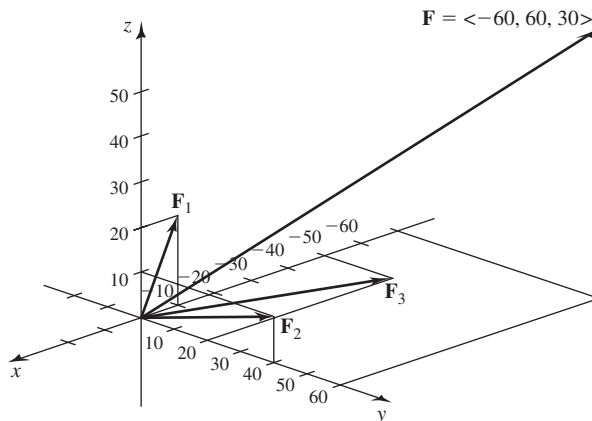
$$22 \quad \text{Completing the square gives } (x^2 - 6x + 9) + (y^2 + 8y + 16) + (z^2 - 2z + 1) = 23 + 9 + 16 + 1 = 49, \text{ or } (x-3)^2 + (y+4)^2 + (z-1)^2 = 7^2. \text{ This is a sphere of radius 7 centered at } (3, -4, 1).$$

$$23 \quad \text{Completing the square gives } (x^2 - x + 1/4) + (y^2 + 4y + 4) + (z^2 - 6z + 9) < -11 + 1/4 + 4 + 9 = \frac{9}{4}, \text{ or } (x-1/2)^2 + (y+2)^2 + (z-3)^2 < (3/2)^2. \text{ This is a ball centered at } (1/2, -2, 3) \text{ of radius } 3/2.$$

$$24 \quad \text{Completing the square gives } x^2 + (y^2 - 10y + 25) + (z^2 - 6z + 9) = -34 + 25 + 9, \text{ or } x^2 + (y-5)^2 + (z-3)^2 = 0. \text{ This consists of the single point } (0, 5, 3).$$

$$25 \quad \text{Completing the square gives } (x^2 - 6x + 9) + y^2 + (z^2 - 20z + 100) > -9 + 9 + 100, \text{ or } (x-3)^2 + y^2 + (z-10)^2 > 10^2. \text{ These are the points outside of a sphere of radius 10 centered at } (3, 0, 10).$$

26 $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle -60, 60, 30 \rangle$.
 $|\mathbf{F}| = 30\sqrt{4 + 4 + 1} = 90$.

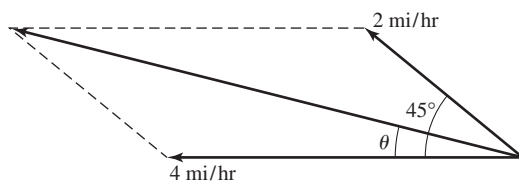


27 The magnitude of $\langle 0, 4, -50 \rangle$ is $\sqrt{2516} \approx 50.16$ meters per second. The direction is $\cos^{-1}(4/\sqrt{2516}) \approx 85.4$ degrees below the horizontal in the northerly horizontal direction.

28 The plane's original velocity vector is $250 \langle 1, 0, 0 \rangle$, the crosswind's is $40 \langle \sqrt{2}/2, \sqrt{2}/2, 0 \rangle$ and the down-draft's is $25 \langle 0, 0, -1 \rangle$. The resulting velocity vector is $\langle 250 + 20\sqrt{2}, 20\sqrt{2}, -25 \rangle$. The speed is therefore $\sqrt{(250 + 20\sqrt{2})^2 + (20\sqrt{2})^2 + (-25)^2} \approx 280.8$ mph.

29 The magnitude of the net force is $|\mathbf{F}| = \sqrt{40^2 + 30^2} = 50$ pounds. $\alpha = \tan^{-1}\left(\frac{3}{4}\right) \approx 0.6435$ radians or 36.87 degrees. The net force has magnitude 50 pounds in the direction 36.9 degrees north of east.

30 The canoe's vector $\mathbf{u}$ is given by $\mathbf{u} = -4\mathbf{i} + (-\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}) = -(4 + \sqrt{2})\mathbf{i} + \sqrt{2}\mathbf{j}$. The magnitude of $\mathbf{u}$ is given by $\sqrt{(4 + \sqrt{2})^2 + (\sqrt{2})^2} \approx 5.5958$ miles per hour. $\theta = \tan^{-1}\left(\frac{\sqrt{2}}{(4 + \sqrt{2})}\right) \approx 0.2555$ radians, or about 14.64 degrees. The canoe has speed about 5.6 miles per hour in the direction 14.64 degrees north of west.



31 This is a circle of radius one centered at $(0, 2, 0)$ and sitting in the plane $y = 2$.

32

a. $\mathbf{u} \cdot \mathbf{v} = 0 - 3 + 20 = 17$. $|\mathbf{u}| = 5$ and $|\mathbf{v}| = \sqrt{42}$. Thus the angle between the vectors is $\cos^{-1}\left(\frac{17}{5\sqrt{42}}\right) \approx 1.02$ radians.

b. $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{17}{42} \langle -4, 1, 5 \rangle$, and $\text{scal}_{\mathbf{v}} \mathbf{u} = \frac{17}{\sqrt{42}}$.

c. $\text{proj}_{\mathbf{u}} \mathbf{v} = \frac{17}{25} \langle 0, -3, 4 \rangle$, and $\text{scal}_{\mathbf{u}} \mathbf{v} = \frac{17}{5}$.

33

a. $\mathbf{u} \cdot \mathbf{v} = -3 + 12 + 12 = 21$, $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 9$, so $\theta = \cos^{-1}\left(\frac{21}{27}\right) \approx 0.68$ radian.

b. $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{21}{81} \langle 3, 6, 6 \rangle = \langle 7/9, 14/9, 14/9 \rangle$, and $\text{scal}_{\mathbf{v}} \mathbf{u} = \frac{21}{9} = \frac{7}{3}$.

c. $\text{proj}_{\mathbf{u}} \mathbf{v} = \frac{21}{9} \langle -1, 2, 2 \rangle = \langle -7/3, 14/3, 14/3 \rangle$, and $\text{scal}_{\mathbf{u}} \mathbf{v} = \frac{21}{3} = 7$.

34 We need to compute $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. Note that

$$\mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 10 & 2 \\ 4 & -8 & 8 \end{vmatrix} = \langle 96, 56, 8 \rangle.$$

Then $\mathbf{u} \cdot \langle 96, 56, 8 \rangle = 2(96) + 4(56) - 5(8) = 376$.

35 The work will be $25 \cdot 20 \cdot \cos 45^\circ = 250\sqrt{2}$ ft-lbs.

36 The work will be $\langle 2, 3, 4 \rangle \cdot \langle 2, 1, 6 \rangle = 31\text{J}$.

37 $|\mathbf{F}_{\text{perp}}| = 180 \cos 30^\circ = 90\sqrt{3}$. $|\mathbf{F}_{\text{par}}| = 180 \sin 30^\circ = 90$.

38 Work = $90 \cdot 10 = 900$ foot-lbs.

39 Two adjacent sides of the parallelogram are given by $\langle 0, -2, 3 \rangle$ and $\langle 3, 0, 1 \rangle$, so the area is

$$|\langle 0, -2, 3 \rangle \times \langle 3, 0, 1 \rangle| = |\langle -2, 9, 6 \rangle| = 11.$$

40 Two adjacent sides of the triangle are given by $\langle 4, 0, -4 \rangle$ and $\langle -1, 2, -5 \rangle$, so the area is

$$\frac{1}{2} |\langle 4, 0, -4 \rangle \times \langle -1, 2, -5 \rangle| = \frac{1}{2} |\langle 8, 24, 8 \rangle| = 4\sqrt{11}.$$

41 $\langle 2, -6, 9 \rangle \times \langle -1, 0, 6 \rangle = \langle -36, -21, -6 \rangle$. The length of this vector is $3\sqrt{12^2 + 7^2 + 2^2} = 3\sqrt{197}$. Thus the unit normals are $\frac{\pm 1}{\sqrt{197}} \langle 12, 7, 2 \rangle$.

42

a. The angle is given by $\cos^{-1} \left(\frac{\langle 2, 0, -2 \rangle \cdot \langle 2, 2, 0 \rangle}{2\sqrt{2} \cdot 2\sqrt{2}} \right) = \cos^{-1}(1/2) = \pi/3$.

b. The angle is also given by $\sin^{-1} \left(\frac{|\langle 2, 0, -2 \rangle \times \langle 2, 2, 0 \rangle|}{2\sqrt{2} \cdot 2\sqrt{2}} \right) = \sin^{-1} \left(\frac{|\langle 4, -4, 4 \rangle|}{8} \right) = \sin^{-1}(4\sqrt{3}/8) = \pi/3$.

43 The torque is $\langle 3, 2, 1 \rangle \times \langle 10, 10, 0 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 10 & 10 & 0 \end{vmatrix} = \langle -10, 10, 10 \rangle$.

44 The magnitude of the torque is $50 \cdot 0.25 \sin 60^\circ = 6.25\sqrt{3}$ N-m.

45 $T(\theta) = (0.4) \cdot 98 \cdot \sin \theta \approx 39.2 \sin \theta$ Newton-meters. This has a maximum of 39.2 when $\sin \theta = 1$ (at $\theta = \pi/2$) and a minimum of 0 at $\theta = 0$. The direction of the torque does not change as the knee is lifted.

46 The direction is $\langle -6 - 2, 4 - 6, 0 - (-1) \rangle = \langle -8, -2, 1 \rangle$. So the line is described by $\mathbf{r} = \langle 2, 6, -1 \rangle + t \langle 8, 2, -1 \rangle$, $-\infty < t < \infty$.

47 The direction of the line segment is $\langle 2 - 0, -8 - (-3), 1 - 9 \rangle = \langle 2, -5, -8 \rangle$. The line segment is described by $\mathbf{r} = \langle 0, -3, 9 \rangle + t \langle 2, -5, -8 \rangle$, $0 \leq t \leq 1$.

48 The direction is $\langle 2, -5, 6 \rangle$, so the line is given by $\mathbf{r} = \langle 0, 1, 1 \rangle + t \langle 2, -5, 6 \rangle = \langle 2t, 1 - 5t, 1 + 6t \rangle$.

49 The direction is given by $\langle 0, -1, 3 \rangle \times \langle 2, -1, 2 \rangle = \langle 1, 6, 2 \rangle$. Thus the line is given by $\mathbf{r} = \langle 0, 1, 1 \rangle + t \langle 1, 6, 2 \rangle = \langle t, 1 + 6t, 1 + 2t \rangle$.

50 The direction is given by $\langle -2, 1, 7 \rangle \times \langle 0, 1, 0 \rangle = \langle -7, 0, -2 \rangle$. Thus the line is given by $\mathbf{r} = \langle 0, 1, 4 \rangle + t \langle -7, 0, -2 \rangle = \langle -7t, 1, 4 - 2t \rangle$.

51

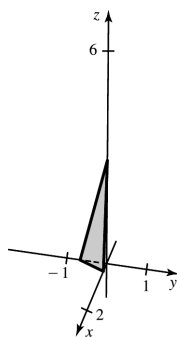
a. Let $P = (0, 0, 3)$, $Q = (1, 0, -6)$ and $R = (1, 2, 3)$. Then the vectors $\overrightarrow{PQ} = \langle 1, 0, -9 \rangle$ and $\overrightarrow{PR} = \langle 1, 2, 0 \rangle$

lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -9 \\ 1 & 2 & 0 \end{vmatrix} = 18\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has

equation $18x - 9y + 2(z - 3) = 0$, which simplifies to $18x - 9y + 2z = 6$.

b. The x -intercept is found by setting $y = z = 0$ and solving $18x = 6$ to obtain $x = \frac{1}{3}$. Similarly, the y and z -intercepts are $y = -\frac{2}{3}$ and $z = 3$.

c.



52 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 2, 1, -1 \rangle$ and $\mathbf{n}_R = \langle -1, 1, 1 \rangle$, are not multiples of each other; therefore, these planes are not parallel and they intersect in a line ℓ . We need to find a point on ℓ and a vector in the direction of ℓ . Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz plane: $y - z = 0$, $y + z = 1$. Solving these equations simultaneously

gives $y = z = \frac{1}{2}$, so $\left(0, \frac{1}{2}, \frac{1}{2}\right)$ is a point on ℓ . A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} =$

$2\mathbf{i} - \mathbf{j} + 3\mathbf{k} = \langle 2, -1, 3 \rangle$. Therefore ℓ has equation $\mathbf{r}(t) = \left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle + t \langle 2, -1, 3 \rangle = \left\langle 2t, \frac{1}{2} - t, \frac{1}{2} + 3t \right\rangle$, or $x = 2t$, $y = \frac{1}{2} - t$, $z = \frac{1}{2} + 3t$.

53 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle -3, 1, 2 \rangle$ and $\mathbf{n}_R = \langle 3, 3, 4 \rangle$ are not multiples of each other; therefore, these planes are not parallel and they intersect in a line ℓ . We need to find a point on ℓ and a vector in the direction of ℓ . Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz plane: $y + 2z = 0$, $3y + 4z = 12$. Solving these equations simultaneously

gives $y = 12$, $z = -6$, so $(0, 12, -6)$ is a point on ℓ . A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 2 \\ 3 & 3 & 4 \end{vmatrix} =$

$-2\mathbf{i} + 18\mathbf{j} - 12\mathbf{k} = -2\langle 1, -9, 6 \rangle$. Therefore, ℓ has equation $\mathbf{r}(t) = \langle 0, 12, -6 \rangle + t \langle 1, -9, 6 \rangle = \langle t, 12 - 9t, -6 + 6t \rangle$, or $x = t$, $y = 12 - 9t$, $z = -6 + 6t$.

54 A normal to the plane we are looking for is $\langle 2, 1, -1 \rangle$, because planes with the same normal are parallel. The desired plane is therefore $2(x - 5) + 1(y - 0) - 1(z - 2) = 0$, or $2x + y - z = 8$.

55 The plane we are looking for has normal $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 4 & 5 & -2 \end{vmatrix} = 4\mathbf{i} + 2\mathbf{j} + 13\mathbf{k}$. A point on both lines is $(4, 5, 1)$ which occurs for $s = 1$ and $t = -1$. The desired plane is therefore $4(x - 4) + 2(y - 5) + 13(z - 1) = 0$, or $4x + 2y + 13z = 39$.

56 The line has direction $\mathbf{v} = \langle 1, 3, -3 \rangle$, so the desired plane has equation $1(x - 2) + 3(y + 3) - 3(z - 1) = 0$, which simplifies to $x + 3y - 3z = -10$.

57 Let $P = (-2, 3, 1)$, $Q = (1, 1, 0)$ and $R = (-1, 0, 1)$. Then the vectors $\overrightarrow{PQ} = \langle 3, -2, -1 \rangle$ and $\overrightarrow{PR} = \langle 1, -3, 0 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 1 & -3 & 0 \end{vmatrix} = -(3\mathbf{i} + \mathbf{j} + 7\mathbf{k})$ is normal to the plane. The plane has equation $3(x + 2) + 1(y - 3) + 7(z - 1) = 0$, which simplifies to $3x + y + 7z = 4$.

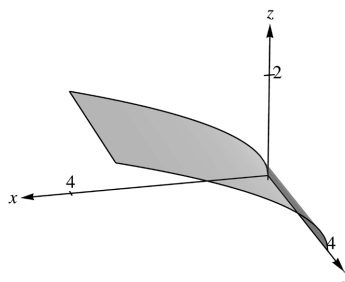
58 Let the point on the line be $P(2, 3, 1)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle 1, 1, -2 \rangle$. Then

$$d = \frac{|\langle 1, 1, -2 \rangle \times \langle 1, 0, -3 \rangle|}{|\langle 1, 0, -3 \rangle|} = \frac{|\langle -3, 1, -1 \rangle|}{|\langle 1, 0, -3 \rangle|} = \frac{\sqrt{11}}{\sqrt{10}}.$$

59 Let $Q(2, 2, 2)$ be the given point and let $P(1, 0, 0)$ be a point on the plane. Then $\overrightarrow{PQ} = \langle 1, 2, 2 \rangle$ is normal to the plane. So the distance from the point to the plane is $|\overrightarrow{PQ}| = |\langle 1, 2, 2 \rangle| = \sqrt{9} = 3$.

60

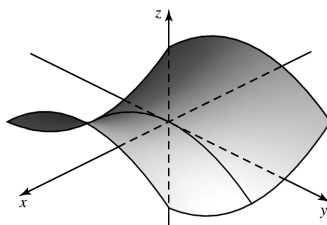
- This surface is a cylinder consisting of lines parallel to the y -axis passing through the curve $z = \sqrt{x}$ in the xz -plane.
- The xy -trace is found by setting $z = 0$ in the equation $z = \sqrt{x}$, which gives all points $(0, y, 0)$ (the y -axis). Similarly, we see that the xz -trace is the curve $z = \sqrt{x}$ and the yz -trace is the line $z = 0$.
- The x -intercept is found by setting $y = z = 0$ in the equation $z = \sqrt{x}$, which gives the point $(0, 0, 0)$. The y -intercepts consist of all points on the y -axis, and the z -intercept is the point $(0, 0, 0)$.
-



61

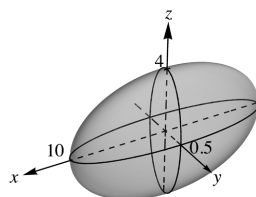
- This surface is a hyperbolic paraboloid.

- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm 2x$ (two lines intersecting at the origin). Similarly, we see that the xz -trace is the parabola $z = \frac{x^2}{36}$ and the yz -trace is the parabola $z = -\frac{y^2}{144}$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y - and z -intercepts are also $(0, 0, 0)$.
- d.



62

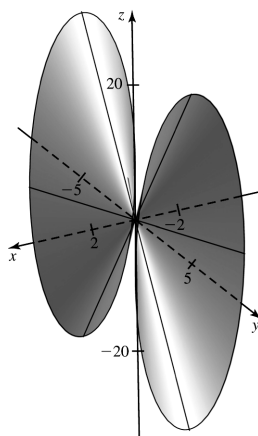
- a. This surface is an ellipsoid.
- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{100} + 4y^2 = 1$. Similarly, we see that the xz -trace is the ellipse $\frac{x^2}{100} + \frac{z^2}{16} = 1$, and the yz -trace is the ellipse $4y^2 + \frac{z^2}{16} = 1$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 10, 0, 0)$. Similarly, we see that the y -intercepts are the points $(0, \pm \frac{1}{2}, 0)$, and the z -intercepts are the points $(0, 0, \pm 4)$.
- d.



63

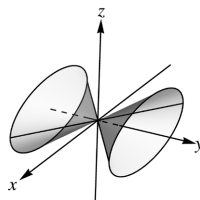
- a. This surface is an elliptic cone.
- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm 2x$ (two lines intersecting at the origin). Similarly, we see that the xz -trace is $(0, 0, 0)$ and the yz -trace is $z = \pm 5y$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y - and z -intercepts are also $(0, 0, 0)$.

d.



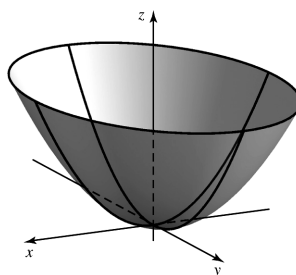
64

- a. This surface is an elliptic cone.
- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm \frac{2x}{3}$ (two lines intersecting at the origin). Similarly, we see that the xz -trace is $(0, 0, 0)$, and the yz -trace is $y = \pm \frac{3z}{2}$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y - and z -intercepts are also $(0, 0, 0)$.
- d.



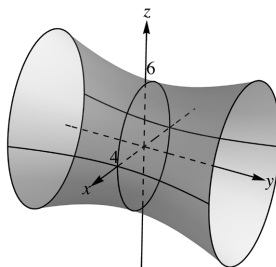
65

- a. This surface is an elliptic paraboloid.
- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives $(0, 0, 0)$. Similarly, we see that the xz -trace is the parabola $z = \frac{x^2}{16}$, and the yz -trace is the parabola $z = \frac{y^2}{36}$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y - and z -intercepts are also $(0, 0, 0)$.
- d.



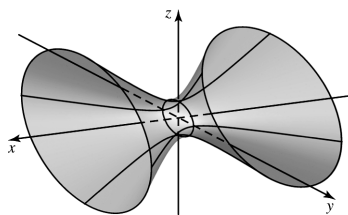
66

- a. This surface is a hyperboloid of one sheet.
- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the hyperbola $\frac{x^2}{16} - \frac{y^2}{100} = 1$. Similarly, we see that the xz -trace is the ellipse $\frac{x^2}{16} + \frac{z^2}{36} = 1$, and the yz -trace is the hyperbola $\frac{z^2}{36} - \frac{y^2}{100} = 1$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that there is no y -intercept, and the z -intercepts are $(0, 0, \pm 6)$.
- d.



67

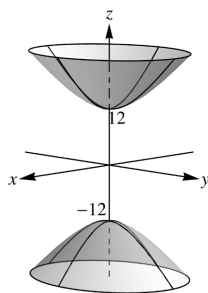
- a. This surface is a hyperboloid of one sheet.
- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the hyperbola $y^2 - 2x^2 = 1$. Similarly, we see that the xz -trace is the hyperbola $4z^2 - 2x^2 = 1$, and the yz -trace is the ellipse $y^2 + 4z^2 = 1$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that the y -intercepts are $(0, \pm 1, 0)$, and the z -intercepts are $(0, 0, \pm \frac{1}{2})$.
- d.



68

- a. This surface is a hyperboloid of two sheets.
- b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that the xz -trace is the hyperbola $-\frac{x^2}{16} + \frac{z^2}{36} = 4$, and the yz -trace is the hyperbola $\frac{z^2}{36} - \frac{y^2}{25} = 4$.
- c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that there is no y -intercept, and the z -intercepts are $(0, 0, \pm 12)$.

d.



69

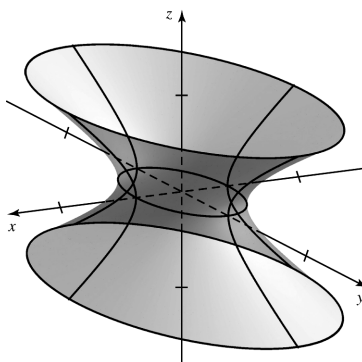
a. This surface is a hyperboloid of one sheet.

b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 4$.

Similarly, we see that the xz -trace is the hyperbola $\frac{x^2}{4} - z^2 = 4$, and the yz -trace is the hyperbola $\frac{y^2}{16} - z^2 = 4$.

c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that the y -intercepts are $(0, \pm 8, 0)$, and there are no z -intercepts.

d.



70

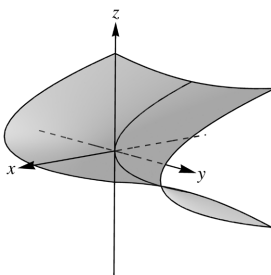
a. This surface is a hyperbolic paraboloid.

b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the parabola $x = \frac{y^2}{64}$.

Similarly, we see that the xz -trace is the parabola $x = -\frac{z^2}{9}$ and the yz -trace is the $y = \pm \frac{8z}{3}$ (two lines intersecting at the origin).

c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y - and z -intercepts are also $(0, 0, 0)$.

d.



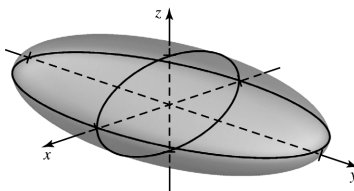
71

a. This surface is an ellipsoid.

b. The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 4$. Similarly, we see that the xz -trace is the ellipse $\frac{x^2}{4} + z^2 = 4$, and the yz -trace is the ellipse $\frac{y^2}{16} + z^2 = 4$.

c. The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that the y -intercepts are the points $(0, \pm 8, 0)$, and the z -intercepts are the points $(0, 0, \pm 2)$.

d.



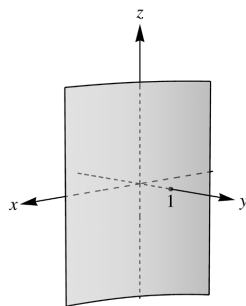
72

a. This surface is a cylinder consisting of lines parallel to the z -axis passing through the curve $y = e^{-x}$ in the xy -plane.

b. The xy -trace is found by setting $z = 0$ in the equation $y = e^{-x}$, which gives the curve $y = e^{-x}$. Similarly, we see that there is no xz -trace, and the yz -trace is the line consisting of all points $(0, 1, z)$.

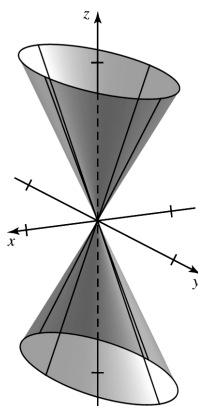
c. The x -intercept is found by setting $y = z = 0$ in the equation $y = e^{-x}$, which has no solutions. Similarly, we see that the y -intercept is the point $(0, 1, 0)$, and there are no z -intercepts.

d.



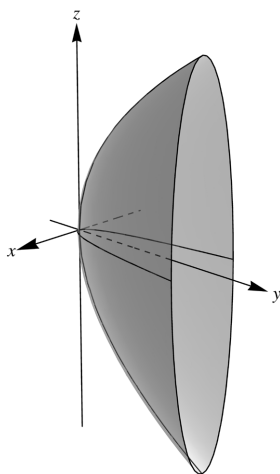
73

- This surface is an elliptic cone.
- The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the origin $(0, 0, 0)$. Similarly, we see that the xz -trace is $z = \pm \frac{8x}{3}$ (two lines intersecting at the origin), and the yz -trace is $y = \pm \frac{7z}{8}$.
- The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y and z -intercepts are also $(0, 0, 0)$.
-



74

- This surface is an elliptic paraboloid.
- The xy -trace is found by setting $z = 0$ in the equation of the surface, which gives the parabola $y = 4x^2$. Similarly, we see that the xz -trace is the origin, and the yz -trace is the parabola $y = \frac{z^2}{9}$.
- The x -intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y - and z -intercepts are also $(0, 0, 0)$.
-



75

- a. The graph of this function is part of a hyperboloid of two sheets and contains the origin, which matches A.
- b. The graph of this function is a cylinder, which matches D.
- c. The graph of this function is a hyperbolic paraboloid, which matches C.
- d. The graph of this function is part of a hyperboloid of one sheet, which matches B.

76 The equation of the cylinder is $x^2 + y^2 = 4$, $0 \leq z \leq 6$. Perhaps the easiest approach to forming the top of the water bottle is to shift the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ up 8 units to obtain $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{(z-8)^2}{c^2} = 1$ so that the top of the bottle corresponds to the narrowest portion of the hyperboloid before it is truncated. For $z = 8$, the equation should simplify to $x^2 + y^2 = \frac{1}{4}$, which implies that $a^2 = b^2 = \frac{1}{4}$. So we have $4x^2 + 4y^2 - \frac{(z-8)^2}{c^2} = 1$ or $x^2 + y^2 - \frac{(z-8)^2}{4c^2} = \frac{1}{4}$. For $z = 6$, the equation simplifies to $x^2 + y^2 = 4$, which implies that $\frac{(6-8)^2}{4c^2} + \frac{1}{4} = 4$, or $c^2 = \frac{4}{15}$. Putting this all together, we have the equation $x^2 + y^2 - \frac{15(z-8)^2}{16} = \frac{1}{4}$ for the top part of the bottle. So the answer is $x^2 + y^2 = 4$, $0 \leq z \leq 6$ and $x^2 + y^2 - \frac{15(z-8)^2}{16} = \frac{1}{4}$, $6 \leq z \leq 8$. Note: Answers vary depending upon which portion of a truncated hyperboloid is used to form the top part of the bottle.

Chapter 14

Vector-Valued Functions

14.1 Vector-Valued Functions

14.1.1 It has one, namely t .

14.1.2 It has three, namely $x = f(t)$, $y = g(t)$, and $z = h(t)$.

14.1.3 For every real number t that is put into the function, the output is a vector $\mathbf{r}(t)$.

14.1.4 It lies in the xz -plane.

14.1.5 Compute $\lim_{t \rightarrow a} f(t) = L_1$, $\lim_{t \rightarrow a} g(t) = L_2$, and $\lim_{t \rightarrow a} h(t) = L_3$. Then $\lim_{t \rightarrow a} \mathbf{r}(t) = \langle L_1, L_2, L_3 \rangle$.

14.1.6 It is continuous at a exactly when the three component functions $x = f(t)$, $y = g(t)$ and $z = h(t)$ are continuous at a .

14.1.7 Note that $\langle 1, 2, 3 \rangle$ is a direction vector for the line. The equation is $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$.

14.1.8 $x = \cos t$, $y = \sin t$, and $z = 10$, so $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 10\mathbf{k}$.

14.1.9 A direction vector for the line is $\overrightarrow{PQ} = \langle 2, 3, -4 \rangle$, so the line is $\mathbf{r}(t) = \langle 2 + 2t, 3 + 3t, 7 - 4t \rangle$.

14.1.10 A direction vector for the line is $\langle 0, -1, 1 \rangle$, so the line is $\mathbf{r}(t) = \langle 0, -3 - t, 2 + t \rangle$.

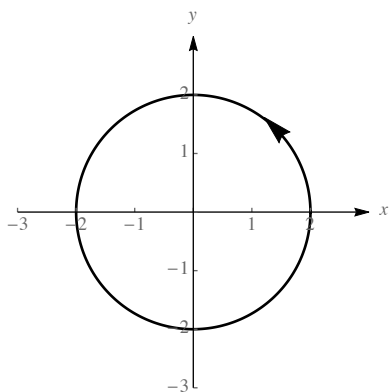
14.1.11 A direction vector for the line is a normal to the plane, which could be $\langle 2, 0, -1 \rangle$, so the line is $\mathbf{r}(t) = \langle 3 + 2t, 4, 5 - t \rangle$.

14.1.12 A direction vector for the line of intersection is obtained by crossing the normals to the planes. So we calculate $\langle 2, 3, 4 \rangle \times \langle 2, 3, 5 \rangle = \langle 3, -2, 0 \rangle$. So the line is given by $\mathbf{r}(t) = \langle 3t, 1 - 2t, 1 \rangle$.

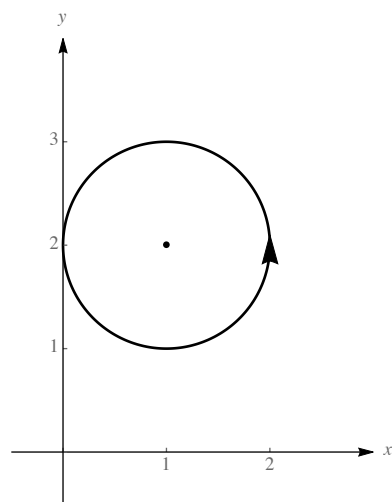
14.1.13 The line segment from P to Q is given by $(1 - t)\overrightarrow{OP} + t\overrightarrow{OQ} = (1 - t)\langle 1, 2, 1 \rangle + t\langle 0, 2, 3 \rangle = \langle 1 - t, 2, 1 + 2t \rangle$, $0 \leq t \leq 1$.

14.1.14 The line segment from P to Q is given by $(1 - t)\overrightarrow{OP} + t\overrightarrow{OQ} = (1 - t)\langle -4, -2, 1 \rangle + t\langle -2, -2, 3 \rangle = \langle -4 + 2t, -2, 1 + 2t \rangle$, $0 \leq t \leq 1$.

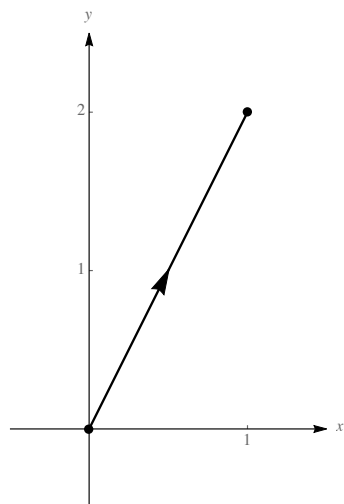
14.1.15



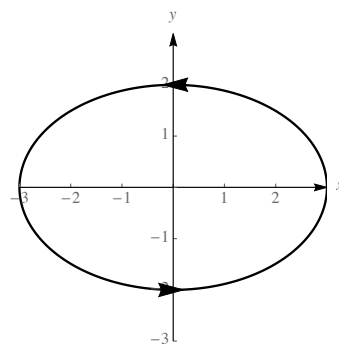
14.1.16



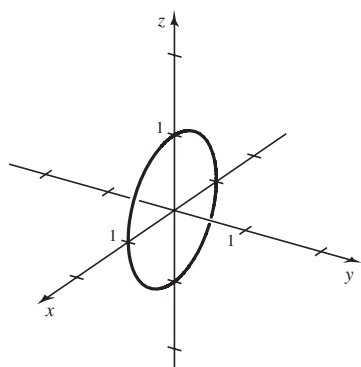
14.1.17



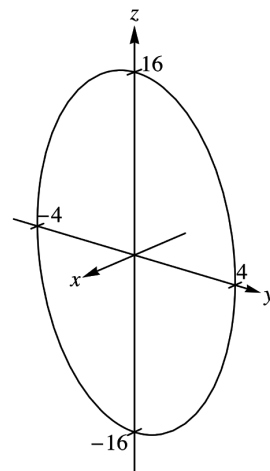
14.1.18



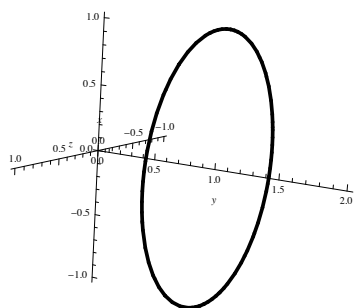
14.1.19



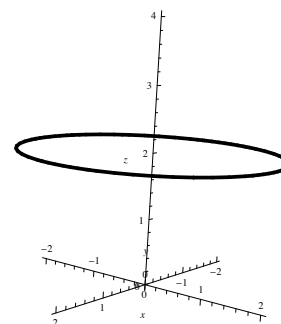
14.1.20



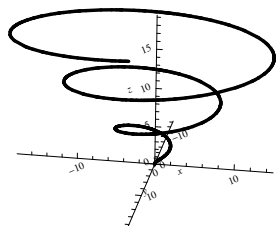
14.1.21



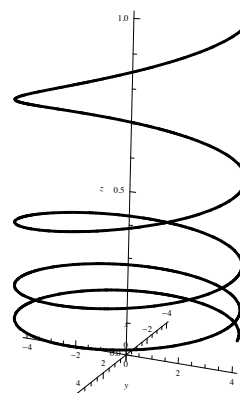
14.1.22



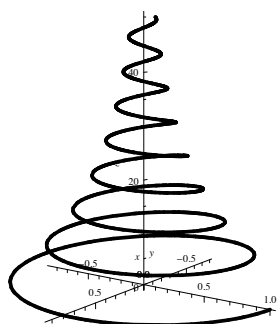
14.1.23



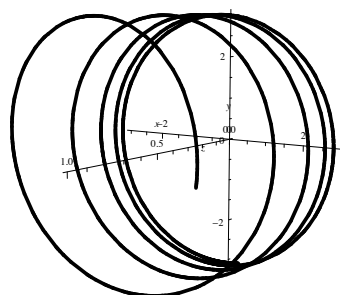
14.1.24



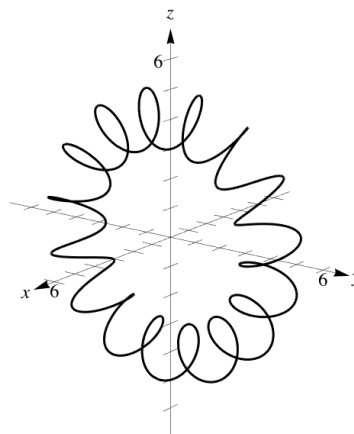
14.1.25



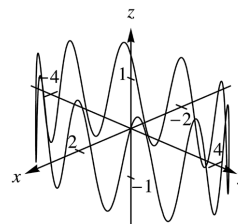
14.1.26



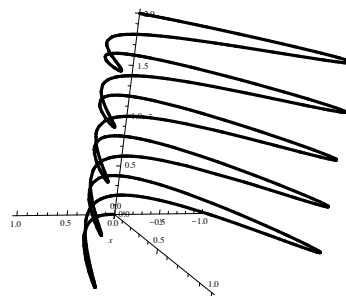
- 14.1.27** Note that the curve is closed (the initial point and the terminal point coincide), and is very “wavy.”



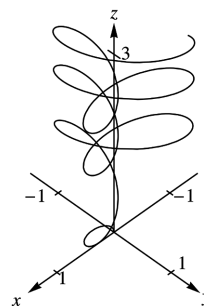
- 14.1.28** The projection onto the xy -plane is an ellipse elongate in the y direction. The curve oscillates in a sinusoidal wave in the z direction.



- 14.1.29** When viewed from the top, the curve looks parabolic.



- 14.1.30** When viewed from the top, the curve appears as a 3-petaled rose.



14.1.31 $\lim_{t \rightarrow \pi/2} \left\langle \cos 2t, -4 \sin t, \frac{2t}{\pi} \right\rangle = \left\langle \cos \pi, -4 \sin \pi/2, \frac{2 \cdot \pi/2}{\pi} \right\rangle = \langle -1, -4, 1 \rangle.$

$$14.1.132 \quad \lim_{t \rightarrow \ln 2} \langle 2e^t, 6e^{-t}, -4e^{-2t} \rangle = \langle 2e^{\ln 2}, 6e^{-\ln 2}, -4e^{-2\ln 2} \rangle = \langle 4, 3, -1 \rangle.$$

$$14.1.133 \quad \lim_{t \rightarrow \infty} \left\langle e^{-t}, -\frac{2t}{t+1}, \tan^{-1} t \right\rangle = \langle 0, -2, \pi/2 \rangle.$$

$$14.1.134 \quad \lim_{t \rightarrow 2} \left\langle \frac{t}{t^2+1}, -4e^{-t} \sin \pi t, \frac{1}{\sqrt{4t+1}} \right\rangle = \left\langle \frac{2}{5}, 0, \frac{1}{3} \right\rangle.$$

14.1.135 Using l'Hôpital's rule (once in the first two components, twice in the third component):

$$\lim_{t \rightarrow 0} \left\langle \frac{\sin t}{t}, -\frac{e^t - t - 1}{t}, \frac{\cos t + t^2/2 - 1}{t^2} \right\rangle = \lim_{t \rightarrow 0} \left\langle \cos t, 1 - e^t, \frac{-\cos t + 1}{2} \right\rangle = \langle 1, 0, 0 \rangle = \mathbf{i}.$$

14.1.136 Using l'Hôpital's rule (once in the first two components, but not in the 3rd component):

$$\lim_{t \rightarrow 0} \left\langle \frac{\tan t}{t}, \frac{-3t}{\sin t}, \sqrt{t+1} \right\rangle = \lim_{t \rightarrow 0} \left\langle \sec^2 t, \frac{-3}{\cos t}, \sqrt{t+1} \right\rangle = \langle 1, -3, 1 \rangle = \mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

14.1.137

- True. Letting $y = 0$ we see that $z = t^2 = x^2$, which is a parabola.
- False. $x^2 + y^2 + z^2 = \sin^2 t + \cos^2 t + \sin^2 t = 1 + \sin^2 t \neq 1$.
- True. The first component function approaches 0 as $t \rightarrow \infty$, while the others are periodic. The parametric equations $y = \sin t$ and $z = -\cos t$ form a circle in the yz -plane.
- True. Both have limit $\langle 0, 0, 0 \rangle$.

14.1.138 The first component function has domain $(-\infty, 1) \cup (1, \infty)$, and the second has domain $(-\infty, -2) \cup (-2, \infty)$, so the domain of $\mathbf{r}(t)$ is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

14.1.139 The first component function has domain $[-2, \infty)$ and the second has domain $(-\infty, 2]$, so the domain of $\mathbf{r}(t)$ is $[-2, 2]$.

14.1.140 The first component function is defined everywhere, the second has domain $[0, \infty)$, and the third has domain $(-\infty, 0) \cup (0, \infty)$, so the domain of $\mathbf{r}(t)$ is $(0, \infty)$.

14.1.141 The first component function has domain $[-2, 2]$, the second has domain $[0, \infty)$, and the third has domain $(-1, \infty)$, so the domain of $\mathbf{r}(t)$ is $[0, 2]$.

14.1.142 The intersection occurs for $y = 1 = 2 \sin t$, so for $t = \pi/6$ and $t = 5\pi/6$. The points of intersection are $(5\sqrt{3}, 1, 1)$ and $(-5\sqrt{3}, 1, 1)$.

14.1.143 The intersection occurs for $z = 16 = 4 + 3t$, so for $t = 4$. The point of intersection is $(4, 8, 16)$.

14.1.144 The intersection occurs for $x + y = \cos t + \sin t = 0$, so for $t = 3\pi/4$, $t = 7\pi/4$, $t = 11\pi/4$, and $t = 15\pi/4$. The points of intersection are $(-\sqrt{2}/2, \sqrt{2}/2, 3\pi/4)$, $(\sqrt{2}/2, -\sqrt{2}/2, 7\pi/4)$, $(-\sqrt{2}/2, \sqrt{2}/2, 11\pi/4)$, and $(\sqrt{2}/2, -\sqrt{2}/2, 15\pi/4)$.

14.1.145

- This matches graph E. (It is a straight line.)
- This matches graph D. (It is parabolic-like.)
- This matches graph F. (It is a circle.)
- This matches graph C. (It is a circular helix, elongated along the x -axis.)
- This matches graph A. (It is a closed curve which isn't a circle.)

f. This matches graph B. (It is a circular helix, elongated along the y -axis.)

14.1.46

a. $\mathbf{r}(0) = \langle 50, 0, 0 \rangle$.

b. $\lim_{t \rightarrow \infty} \frac{50 \cos t}{e^t} = 0$ by the squeeze theorem, and likewise $\lim_{t \rightarrow \infty} \frac{50 \sin t}{e^t} = 0$. Also, $\lim_{t \rightarrow \infty} \left(5 - \frac{5}{e^t}\right) = 5$.
Thus we have $\lim_{t \rightarrow \infty} \mathbf{r}(t) = \langle 0, 0, 5 \rangle$.

c. Let $x = 50e^{-t} \cos t$ and $y = 50e^{-t} \sin t$ and $z = 5 - 5e^{-t}$. Note that $x^2 + y^2 = 2500e^{-2t}$, so $r = 50e^{-t}$.
We have $z = 5 - 5e^{-t} = 5 - \frac{r}{10}$.

14.1.47 We have $z = 4 = x^2 + y^2$, so we have a circle of radius 2 at a height of 4. This can be describe by $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 4 \rangle$.

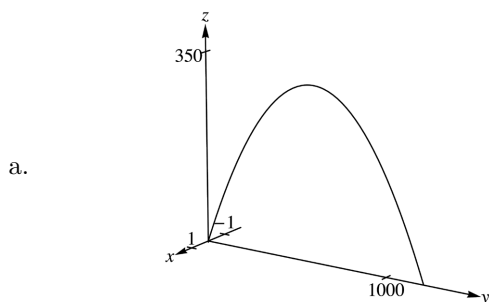
14.1.48 We have $z = 3x^2 + y^2 + 1 = 5 - x^2 - 3y^2$, so $4x^2 + 4y^2 = 4$, or $x^2 + y^2 = 1$. Then $z = 3x^2 + y^2 + 1 = 2x^2 + x^2 + y^2 + 1 = 2x^2 + 2$. This can be described by $\mathbf{r}(t) = \langle \cos t, \sin t, 2 + 2 \cos^2 t \rangle$.

14.1.49 Letting $x = 5 \cos t$ and $y = 5 \sin t$ we have $x^2 + y^2 = 25$. So we have

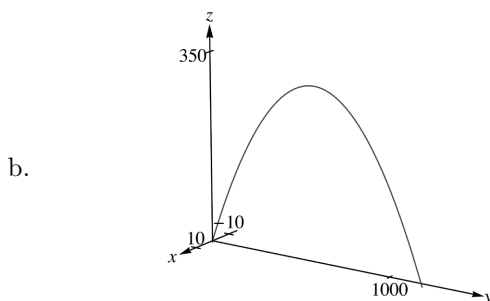
$$\mathbf{r}(t) = \langle 5 \cos t, 5 \sin t, 10 \cos t + 10 \sin t \rangle.$$

14.1.50 If $z = y + 1 = x^2 + 1$, then $y = x^2$. If we let $x = t$, we have $y = t^2$ and $z = t^2 + 1$, so $\mathbf{r}(t) = \langle t, t^2, t^2 + 1 \rangle$.

14.1.51



With $a = 0$, we have $\mathbf{r}(t) = \langle 0, 75t, -5t^2 + 80t \rangle$.
Note that $z = 0$ when $t = 0$ and when $-5t + 80 = 0$, or $t = 16$. At this time, $y = 75 \cdot 16 = 1200$ feet.



With $a = 0.2$, we have $\mathbf{r}(t) = \langle .2t, (75 - .02)t, -5t^2 + 80t \rangle$. Note that $z = 0$ when $t = 0$ and when $-5t + 80 = 0$, or $t = 16$. At this time, $y = (75 - .02) \cdot 16 = 1199.7$ feet.

c. With $a = 2.5$, the ball travels $(75 - .025) \cdot 16 = 1196$ feet.

14.1.52 This is an elliptic cone. Note that

$$x^2 + y^2 = t^2 \cos^2 t + t^2 \sin^2 t = t^2 (\cos^2 t + \sin^2 t) = t^2 = z^2.$$

14.1.53 This is a hyperboloid of one sheet. Note that

$$x^2 + y^2 - z^2 = (t^2 + 1) \cos^2 t + (t^2 + 1) \sin^2 t - t^2 = (t^2 + 1)(\cos^2 t + \sin^2 t) - t^2 = (t^2 + 1) - t^2 = 1.$$

14.1.54 This is an elliptic paraboloid. Note that

$$x^2 + y^2 = t \cos^2 t + t \sin^2 t = t(\cos^2 t + \sin^2 t) = t = z.$$

14.1.55 This is an ellipsoid. Note that

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 0 + \frac{4 \cos^2 t}{4} + \frac{9 \sin^2 t}{9} = \cos^2 t + \sin^2 t = 1.$$

14.1.56 This is a torus. Note that

$$\begin{aligned} (3 - \sqrt{x^2 + y^2})^2 + z^2 &= \left(3 - \sqrt{(3 + \cos 15t)^2 \cos^2 t + (3 + \cos 15t)^2 \sin^2 t}\right)^2 + \sin^2 15t \\ &= \left(3 - \sqrt{(3 + \cos 15t)^2 (\cos^2 t + \sin^2 t)}\right)^2 + \sin^2 15t \\ &= (3 - (3 + \cos 15t))^2 + \sin^2 15t = \cos^2 15t + \sin^2 15t = 1. \end{aligned}$$

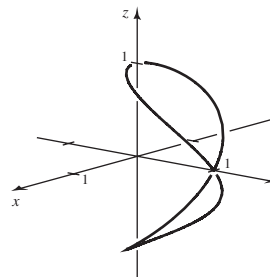
14.1.57 Consider $s(t) = \sqrt{(t^2 - 1)^2 + (t - 1)^2 + (t - 15)^2} = \sqrt{t^4 - 32t + 227}$. Computing $s'(t)$ reveals $s'(t) = \frac{2(t^3 - 8)}{\sqrt{t^4 - 32t + 227}}$. This is zero for $t = 2$. So the point closest to P_0 is $(4, 2, 2)$. The distance between the two points is $\sqrt{179}$.

14.1.58 Consider

$$\begin{aligned} s(t) &= \sqrt{(\cos t - 1)^2 + (\sin t - 1)^2 + (t - 3)^2} \\ &= \sqrt{t^2 - 6t + \sin^2(t) - 2\sin(t) + \cos^2(t) - 2\cos(t) + 11} \\ &= \sqrt{t^2 - 6t - 2\sin(t) - 2\cos(t) + 12}. \end{aligned}$$

Note that $s'(t) = \frac{t + \sin(t) - \cos(t) - 3}{\sqrt{t^2 - 6t - 2\sin(t) - 2\cos(t) + 12}}$ which is zero for approximately $t = 1.8$. The closest point is approximately $(-0.23, 0.97, 1.8)$ and the distance is approximately 1.72.

14.1.59 First note that $x = \frac{1}{2} \sin 2t = \sin t \cos t$ and $y = \frac{1}{2}(1 - \cos 2t) = \sin^2 t$. Then $x^2 + y^2 + z^2 = \sin^2 t \cos^2 t + \sin^4 t + \cos^2 t = \sin^2 t(\cos^2 t + \sin^2 t) + \cos^2 t = \sin^2 t \cdot 1 + \cos^2 t = 1$. So all points on the curve are equidistant from the origin, so they lie on the sphere of radius one centered at the origin.

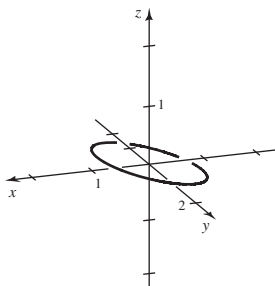


14.1.60 Suppose $a^2 = b^2 = c^2$. We have $x^2 + y^2 + z^2 = a^2 \sin^2 mt \cdot \cos^2 nt + b^2 \sin^2 mt \cdot \sin^2 nt + c^2 \cos^2 mt = a^2(\cos^2 nt + \sin^2 nt) \sin^2 mt + a^2 \cos^2 mt = a^2 \sin^2 mt + a^2 \cos^2 mt = a^2$. Thus we have $x^2 + y^2 + z^2 = a^2$. So the curve lies on a sphere.

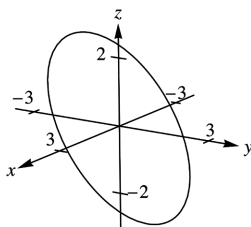
14.1.61 In order for $\sin(mt + mT) \cos(nt + nT) = \sin mt \cos nt$ and $\sin(mt + mT) \sin(nt + nT) = \sin mt \sin nt$ and $\cos(mt + mT) = \cos mt$ we would need $T = \frac{2\pi}{m}$ or a multiple of it, and then it would be necessary for $\sin(nt + nT) = \sin nt$, which would require $T = \frac{2\pi}{n}$, or a multiple of it. Thus, the smallest such T would be $\frac{2\pi}{(m, n)}$, where (m, n) represents the greatest common factor of m and n .

14.1.62 If $x = a \cos t + b \sin t$ and $y = c \cos t + d \sin t$ and $z = e \cos t + f \sin t$ then $x^2 + y^2 + z^2 = (a \cos t + b \sin t)^2 + (c \cos t + d \sin t)^2 + (e \cos t + f \sin t)^2 = (a^2 + c^2 + e^2) \cos^2 t + (b^2 + d^2 + f^2) \sin^2 t + 2(ab + cd + ef) \sin t \cos t$. If $ab + cd + ef = 0$ and if $a^2 + c^2 + e^2 = R^2 = b^2 + d^2 + f^2$, then we have $x^2 + y^2 + z^2 = R^2$, so all the points on the curve lie at a distance of R from the origin, so (because the curve lies in a plane) it is a circle of radius R centered at the origin.

14.1.63 This has the form mentioned in exercise 62, with $a = 1/\sqrt{2}$, $b = 1/\sqrt{3}$, $c = -1/\sqrt{2}$, $d = 1/\sqrt{3}$, $e = 0$, and $f = 1/\sqrt{3}$. Note that $a^2 + c^2 + e^2 = \frac{1}{2} + \frac{1}{2} + 0 = 1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = b^2 + d^2 + f^2$, and $ab + cd + ef = 0$. So this is a circle of radius 1 centered at the origin.



14.1.64 Using the notation of exercise 62, we have $ab + cd + ef = 0$. However, we have $a^2 + c^2 + e^2 = 6$ and $b^2 + d^2 + f^2 = 12$. Thus $x^2 + y^2 + z^2 = 6 \cos^2 t + 12 \sin^2 t = 6 + 6 \sin^2 t$. The curve is an oval-shaped closed curve.



14.1.65 Note that $\mathbf{r}(0) = \langle a, c, e \rangle$ and $\mathbf{r}(\pi/2) = \langle b, d, f \rangle$, and $\mathbf{r}(\pi) = \langle -a, -c, -e \rangle$ have their terminal points on the curve. So $\langle a, c, e \rangle - \langle -a, -c, -e \rangle = \langle 2a, 2c, 2e \rangle = 2\mathbf{r}(0)$ lies in the plane containing the curve, which implies that $\mathbf{r}(0)$ lies in the plane containing the curve, and that implies that the point $(0, 0, 0)$ is in the plane containing the curve. So a normal to the curve is $\mathbf{r}(0) \times \mathbf{r}(\pi/2) = \langle a, c, e \rangle \times \langle b, d, f \rangle = \langle cf - de, be - af, ad - bc \rangle$.

14.1.66

- a. Assume that $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L} = \langle L_1, L_2, L_3 \rangle$. Then for any $\epsilon > 0$, there exists $\delta > 0$ so that $|\mathbf{r}(t) - \mathbf{L}| = \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2} < \epsilon$ for $|t - a| < \delta$.

Note that $|f(t) - L_1| \leq \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2}$, and likewise for $|g(t) - L_2|$ and $|h(t) - L_3|$. So given $\epsilon > 0$, choosing the δ mentioned in the paragraph above guarantees that the absolute values of the differences of the corresponding coordinate functions and the L_i are less than epsilon whenever $|t - a| < \delta$, as desired.

- b. Assume $\lim_{t \rightarrow a} f(t) = L_1$, $\lim_{t \rightarrow a} g(t) = L_2$, and $\lim_{t \rightarrow a} h(t) = L_3$. Then for any $\epsilon > 0$, there exists $\delta_1 > 0$ so that $|f(t) - L_1| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_1$, and there exists δ_2 so that $|g(t) - L_2| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_2$, and there exists δ_3 so that $|h(t) - L_3| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_3$.

Now let $\epsilon > 0$ be given, and let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$|\mathbf{r}(t) - \langle L_1, L_2, L_3 \rangle| = \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2} \leq \sqrt{3 \cdot \epsilon^2/3} = \sqrt{\epsilon^2} = \epsilon,$$

as desired.

14.2 Calculus of Vector-Valued Functions

14.2.1 It is $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

14.2.2 $\mathbf{r}'(t)$ is a vector tangent to the curve $\mathbf{r}(t)$.

14.2.3 Divide the vector by its length, so if the vector is $\mathbf{r}'(t)$, form $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

14.2.4 $\mathbf{r}'(t) = \langle 10t^9, 8, -\sin t \rangle$, so $\mathbf{r}''(t) = \langle 90t^8, 0, -\cos t \rangle$.

14.2.5 Compute the indefinite integral of each of the component functions, and then

$$\int \mathbf{r}(t) dt = \left\langle \int f(t) dt, \int g(t) dt, \int h(t) dt \right\rangle.$$

14.2.6 $\int_a^b \mathbf{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle.$

14.2.7 $\mathbf{r}(0) = \langle 1, 3, 10 \rangle + C$, so if $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, then we must have $C = \langle -1, -3, -10 \rangle$.

14.2.8 $\mathbf{r}'(0) = \langle 6, 2, 3 \rangle$, so $\frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \left\langle \frac{6}{7}, \frac{2}{7}, \frac{3}{7} \right\rangle.$

14.2.9 $\mathbf{r}'(t) = \langle -\sin t, 2t, \cos t \rangle$.

14.2.10 $\mathbf{r}'(t) = \langle 4e^t, 0, 1/t \rangle$.

14.2.11 $\mathbf{r}'(t) = \langle 6t, 3/\sqrt{t}, -3/t^2 \rangle$.

14.2.12 $\mathbf{r}'(t) = \langle 0, -6\sin 2t, 6\cos 3t \rangle$.

14.2.13 $\mathbf{r}'(t) = \langle e^t, -2e^{-t}, -8e^{2t} \rangle$.

14.2.14 $\mathbf{r}'(t) = \langle \sec^2 t, \sec t \tan t, -\sin 2t \rangle$.

14.2.15 $\mathbf{r}'(t) = \langle e^{-t}(1-t), 1 + \ln t, \cos t - t \sin t \rangle$.

14.2.16 $\mathbf{r}'(t) = \langle -(t+1)^{-2}, (t^2+1)^{-1}, (t+1)^{-1} \rangle$.

14.2.17 $\mathbf{r}'(t) = \langle 1, 6t, 3t^2 \rangle$, so $\mathbf{r}'(1) = \langle 1, 6, 3 \rangle$.

14.2.18 $\mathbf{r}'(t) = \langle e^t, 3e^{3t}, 5e^{5t} \rangle$, so $\mathbf{r}'(0) = \langle 1, 3, 5 \rangle$.

14.2.19 $\mathbf{r}'(t) = \langle 1, -2\sin 2t, 2\cos t \rangle$, so $\mathbf{r}'(\pi/2) = \langle 1, 0, 0 \rangle$.

14.2.20 $\mathbf{r}'(t) = \left\langle 2\cos t, -3\sin t, \frac{1}{2}\cos(t/2) \right\rangle$, so $\mathbf{r}'(\pi) = \langle -2, 0, 0 \rangle$.

14.2.21 $\mathbf{r}'(t) = \langle 8t^3, 9\sqrt{t}, -10/t^2 \rangle$, so $\mathbf{r}'(1) = \langle 8, 9, -10 \rangle$.

14.2.22 $\mathbf{r}'(t) = \langle 2e^t, -2e^{-2t}, 8e^{2t} \rangle$, so $\mathbf{r}'(\ln 3) = \langle 6, -2/9, 72 \rangle$.

14.2.23 $\mathbf{r}'(t) = \langle 2, 2, 1 \rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{3} \langle 2, 2, 1 \rangle = \langle 2/3, 2/3, 1/3 \rangle.$$

14.2.24 $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{\sin^2 t + \cos^2 t + 0}} \langle -\sin t, \cos t, 0 \rangle = \langle -\sin t, \cos t, 0 \rangle.$$

14.2.25 $\mathbf{r}'(t) = \langle 0, -2 \sin 2t, 4 \cos 2t \rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4 \sin^2 2t + 16 \cos^2 2t}} \langle 0, -2 \sin 2t, 4 \cos 2t \rangle = \frac{1}{\sqrt{1 + 3 \cos^2 2t}} \langle 0, -\sin 2t, 2 \cos 2t \rangle.$$

14.2.26 $\mathbf{r}'(t) = \langle \cos t, -\sin t, -\sin t \rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{\cos^2 t + \sin^2 t + \sin^2 t}} \langle \cos t, -\sin t, -\sin t \rangle = \frac{1}{\sqrt{1 + \sin^2 t}} \langle \cos t, -\sin t, -\sin t \rangle.$$

14.2.27 $\mathbf{r}'(t) = \langle 1, 0, -2/t^2 \rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1 + (4/t^4)}} \langle 1, 0, -2/t^2 \rangle = \frac{1}{\sqrt{t^4 + 4}} \langle t^2, 0, -2 \rangle.$$

14.2.28 $\mathbf{r}'(t) = \langle 2e^{2t}, 4e^{2t}, -6e^{-3t} \rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{20e^{4t} + 36e^{-6t}}} \langle 2e^{2t}, 4e^{2t}, -6e^{-3t} \rangle = \frac{1}{\sqrt{5e^{4t} + 9e^{-6t}}} \langle e^{2t}, 2e^{2t}, -3e^{-3t} \rangle.$$

14.2.29 $\mathbf{r}'(t) = \langle -2 \sin 2t, 0, 6 \cos 2t \rangle$, so at $t = \pi/2$, we have $\mathbf{r}'(\pi/2) = \langle 0, 0, -6 \rangle$. Thus, the unit tangent at $\pi/2$ is $\langle 0, 0, -1 \rangle$.

14.2.30 $\mathbf{r}'(t) = \langle \cos t, -\sin t, -e^{-t} \rangle$, so at $t = 0$, we have $\mathbf{r}'(0) = \langle 1, 0, -1 \rangle$. Thus, the unit tangent at 0 is $\langle 1/\sqrt{2}, 0, -1/\sqrt{2} \rangle$.

14.2.31 $\mathbf{r}'(t) = \langle 6, 0, -3/t^2 \rangle$, so at $t = 1$, we have $\mathbf{r}'(1) = \langle 6, 0, -3 \rangle$. Thus, the unit normal at 1 is $\langle 2/\sqrt{5}, 0, -1/\sqrt{5} \rangle$.

14.2.32 $\mathbf{r}'(t) = \langle \sqrt{7}e^t, 3e^t, 3e^t \rangle$, so at $t = \ln 2$, we have $\mathbf{r}'(\ln 2) = \langle 2\sqrt{7}, 6, 6 \rangle$. Thus, the unit normal at $\ln 2$ is $\langle \sqrt{7}/5, 3/5, 3/5 \rangle$.

14.2.33

$$\begin{aligned} (t^{12} + 3t)\mathbf{u}'(t) + \mathbf{u}(t)(12t^{11} + 3) &= (t^{12} + 3t) \langle 6t^2, 2t, 0 \rangle + \langle 2t^3, (t^2 - 1), -8 \rangle (12t^{11} + 3) \\ &= \langle 30t^{14} + 24t^3, 14t^{13} - 12t^{11} + 9t^2 - 3, -96t^{11} - 24 \rangle. \end{aligned}$$

14.2.34 $(4t^8 - 6t^3)\mathbf{v}'(t) + \mathbf{v}(t)(32t^7 - 18t^2) = (4t^8 - 6t^3) \langle e^t, -2e^{-t}, -2e^{2t} \rangle + \langle e^t, 2e^{-t}, -e^{2t} \rangle (32t^7 - 18t^2) = \langle (4t^8 + 32t^7 - 6t^3 - 18t^2)e^t, (64t^7 - 36t^2 - 8t^8 + 12t^3)e^{-t}, (-8t^8 - 32t^7 + 12t^3 + 18t^2)e^{2t} \rangle$.

14.2.35

$$\begin{aligned} \mathbf{u}'(t^4 - 2t) \cdot (4t^3 - 2) &= \langle 6(t^4 - 2t)^2, 2(t^4 - 2t), 0 \rangle (4t^3 - 2) = \langle 6(t^4 - 2t)^2(4t^3 - 2), 2(t^4 - 2t)(4t^3 - 2), 0 \rangle \\ &= 4t(2t^3 - 1)(t^3 - 2) \langle 3t(t^3 - 2), 1, 0 \rangle. \end{aligned}$$

14.2.36 $\mathbf{v}'(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = \langle e^{\sqrt{t}}, -2e^{-\sqrt{t}}, -2e^{2\sqrt{t}} \rangle \cdot \frac{1}{2\sqrt{t}} = \left\langle \frac{e^{\sqrt{t}}}{2\sqrt{t}}, -\frac{1}{\sqrt{t}e^{\sqrt{t}}}, -\frac{e^{2\sqrt{t}}}{\sqrt{t}} \right\rangle$.

14.2.37 $\mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{v}(t) \cdot \mathbf{u}'(t) = \langle 2t^3, (t^2 - 1), -8 \rangle \cdot \langle e^t, -2e^{-t}, -2e^{2t} \rangle + \langle e^t, 2e^{-t}, -e^{2t} \rangle \cdot \langle 6t^2, 2t, 0 \rangle = 2t^3e^t - 2(t^2 - 1)e^{-t} + 16e^{2t} + 6t^2e^t + 4te^{-t} + 0 = e^t(2t^3 + 6t^2) - 2e^{-t}(t^2 - 2t - 1) + 16e^{2t}$.

$$\begin{aligned} 14.2.38 \quad \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t) &= \langle 2t^3, (t^2 - 1), -8 \rangle \times \langle e^t, -2e^{-t}, -2e^{2t} \rangle + \langle 6t^2, 2t, 0 \rangle \times \langle e^t, 2e^{-t}, -e^{2t} \rangle = \\ &= \langle -2e^{2t}t^2 - 16e^{-t} + 2e^{2t}, 4e^{2t}t^3 - 8e^t, -4e^{-t}t^3 - e^t t^2 + e^t \rangle + \langle -2e^{2t}t, 6e^{2t}t^2, 12e^{-t}t^2 - 2e^t t \rangle = \\ &= \langle -2e^{2t}t^2 - 2e^{2t}t - 16e^{-t} + 2e^{2t}, 4e^{2t}t^3 + 6e^{2t}t^2 - 8e^t, -4e^{-t}t^3 + 12e^{-t}t^2 - e^t t^2 - 2e^t t + e^t \rangle. \end{aligned}$$

$$14.2.39 \quad \text{This is equal to } \mathbf{u}(0) \cdot \mathbf{v}'(0) + \mathbf{u}'(0) \cdot \mathbf{v}(0) = \langle 0, 1, 1 \rangle \cdot \langle 1, 1, 2 \rangle + \langle 0, 7, 1 \rangle \cdot \langle 0, 1, 1 \rangle = 1 + 2 + 7 + 1 = 11.$$

$$14.2.40 \quad \text{This is equal to } \mathbf{u}(0) \times \mathbf{v}'(0) + \mathbf{u}'(0) \times \mathbf{v}(0) = \langle 0, 1, 1 \rangle \times \langle 1, 1, 2 \rangle + \langle 0, 7, 1 \rangle \times \langle 0, 1, 1 \rangle = \langle 1, 1, -1 \rangle + \langle 6, 0, 0 \rangle = \langle 7, 1, -1 \rangle.$$

$$14.2.41 \quad \text{This is equal to } -\sin(0) \cdot \mathbf{u}(0) + \cos(0) \cdot \mathbf{u}'(0) = \langle 0, 7, 1 \rangle.$$

$$14.2.42 \quad \text{This is equal to } \mathbf{u}'(0) \cos 0 = \langle 0, 7, 1 \rangle.$$

$$14.2.43 \quad \mathbf{v}'(e^t) \cdot e^t = e^t \langle 2e^t, -2, 0 \rangle = \langle 2e^{2t}, -2e^t, 0 \rangle.$$

$$14.2.44 \quad \mathbf{u}'(t^3) \cdot 3t^2 = 3t^2 \langle 0, 1, 2t^3 \rangle = \langle 0, 3t^2, 6t^5 \rangle.$$

$$14.2.45 \quad \mathbf{v}'(g(t))g'(t) = \langle 4\sqrt{t}, -2, 0 \rangle \left(\frac{1}{\sqrt{t}} \right) = \langle 4, -2/\sqrt{t}, 0 \rangle.$$

$$14.2.46 \quad g(t)\mathbf{v}'(t) + \mathbf{v}(t)g'(t) = (2\sqrt{t}) \langle 2t, -2, 0 \rangle + \langle t^2, -2t, 1 \rangle \left(\frac{1}{\sqrt{t}} \right) = \langle 5t^{3/2}, -6\sqrt{t}, t^{-1/2} \rangle.$$

$$14.2.47 \quad \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t) = \langle 1, t, t^2 \rangle \times \langle 2t, -2, 0 \rangle + \langle 0, 1, 2t \rangle \times \langle t^2, -2t, 1 \rangle = \langle 2t^2, 2t^3, -2t^2 - 2 \rangle + \langle 4t^2 + 1, 2t^3, -t^2 \rangle = \langle 6t^2 + 1, 4t^3, -3t^2 - 2 \rangle$$

$$14.2.48 \quad \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t) = \langle 1, t, t^2 \rangle \cdot \langle 2t, -2, 0 \rangle + \langle 0, 1, 2t \rangle \cdot \langle t^2, -2t, 1 \rangle = (2t - 2t + 0) + 0 - 2t + 2t = 0.$$

$$14.2.49 \quad \langle t^2, 2t^2, -2t^3 \rangle \cdot \langle e^t, 2e^t, 3e^{-t} \rangle + \langle 2t, 4t, -6t^2 \rangle \cdot \langle e^t, 2e^t, -3e^{-t} \rangle = t^2 e^t + 4t^2 e^t - 6t^3 e^{-t} + 2te^t + 8te^t + 18t^2 e^{-t} = 5t^2 e^t + 10te^t - 6t^3 e^{-t} + 18t^2 e^{-t}.$$

14.2.50

$$\begin{aligned} &\langle t^3, -2t, -2 \rangle \times \langle 1, -2t, -3t^2 \rangle + \langle 3t^2, -2, 0 \rangle \times \langle t, -t^2, -t^3 \rangle \\ &= \langle 6t^3 - 4t, 3t^5 - 2, 2t - 2t^4 \rangle + \langle 2t^3, 3t^5, 2t - 3t^4 \rangle = \langle 8t^3 - 4t, 6t^5 - 2, 4t - 5t^4 \rangle. \end{aligned}$$

$$14.2.51 \quad \langle 3t^2, \sqrt{t}, -2/t \rangle \cdot \langle -\sin t, 2 \cos 2t, -3 \rangle + \langle 6t, 1/(2\sqrt{t}), 2/t^2 \rangle \cdot \langle \cos t, \sin 2t, -3t \rangle = -3t^2 \sin t + 2\sqrt{t} \cos 2t + 6t \cos t + \sin(2t)/(2\sqrt{t}).$$

$$\begin{aligned} 14.2.52 \quad &\langle t^3, 6, -2\sqrt{t} \rangle \times \langle 3, -24t, 12t^{-3} \rangle + \langle 3t^2, 0, -1/\sqrt{t} \rangle \times \langle 3t, -12t^2, -6t^{-2} \rangle = \\ &\left\langle \frac{72}{t^3} - 48t^{3/2}, -6\sqrt{t} - 12, -24t^4 - 18 \right\rangle + \langle -12t^{3/2}, 18 - 3\sqrt{t}, -36t^4 \rangle = \left\langle \frac{72}{t^3} - 60t^{3/2}, 6 - 9\sqrt{t}, -60t^4 - 18 \right\rangle. \end{aligned}$$

$$14.2.53 \quad \mathbf{r}'(t) = \langle 2t, 1, 0 \rangle. \quad \mathbf{r}''(t) = \langle 2, 0, 0 \rangle. \quad \mathbf{r}'''(t) = \langle 0, 0, 0 \rangle.$$

$$14.2.54 \quad \mathbf{r}'(t) = \langle 36t^{11} - 2t, 8t^7 + 3t^2, -4t^{-5} \rangle. \quad \mathbf{r}''(t) = \langle 396t^{10} - 2, 56t^6 + 6t, 20t^{-6} \rangle. \\ \mathbf{r}'''(t) = \langle 3960t^9, 336t^5 + 6, -120t^{-7} \rangle.$$

$$14.2.55 \quad \mathbf{r}'(t) = \langle -3 \sin 3t, 4 \cos 4t, -6 \sin 6t \rangle. \quad \mathbf{r}''(t) = \langle -9 \cos 3t, -16 \sin 4t, -36 \cos 6t \rangle. \\ \mathbf{r}'''(t) = \langle 27 \sin 3t, -64 \cos 4t, 216 \sin 6t \rangle.$$

$$14.2.56 \quad \mathbf{r}'(t) = \langle 4e^{4t}, -8e^{-4t}, -2e^{-t} \rangle. \quad \mathbf{r}''(t) = \langle 16e^{4t}, 32e^{-4t}, 2e^{-t} \rangle. \quad \mathbf{r}'''(t) = \langle 64e^{4t}, -128e^{-4t}, -2e^{-t} \rangle.$$

$$\begin{aligned} 14.2.57 \quad &\mathbf{r}'(t) = \left\langle \frac{1}{2\sqrt{t+4}}, \frac{1}{(t+1)^2}, 2e^{-t^2} t \right\rangle. \quad \mathbf{r}''(t) = \left\langle -\frac{1}{4(t+4)^{3/2}}, -\frac{2}{(t+1)^3}, e^{-t^2} (2 - 4t^2) \right\rangle. \\ &\mathbf{r}'''(t) = \left\langle \frac{3}{8(t+4)^{5/2}}, \frac{6}{(t+1)^4}, 4e^{-t^2} t (2t^2 - 3) \right\rangle. \end{aligned}$$

$$\begin{aligned} 14.2.58 \quad \mathbf{r}'(t) &= \left\langle \sec^2(t), 1 - \frac{1}{t^2}, -\frac{1}{t+1} \right\rangle, \quad \mathbf{r}''(t) = \left\langle 2 \tan(t) \sec^2(t), \frac{2}{t^3}, \frac{1}{(t+1)^2} \right\rangle. \\ \mathbf{r}'''(t) &= \left\langle 2 \sec^4(t) + 4 \tan^2(t) \sec^2(t), -\frac{6}{t^4}, -\frac{2}{(t+1)^3} \right\rangle. \end{aligned}$$

$$14.2.59 \quad \int \langle t^4 - 3t, 2t - 1, 10 \rangle dt = \langle t^5/5 - 3t^2/2, t^2 - t, 10t \rangle + \mathbf{C}.$$

$$14.2.60 \quad \int \langle 5t^{-4} - t^2, t^6 - 4t^3, 2/t \rangle dt = \langle (-5/3)t^{-3} - t^3/3, t^7/7 - t^4, 2 \ln |t| \rangle + \mathbf{C}.$$

$$14.2.61 \quad \int \langle 2 \cos t, 2 \sin 3t, 4 \cos 8t \rangle dt = \langle 2 \sin t, (-2/3) \cos 3t, (1/2) \sin 8t \rangle + \mathbf{C}.$$

$$14.2.62 \quad \int \left\langle te^t, t \sin t^2, -\frac{2t}{\sqrt{t^2+4}} \right\rangle dt = \left\langle (t-1)e^t, -\frac{1}{2} \cos t^2, -2\sqrt{t^2+4} \right\rangle + \mathbf{C}.$$

$$14.2.63 \quad \int \left\langle e^{3t}, \frac{1}{1+t^2}, \frac{-1}{\sqrt{2t}} \right\rangle dt = \left\langle e^{3t}/3, \tan^{-1} t, -\sqrt{2t} \right\rangle + \mathbf{C}.$$

$$14.2.64 \quad \int \left\langle 2^t, \frac{1}{1+2t}, \ln t \right\rangle dt = \left\langle 2^t/(\ln 2), \frac{1}{2} \ln |1+2t|, t \ln t - t \right\rangle + \mathbf{C}.$$

$$14.2.65 \quad \int \langle e^t, \sin t, \sec^2 t \rangle dt = \langle e^t, -\cos t, \tan t \rangle + \mathbf{C}. \text{ Because } \mathbf{r}(0) = \langle 2, 2, 2 \rangle = \langle 1, -1, 0 \rangle + \mathbf{C}, \text{ we have } \mathbf{C} = \langle 1, 3, 2 \rangle. \text{ Thus, } \mathbf{r}(t) = \langle e^t, -\cos t, \tan t \rangle + \langle 1, 3, 2 \rangle = \langle 1 + e^t, 3 - \cos t, 2 + \tan t \rangle.$$

$$14.2.66 \quad \int \langle 0, 2, 2t \rangle dt = \langle 0, 2t, t^2 \rangle + \mathbf{C}. \text{ Because } \mathbf{r}(1) = \langle 4, 3, -5 \rangle, \text{ we have } \mathbf{r}(1) = \langle 0, 2, 1 \rangle + \mathbf{C} = \langle 4, 3, -5 \rangle, \text{ so } \mathbf{C} = \langle 4, 1, -6 \rangle. \text{ Thus, } \mathbf{r}(t) = \langle 4, 2t + 1, t^2 - 6 \rangle.$$

$$14.2.67 \quad \int \langle 1, 2t, 3t^2 \rangle dt = \langle t, t^2, t^3 \rangle + \mathbf{C}. \text{ Because } \mathbf{r}(1) = \langle 4, 3, -5 \rangle, \text{ we have } \langle 1, 1, 1 \rangle + \mathbf{C} = \langle 4, 3, -5 \rangle, \text{ so } \mathbf{C} = \langle 3, 2, -6 \rangle, \text{ and } \mathbf{r}(t) = \langle t + 3, t^2 + 2, t^3 - 6 \rangle.$$

$$14.2.68 \quad \int \langle \sqrt{t}, \cos \pi t, 4/t \rangle dt = \left\langle (2/3)t^{3/2}, (\sin \pi t)/\pi, 4 \ln |t| \right\rangle + \mathbf{C}. \text{ Because } \mathbf{r}(1) = \langle 2, 3, 4 \rangle, \text{ we have } \langle 2/3, 0, 0 \rangle + \mathbf{C} = \langle 2, 3, 4 \rangle, \text{ so } \mathbf{C} = \langle 4/3, 3, 4 \rangle, \text{ and } \mathbf{r}(t) = \left\langle (2/3)t^{4/2} + 1/3, (\sin \pi t)/\pi + 3, 4 \ln |t| + 4 \right\rangle.$$

$$14.2.69 \quad \int \langle e^{2t}, 1 - 2e^{-t}, 1 - 2e^t \rangle dt = \langle e^{2t}/2, t + 2e^{-t}, t - 2e^t \rangle + \mathbf{C}. \text{ Because } \mathbf{r}(0) = \langle 1, 1, 1 \rangle, \text{ we have } \langle 1/2, 2, -2 \rangle + \mathbf{C} = \langle 1, 1, 1 \rangle, \text{ so } \mathbf{C} = \langle 1/2, -1, 3 \rangle, \text{ and } \mathbf{r}(t) = \langle e^{2t}/2 + 1/2, t + 2e^{-t} - 1, t - 2e^t + 3 \rangle.$$

$$14.2.70 \quad \int \left\langle \frac{t}{(t^2+1)}, te^{-t^2}, -\frac{2t}{\sqrt{t^2+4}} \right\rangle dt = \left\langle \frac{1}{2} \ln(t^2+1), -\frac{e^{-t^2}}{2}, -2\sqrt{t^2+4} \right\rangle + \mathbf{C}. \text{ Because } \mathbf{r}(0) = \langle 1, 3/2, -3 \rangle, \text{ we have } \langle 0, -1/2, -4 \rangle + \mathbf{C} = \langle 1, 3/2, -3 \rangle, \text{ so } \mathbf{C} = \langle 1, 2, 1 \rangle, \text{ and } \mathbf{r}(t) = \left\langle \frac{1}{2} \ln(t^2+1) + 1, -\frac{e^{-t^2}}{2} + 2, -2\sqrt{t^2+4} + 1 \right\rangle.$$

$$14.2.71 \quad \int_{-1}^1 \langle 1, t, 3t^2 \rangle dt = \langle t, t^2/2, t^3 \rangle \Big|_{-1}^1 = \langle 2, 0, 2 \rangle.$$

$$14.2.72 \quad \int_1^4 \langle 6t^2, 8t^3, 9t^2 \rangle dt = \langle 2t^3, 2t^4, 3t^3 \rangle \Big|_1^4 = \langle 128, 512, 192 \rangle - \langle 2, 2, 3 \rangle = \langle 126, 510, 189 \rangle.$$

$$14.2.73 \int_0^{\ln 2} \langle e^t, e^t \cos \pi e^t, 0 \rangle dt = \left\langle e^t, \frac{\sin \pi e^t}{\pi}, 0 \right\rangle \bigg|_0^{\ln 2} = \langle 2, 0, 0 \rangle - \langle 1, 0, 0 \rangle = \langle 1, 0, 0 \rangle = \mathbf{i}.$$

14.2.74

$$\begin{aligned} \int_{1/2}^1 \left\langle \frac{3}{1+2t}, 0, -\pi \csc^2(\pi t/2) \right\rangle dt &= \left\langle \frac{3}{2} \ln(1+2t), 0, 2 \cot(\pi t/2) \right\rangle \bigg|_{1/2}^1 \\ &= \langle (3/2) \ln 3, 0, 0 \rangle - \langle (3/2) \ln 2, 0, 2 \rangle = \langle (3/2) \ln(3/2), 0, -2 \rangle. \end{aligned}$$

$$14.2.75 \int_{-\pi}^{\pi} \langle \sin t, \cos t, 2t \rangle dt = \langle -\cos t, \sin t, t^2 \rangle \bigg|_{-\pi}^{\pi} = \langle 0, 0, 0 \rangle.$$

$$14.2.76 \int_0^{\ln 2} \langle e^{-t}, 2e^{2t}, -4e^t \rangle dt = \langle -e^{-t}, e^{2t}, -4e^t \rangle \bigg|_0^{\ln 2} = \langle 1/2, 3, -4 \rangle.$$

$$\begin{aligned} 14.2.77 \int_0^2 \langle te^t, 2te^t, -te^t \rangle dt &= \langle (t-1)e^t, 2(t-1)e^t, -(t-1)e^t \rangle \bigg|_0^2 = \langle e^2 + 1, 2e^2 + 2, -e^2 - 1 \rangle \\ &= (e^2 + 1) \langle 1, 2, -1 \rangle. \end{aligned}$$

$$14.2.78 \int_0^{\pi/4} \langle \sec^2 t, -2 \cos t, -1 \rangle dt = \langle \tan t, -2 \sin t, -t \rangle \bigg|_0^{\pi/4} = \langle 1, -\sqrt{2}, -\pi/4 \rangle.$$

14.2.79

- False. For example, if $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, then $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ is not parallel to $\mathbf{r}(t)$, and is in fact perpendicular to it.
- True. $\mathbf{r}'(t) = \langle 1, 2t - 2, -\pi \sin \pi t \rangle \neq \langle 0, 0, 0 \rangle$. Each component function is differentiable, and the derivative is never $\langle 0, 0, 0 \rangle$, so the function is smooth by definition.
- True. This follows because $\int_{-a}^a o(x) dx = 0$ for any odd function $o(x)$.

14.2.80 $\mathbf{r}'(t) = \langle e^t, 2e^{2t}, 3e^{3t} \rangle$, so $\mathbf{r}'(0) = \langle 1, 2, 3 \rangle$. We have $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$, so the tangent line is given by $\langle 1+t, 1+2t, 1+3t \rangle$.

14.2.81 $\mathbf{r}'(t) = \langle -\sin t, 2 \cos 2t, 1 \rangle$, so $\mathbf{r}'(\pi/2) = \langle -1, -2, 1 \rangle$. We have $\mathbf{r}(\pi/2) = \langle 2, 3, \pi/2 \rangle$, so the tangent line is given by $\langle 2-t, 3-2t, \pi/2+t \rangle$.

14.2.82 $\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{2t+1}}, \pi \cos \pi t, 0 \right\rangle$, so $\mathbf{r}'(4) = \langle 1/3, \pi, 0 \rangle$. We have $\mathbf{r}(4) = \langle 3, 0, 4 \rangle$, so the tangent line is given by $\langle 3 + (1/3)t, \pi t, 4 \rangle$.

14.2.83 $\mathbf{r}'(t) = \langle 3, 7, 2t \rangle$, so $\mathbf{r}'(1) = \langle 3, 7, 2 \rangle$. We have $\mathbf{r}(1) = \langle 2, 9, 1 \rangle$, so the tangent line is given by $\langle 2+3t, 9+7t, 1+2t \rangle$.

14.2.84 $\mathbf{r}'(t) = \langle -a \sin t, a \cos t \rangle$, and $\mathbf{r}(t) \cdot \mathbf{r}'(t) = \langle a \cos t, a \sin t \rangle \cdot \langle -a \sin t, a \cos t \rangle = -a^2 \cos t \sin t + a^2 \sin t \cos t = 0$. So $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for all t .

14.2.85 $\mathbf{r}'(t) = \langle 2at, 1 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $(2at)(at^2 + 1) + t(1) = 0$, which occurs only for $t = 0$ because $2a^2t^2 + 2a + 1 > 0$ for all t . The corresponding point on the parabola is $(1, 0)$.

14.2.86 $\mathbf{r}'(t) = \langle 1/(2\sqrt{t}), 0, 1 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $1/2 + 0 + t = 0$, so for $t = -1/2$, which isn't in the domain. So the vectors $\mathbf{r}$ and $\mathbf{r}'$ are never orthogonal.

14.2.87 $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $-\sin t \cos t + \sin t \cos t + t = 0$, which occurs only for $t = 0$. So the only point on the helix where these vectors are orthogonal is at $t = 0$. This corresponds to the point $(1, 0, 0)$.

14.2.88 $\mathbf{r}'(t) = \langle -2 \sin t, 8 \cos t, 0 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $-4 \sin t \cos t + 64 \sin t \cos t + 0 = 0$, which occurs when $60 \sin t \cos t = 0$, or $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.

The corresponding points are $(2, 0, 0)$, $(0, 8, 0)$, $(-2, 0, 0)$, and $(0, -8, 0)$.

14.2.89 Note that $\mathbf{r}(t) = \langle a_1 t, a_2 t, a_3 t \rangle = t \langle a_1, a_2, a_3 \rangle$ where the a_i 's are real numbers has this property because $\mathbf{r}'(t) = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{r}(t)$ is a multiple of $\mathbf{r}'(t)$.

Also, $\mathbf{r}(t) = \langle a_1 e^{kt}, a_2 e^{kt}, a_3 e^{kt} \rangle$ where k is a real number has this property, as its derivative is k times itself.

14.2.90 If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ lies on the sphere $x^2 + y^2 + z^2 = a^2$, then (differentiating with respect to t) we have $2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0$, so $\langle x, y, z \rangle \cdot \langle x', y', z' \rangle = 0$, so $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$.

If $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, then $x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt} = 0$, so $2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0$, so by integrating both sides with respect to t , we have $x^2 + y^2 + z^2 = K$ for some constant $K > 0$, so the curve lies on a sphere.

14.2.91

a. $\mathbf{r}(t) \cdot \mathbf{r}'(t) = (a^2 + b^2 + c^2)t = |\mathbf{r}(t)| |\mathbf{r}'(t)| \cos \theta$, so $\cos \theta = \frac{(a^2 + b^2 + c^2)t}{\sqrt{a^2 + b^2 + c^2}t \cdot \sqrt{a^2 + b^2 + c^2}} = 1$, so $\theta = 0$.

b. $\mathbf{r}(t) \cdot \mathbf{r}'(t) = ax_0 + by_0 + cz_0 + (a^2 + b^2 + c^2)t = |\mathbf{r}(t)| |\mathbf{r}'(t)| \cos \theta$, so

$$\cos \theta = \frac{ax_0 + by_0 + cz_0 + (a^2 + b^2 + c^2)t}{\sqrt{(x_0 + at)^2 + (y_0 + bt)^2 + (z_0 + ct)^2} \cdot \sqrt{a^2 + b^2 + c^2}}.$$

Because x_0 , y_0 , and z_0 are not all 0, $\cos \theta$ depends on t .

c. In part a, the curve is a straight line through the origin, so the position vector and the tangent vector are parallel for all t . In part b, the line is not through the origin, so the tangent vector (which is the direction vector for the line) is not parallel to the position vector.

14.2.92

$$\begin{aligned} \frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) &= \frac{d}{dt} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t) \rangle \\ &= \langle u'_1(t) + v'_1(t), u'_2(t) + v'_2(t), u'_3(t) + v'_3(t) \rangle = \mathbf{u}'(t) + \mathbf{v}'(t). \end{aligned}$$

14.2.93

$$\begin{aligned} \frac{d}{dt}(f(t)\mathbf{u}(t)) &= \frac{d}{dt} \langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t) \rangle \\ &= \langle f'(t)u_1(t) + f(t)u'_1(t), f'(t)u_2(t) + f(t)u'_2(t), f'(t)u_3(t) + f(t)u'_3(t) \rangle \\ &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t). \end{aligned}$$

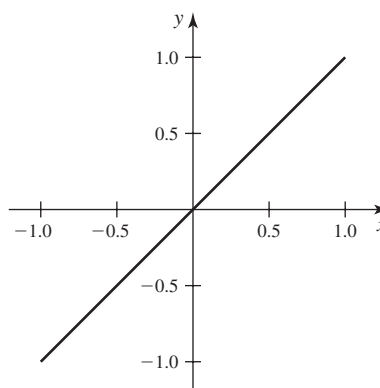
14.2.94

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \frac{d}{dt} \langle u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - u_2v_1 \rangle \\ &= \langle u'_2v_3 + u_2v'_3 - (v'_2u_3 + v_2u'_3), u'_3v_1 + u_3v'_1 - (u'_1v_3 + u_1v'_3), u'_1v_2 + u_1v'_2 - (v'_1u_2 + v_1u'_2) \rangle \\ &= \langle u'_2v_3 - v_2u'_3 + u_2v'_3 - v'_2u_3, u_3v'_1 - u_1v'_3 + u'_3v_1 - u'_1v_3, u'_1v_2 - v_1u'_2 + u_1v'_2 - v'_1u_2 \rangle \\ &= \langle (u'_2v_3 - v_2u'_3), (u_3v'_1 - u_1v'_3), (u'_1v_2 - v_1u'_2) \rangle + \langle (u_2v'_3 - v'_2u_3), (u'_3v_1 - u'_1v_3), (u_1v'_2 - v'_1u_2) \rangle \\ &= (\mathbf{u}'(t) \times \mathbf{v}(t)) + (\mathbf{u}(t) \times \mathbf{v}'(t)) \end{aligned}$$

14.2.95

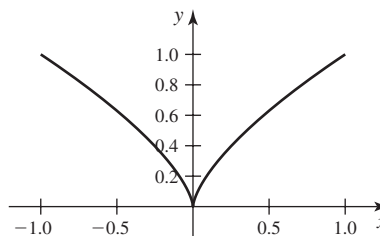
a.

$\mathbf{r}'(t) = \langle 3t^2, 3t^2 \rangle$, so $\mathbf{r}'(0) = \langle 0, 0 \rangle$. There is no cusp because $\lim_{t \rightarrow 0} \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{3t^2} = 1$ exists.



b.

$\mathbf{r}'(t) = \langle 3t^2, 2t \rangle$, so $\mathbf{r}'(0) = \langle 0, 0 \rangle$. There is a cusp because $\lim_{t \rightarrow 0} \frac{dy}{dx} = \lim_{t \rightarrow 0} \frac{dy/dt}{dx/dt} = \lim_{t \rightarrow 0} \frac{2t}{3t^2} = \lim_{t \rightarrow 0} \frac{2}{3t}$ does not exist.



c. The curve $\mathbf{r}(t)$ for $-\infty < t < \infty$ traces out the whole curve $y = x^2$, while the curve $\mathbf{p}(t)$ only traces out the part in the first quadrant, because $x = t^2 > 0$ for all t .

d. $\mathbf{r}'(t) = \langle mt^{m-1}, nt^{n-1} \rangle$, so $\mathbf{r}'(0) = \langle 0, 0 \rangle$.

Assume $m > n$. There is a cusp because

$$\lim_{t \rightarrow 0} \frac{dy}{dx} = \lim_{t \rightarrow 0} \frac{dy/dt}{dx/dt} = \lim_{t \rightarrow 0} \frac{nt^{n-1}}{mt^{m-1}} = \lim_{t \rightarrow 0} (n/m) \frac{1}{t^{m-n}},$$

which does not exist.

Now assume $m < n$. There is a cusp because

$$\lim_{t \rightarrow 0} \frac{dx}{dy} = \lim_{t \rightarrow 0} \frac{dx/dt}{dy/dt} = \lim_{t \rightarrow 0} \frac{mt^{m-1}}{nt^{n-1}} = \lim_{t \rightarrow 0} (m/n) \frac{1}{t^{n-m}},$$

which does not exist.

14.3 Motion in Space

14.3.1 The velocity is the derivative of position, the speed is the magnitude of velocity, and the acceleration is the derivative of velocity.

14.3.2 For the circle $\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle$ for $a > 0$, the two vectors are orthogonal, with the velocity vector tangent to the circle.

14.3.3 $m\mathbf{a}(t) = \mathbf{F}(t)$.

14.3.4 $\mathbf{ma}(t) = \langle mx''(t), my''(t), mz''(t) \rangle = \mathbf{F}(t) = \langle 0, 0, -mg \rangle.$

14.3.5 Integrate the acceleration to find an expression for the velocity plus a constant, and then use the initial velocity condition to find the constant.

14.3.6 Integrate the velocity to find an expression for the position plus a constant, and then use the initial position condition to find the constant.

14.3.7

a. When $96 - 32t = 0$, so $t = 3$.

b. $\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int \langle 60, 96 - 32t \rangle dt = \langle 60t, 96t - 16t^2 \rangle + \mathbf{C}$. Then because $\mathbf{r}(0) = \langle 0, 3 \rangle$, we have $\mathbf{r}(t) = \langle 60t, -16t^2 + 96t + 3 \rangle$.

14.3.8

a. The ball hits the ground when $y = 0$, so for $-16t^2 + 31t + 2 = 0$. This factors as $-(t - 2)(16t + 1) = 0$, so the ball hits the ground for $t = 2$.

b. We need to find x when $t = 2$. Because $x = 40t$, we have $x = 80$ when the ball hits the ground.

14.3.9

a. $\mathbf{v}(t) = \langle 6t, 8t \rangle$, so the speed is $\sqrt{36t^2 + 64t^2} = \sqrt{100t^2} = 10t$.

b. $\mathbf{a}(t) = \langle 6, 8 \rangle$.

14.3.10

a. $\mathbf{v}(t) = \langle 5t, 12t \rangle$, so the speed is $\sqrt{25t^2 + 144t^2} = \sqrt{169t^2} = 13t$.

b. $\mathbf{a}(t) = \langle 5, 12 \rangle$.

14.3.11

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2, -4 \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{20} = 2\sqrt{5}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 0 \rangle$.

14.3.12

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2t, 6t^2 \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{4t^2 + 36t^4} = 2|t|\sqrt{1 + 9t^2}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -2, 12t \rangle$.

14.3.13

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 8 \cos t, -8 \sin t \rangle$, so the speed is $|\mathbf{r}'(t)| = 8$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -8 \sin t, -8 \cos t \rangle$.

14.3.14

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3 \sin t, 4 \cos t \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{9 \sin^2 t + 16 \cos^2 t} = \sqrt{9 + 7 \cos^2 t}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -3 \cos t, -4 \sin t \rangle$.

14.3.15

a. $\mathbf{v}(t) = \langle 2t, 2t, t \rangle$, so the speed is $\sqrt{4t^2 + 4t^2 + t^2} = 3t$.

b. $\mathbf{a}(t) = \langle 2, 2, 1 \rangle$.

14.3.16

a. $\mathbf{v}(t) = \langle 4e^{2t}, 2e^{2t}, 4e^{2t} \rangle$, so the speed is $\sqrt{16e^{4t} + 4e^{4t} + 16e^{4t}} = 6e^{2t}$.

b. $\mathbf{a}(t) = \langle 8e^{2t}, 4e^{2t}, 8e^{2t} \rangle$.

14.3.17

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -4, 6 \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{1 + 16 + 36} = \sqrt{53}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 0, 0 \rangle$.

14.3.18

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3 \cos t, -5 \sin t, 4 \cos t \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{25 \cos^2 t + 25 \sin^2 t} = 5$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -3 \sin t, -5 \cos t, -4 \sin t \rangle$.

14.3.19

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 0, 2t, -e^{-t} \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{4t^2 + e^{-2t}}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 2, e^{-t} \rangle$.

14.3.20

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -26 \sin 2t, 24 \cos 2t, 10 \cos 2t \rangle$, so the speed is

$$|\mathbf{r}'(t)| = \sqrt{676 \sin^2 2t + 576 \cos^2 2t + 100 \cos^2 2t} = 26.$$

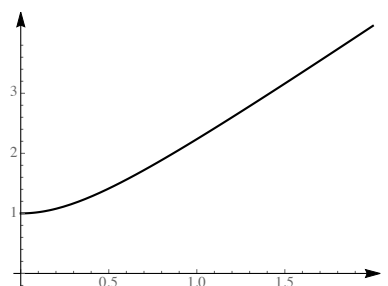
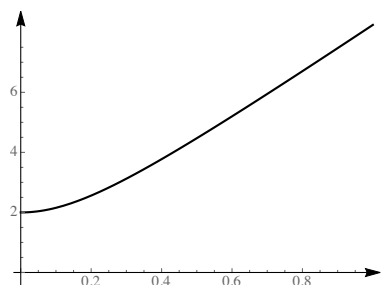
b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -52 \cos 2t, -48 \sin 2t, -20 \sin 2t \rangle$.

14.3.21

a. The interval must be shrunk by a factor of 2, so $[c, d] = [0, 1]$.

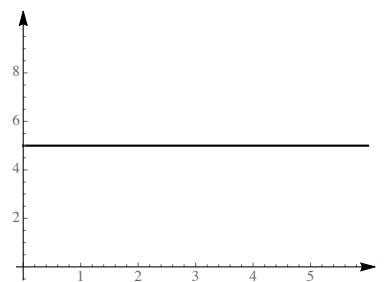
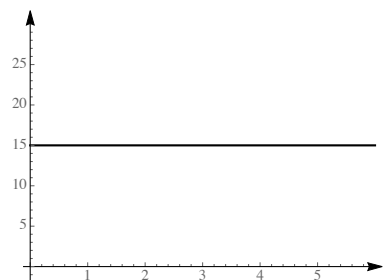
b. $\mathbf{r}'(t) = \langle 1, 2t \rangle$, and $\mathbf{R}'(t) = \langle 2, 8t \rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2}$ and $|\mathbf{R}'(t)| = 2\sqrt{1 + 16t^2}$.

The speed of $\mathbf{r}(t)$.The speed of $\mathbf{R}(t)$.

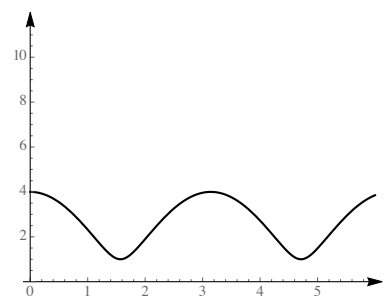
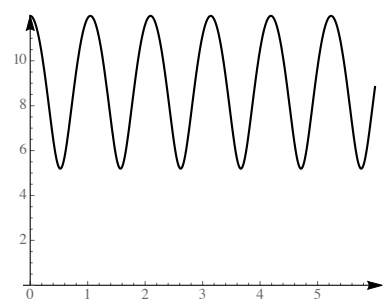
14.3.22

- a. The interval must be shrunk by a factor of 3, so $[c, d] = [0, 2]$.
- b. $\mathbf{r}'(t) = \langle 3, 4 \rangle$, and $\mathbf{R}'(t) = \langle 9, 12 \rangle$.
- c. $|\mathbf{r}'(t)| = \sqrt{9 + 16} = 5$ and $|\mathbf{R}'(t)| = \sqrt{81 + 144} = 15$.

The speed of $\mathbf{r}(t)$.The speed of $\mathbf{R}(t)$.

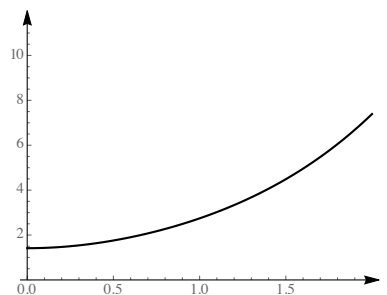
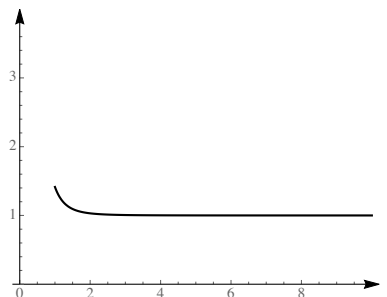
14.3.23

- a. The interval must be shrunk by a factor of $1/3$, so $[c, d] = [0, 2\pi/3]$.
- b. $\mathbf{r}'(t) = \langle -\sin t, 4 \cos t \rangle$, and $\mathbf{R}'(t) = \langle -3 \sin 3t, 12 \cos 3t \rangle$.
- c. $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + 16 \cos^2 t}$ and $|\mathbf{R}'(t)| = 3\sqrt{\sin^2 3t + 16 \cos^2 3t}$.

The speed of $\mathbf{r}(t)$.The speed of $\mathbf{R}(t)$.

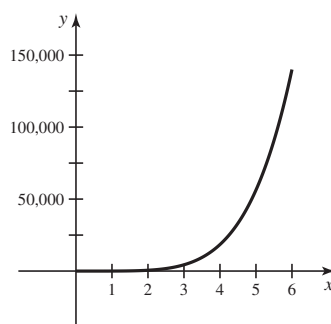
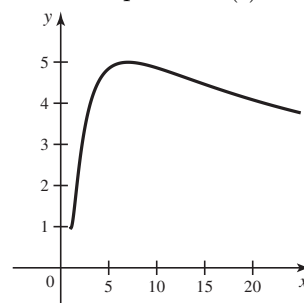
14.3.24

- a. Because $e^0 = 1$ and $e^{\ln 10} = 10$, we have $[c, d] = [1, 10]$.
- b. $\mathbf{r}'(t) = \langle -e^{-t}, e^{-t} \rangle$, and $\mathbf{R}'(t) = \langle -1, 1/t^2 \rangle$.
- c. $|\mathbf{r}'(t)| = \sqrt{e^{2t} + e^{-2t}} = \sqrt{2 \cosh(2t)}$
and $|\mathbf{R}'(t)| = \sqrt{1 + 1/t^4}$.

The speed of $\mathbf{r}(t)$.The speed of $\mathbf{R}(t)$.

14.3.25

- a. Because $e^{0^2} = 1$ and $e^{6^2} = e^{36}$, we have $[c, d] = [1, e^{36}]$.
- b. $\mathbf{r}'(t) = \langle 2t, -8t^3, 18t^5 \rangle$, and $\mathbf{R}'(t) = \langle 1/t, (-4 \ln t)/t, (9 \ln^2 t)/t \rangle$.
- c. $|\mathbf{r}'(t)| = \frac{2t\sqrt{1 + 16t^4 + 81t^8}}{t}$ and $|\mathbf{R}'(t)| = \frac{1}{t}\sqrt{1 + 16 \ln^2 t + 81 \ln^4 t}$.

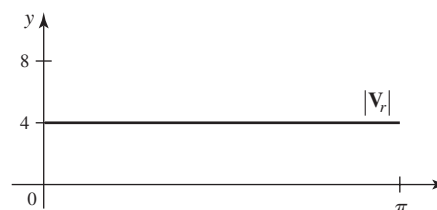
The speed of $\mathbf{r}(t)$.The speed of $\mathbf{R}(t)$.

14.3.26

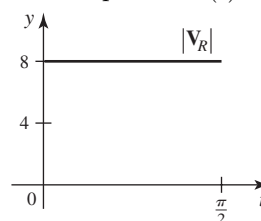
- a. The interval must be shrunk by a factor of $1/2$ so $[c, d] = [0, \pi/2]$.

$$\begin{aligned} \text{b. } \mathbf{r}'(t) &= \langle -4 \sin 2t, 2\sqrt{2} \cos 2t, 2\sqrt{2} \cos 2t \rangle, \\ \mathbf{R}'(t) &= \langle -8 \sin 4t, 4\sqrt{2} \cos 4t, 4\sqrt{2} \cos 4t \rangle. \end{aligned}$$

$$\begin{aligned} \text{c. } |\mathbf{r}'(t)| &= \sqrt{16 \sin^2 2t + 8 \cos^2 2t + 8 \cos^2 2t} = \sqrt{16} = 4, \text{ while we have } |\mathbf{R}'(t)| = \sqrt{64 \sin^2 4t + 32 \cos^2 4t + 32 \cos^2 4t} = 8. \end{aligned}$$



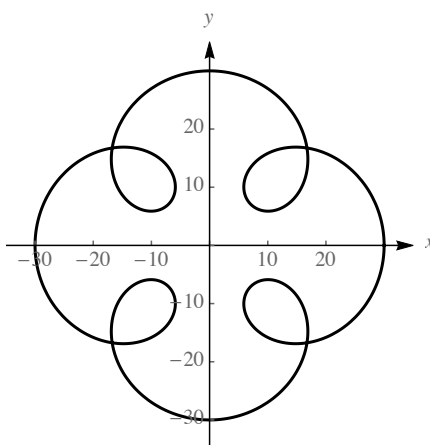
The speed of $\mathbf{r}(t)$.



The speed of $\mathbf{R}(t)$.

14.3.27

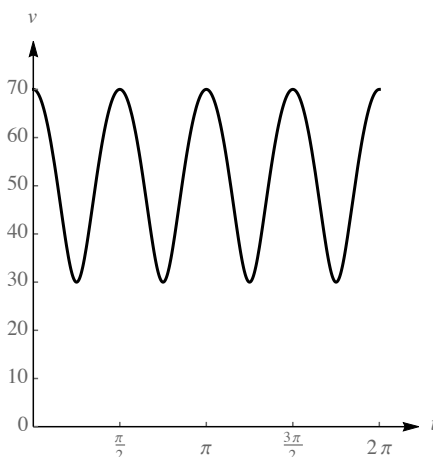
a.



b. $\mathbf{r}'(t) = \langle -20 \sin t - 50 \sin 5t, 20 \cos t + 50 \cos 5t \rangle.$

c.

$$\begin{aligned} |\mathbf{v}(t)| &= \sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)} \\ &= \sqrt{400 \sin^2 t + 2500 \sin^2 5t + 2000 \sin t \sin 5t + 400 \cos^2 t + 2500 \cos^2 5t + 2000 \cos t \cos 5t} \\ &= \sqrt{400 + 2500 + 2000 \sin t \sin 5t + 2000 \cos t \cos 5t} \\ &= 10\sqrt{29 + 20(\sin 5t \sin t + \cos 5t \cos t)} = 10\sqrt{29 + 20 \cos 4t} \end{aligned}$$

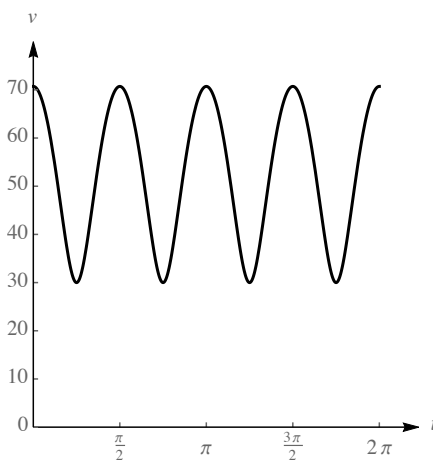


- d. The maximum occurs when $\cos 4t = 1$, and the value is $10\sqrt{49} = 70$, while the minimum occurs when $\cos 4t = -1$ and the value is $10\sqrt{9} = 30$.

14.3.28

- a. Andrea is moving up and down.
- b. $\mathbf{r}'(t) = \langle -20 \sin t - 50 \sin 5t, 20 \cos t + 50 \cos 5t, 10 \cos 2t \rangle$. So, using the result from the previous problem,

$$\begin{aligned} |\mathbf{v}(t)| &= \sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)} \\ &= \sqrt{2900 + 2000 \cos 4t + 100 \cos^2 2t} \\ &= \sqrt{2900 + 2000 \cos 4t + 50 + 50 \cos 4t} = \sqrt{2950 + 2050 \cos 4t} = 5\sqrt{118 + 82 \cos 4t}. \end{aligned}$$



- c. The maximum occurs when $\cos 4t = 1$ and the value is $5\sqrt{200} \approx 70.7$ and the minimum occurs when $\cos 4t = -1$ and the value is $5\sqrt{36} = 30$.

14.3.29 Note that $x^2 + y^2 = 64$, so the trajectory lies on a circle centered at the origin of radius 8. $\mathbf{r}(t) \cdot \mathbf{r}'(t) = \langle 8 \cos 2t, 8 \sin 2t \rangle \cdot \langle -16 \sin 2t, 16 \cos 2t \rangle = -128 \sin 2t \cos 2t + 128 \sin 2t \cos 2t = 0$.

14.3.30 Note that $x^2 + y^2 = 16 \sin^2 t + 4 \cos^2 t = 4 + 12 \sin^2 t$ which is not a constant, so the trajectory does not lie on a circle centered at the origin.

14.3.31 Note that $x^2 + y^2 = (\sin t + \sqrt{3} \cos t)^2 + (\sqrt{3} \sin t - \cos t)^2 = (\sin^2 t + 2 \sin t \sqrt{3} \cos t + 3 \cos^2 t) + (3 \sin^2 t - 2 \cos t \sqrt{3} \sin t + \cos^2 t) = 4$, so the trajectory lies on a circle centered at the origin of radius 2. $\mathbf{r}(t) \cdot \mathbf{r}'(t) = \langle \sin t + \sqrt{3} \cos t, \sqrt{3} \sin t - \cos t \rangle \cdot \langle \cos t - \sqrt{3} \sin t, \sqrt{3} \cos t + \sin t \rangle = (\sin t \cos t - \sqrt{3} \sin^2 t + \sqrt{3} \cos^2 t - 3 \sin t \cos t) + (3 \sin t \cos t + \sqrt{3} \sin^2 t - \sqrt{3} \cos^2 t - \sin t \cos t) = 0$.

14.3.32 $x^2 + y^2 + z^2 = 9 \sin^2 t + 25 \cos^2 t + 16 \sin^2 t = 25$, so the trajectory lies on a sphere centered at the origin of radius 5. $\mathbf{r}(t) \cdot \mathbf{r}'(t) = \langle 3 \sin t, 5 \cos t, 4 \sin t \rangle \cdot \langle 3 \cos t, -5 \sin t, 4 \cos t \rangle = 9 \sin t \cos t - 25 \sin t \cos t + 16 \sin t \cos t = 0$.

14.3.33 $\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} = \sqrt{\frac{25 \sin^2 t + 25 \cos^2 t + 25 \sin^2 2t}{1 + \sin^2 2t}} = \sqrt{\frac{25(1 + \sin^2 2t)}{1 + \sin^2 2t}} = \sqrt{25} = 5$.

14.3.34 $\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} = \sqrt{\frac{16 \cos^2 t + 4t^2 + 16 \sin^2 t}{4 + t^2}} = \sqrt{\frac{4(4 + t^2)}{4 + t^2}} = \sqrt{4} = 2$.

14.3.35 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 0, 1 \rangle dt = \langle 0, t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 2, 3 \rangle$, we have $\mathbf{C} = \langle 2, 3 \rangle$. Thus, $\mathbf{v}(t) = \langle 2, t + 3 \rangle$.

$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 2, t + 3 \rangle dt = \langle 2t, t^2/2 + 3t \rangle + \mathbf{D}$. Because $\mathbf{r}(0) = \langle 0, 0 \rangle$, we have $\mathbf{D} = \langle 0, 0 \rangle$. Therefore, $\mathbf{r}(t) = \langle 2t, t^2/2 + 3t \rangle$.

14.3.36 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 1, 2 \rangle dt = \langle t, 2t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 1 \rangle$, we have $\mathbf{C} = \langle 1, 1 \rangle$. Thus, $\mathbf{v}(t) = \langle t + 1, 2t + 1 \rangle$.

$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle t + 1, 2t + 1 \rangle dt = \langle t^2/2 + t, t^2 + t \rangle + \mathbf{D}$. Because $\mathbf{r}(0) = \langle 2, 3 \rangle$, we have $\mathbf{D} = \langle 2, 3 \rangle$. Therefore, $\mathbf{r}(t) = \langle t^2/2 + t + 2, t^2 + t + 3 \rangle$.

14.3.37 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 0, 10 \rangle dt = \langle 0, 10t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 5 \rangle$, we have $\mathbf{v}(t) = \langle 0, 10t + 5 \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 0, 10t + 5 \rangle dt = \langle 0, 5t^2 + 5t \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 1, -1 \rangle$, we have $\mathbf{r}(t) = \langle 1, 5t^2 + 5t - 1 \rangle$.

14.3.38 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 1, t \rangle dt = \left\langle t, \frac{t^2}{2} \right\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 2, -1 \rangle$, we obtain $\mathbf{v}(t) = \left\langle t + 2, \frac{t^2}{2} - 1 \right\rangle$. Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \left\langle t + 2, \frac{t^2}{2} - 1 \right\rangle dt = \left\langle \frac{t^2}{2} + 2t, \frac{t^3}{6} - t \right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 8 \rangle$, we have $\mathbf{r}(t) = \left\langle \frac{t^2}{2} + 2t, \frac{t^3}{6} - t + 8 \right\rangle$.

14.3.39 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle \cos t, 2 \sin t \rangle dt = \langle \sin t, -2 \cos t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 1 \rangle$, we have $\mathbf{v}(t) = \langle \sin t, 3 - 2 \cos t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle \sin t, 3 - 2 \cos t \rangle dt = \langle -\cos t, 3t - 2 \sin t \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 1, 0 \rangle$, we have $\mathbf{r}(t) = \langle 2 - \cos t, 3t - 2 \sin t \rangle$.

14.3.40 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle e^{-t}, 1 \rangle dt = \langle -e^{-t}, t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 0 \rangle$, we have $\mathbf{v}(t) = \langle 2 - e^{-t}, t \rangle$.

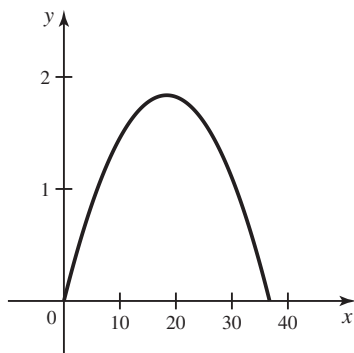
Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 2 - e^{-t}, t \rangle dt = \langle 2t + e^{-t}, t^2/2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0 \rangle$, we have $\mathbf{r}(t) = \langle 2t + e^{-t} - 1, t^2/2 \rangle$.

14.3.41

a. $\mathbf{v}(t) = \int \langle 0, -9.8 \rangle dt = \langle 0, -9.8t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = \langle 30, 6 \rangle$, we have $\mathbf{v}(t) = \langle 30, 6 - 9.8t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 30, 6 - 9.8t \rangle dt = \langle 30t, 6t - 4.9t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0 \rangle$, we have $\mathbf{r}(t) = \langle 30t, 6t - 4.9t^2 \rangle$.

b.



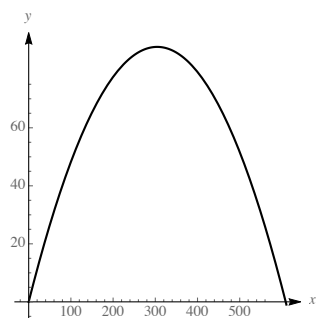
- c. The ball hits the ground when $6t - 4.9t^2 = 0$, which occurs for $t = 6/4.9 \approx 1.22$ seconds. The range of the ball is approximately $30 \cdot 1.22 \approx 36.7$ meters.
- d. The maximum height occurs at time $T \approx 1.22/2 = .61$ seconds, and is $6T - 4.9T^2 \approx 1.84$ meters.

14.3.42

a. $\mathbf{v}(t) = \int \langle 0, -32 \rangle dt = \langle 0, -32t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 150 \langle \sqrt{3}/2, 1/2 \rangle = \langle 75\sqrt{3}, 75 \rangle$, we have $\mathbf{v}(t) = \langle 75\sqrt{3}, -32t + 75 \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 75\sqrt{3}, -32t + 75 \rangle dt = \langle 75\sqrt{3}t, -16t^2 + 75t \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0 \rangle$, we have $\mathbf{r}(t) = \langle 75\sqrt{3}t, -16t^2 + 75t \rangle$.

b.



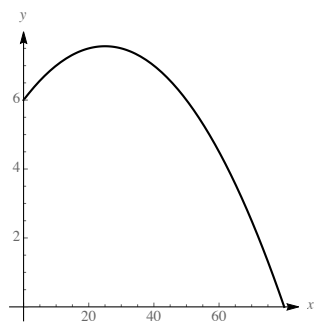
- c. The ball hits the ground when $-16t^2 + 75t = 0$, which occurs for $t = 75/16 = 4.6875$ seconds. The range of the ball is approximately $75\sqrt{3}(4.6875) \approx 608.9$ feet.
- d. The maximum height occurs at time $T \approx 4.6875/2 \approx 2.344$, and is $-16(2.344)^2 + 75(2.344) \approx 87.89$ feet.

14.3.43

a. $\mathbf{v}(t) = \int \langle 0, -32 \rangle dt = \langle 0, -32t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = \langle 80, 10 \rangle$, we have $\mathbf{v}(t) = \langle 80, 10 - 32t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 80, 10 - 32t \rangle dt = \langle 80t, 10t - 16t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 6 \rangle$, we have $\mathbf{r}(t) = \langle 80t, 6 + 10t - 16t^2 \rangle$.

b.



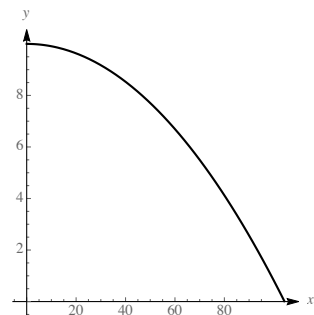
- c. The ball hits the ground when $-16t^2 + 10t + 6 = -2(t - 1)(8t + 3) = 0$, which occurs for $t = 1$ second. The range of the ball is $80 \cdot 1 = 80$ feet.
- d. The maximum height occurs at time $T \approx 10/32 \approx 0.3125$, and is $-16(.3125)^2 + 10(.3125) + 6 \approx 7.56$ feet.

14.3.44

- a. $\mathbf{v}(t) = \int \langle 0, -32 \rangle dt = \langle 0, -32t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 132 \langle 1, 0 \rangle$, we have $\mathbf{v}(t) = \langle 132, -32t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 132, -32t \rangle dt = \langle 132t, -16t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 10 \rangle$, we have $\mathbf{r}(t) = \langle 132t, -16t^2 + 10 \rangle$.

b.



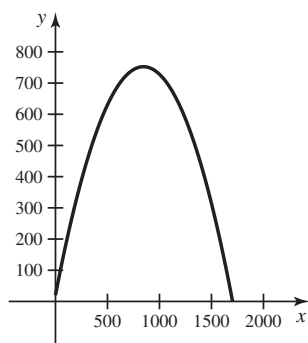
- c. The ball hits the ground when $-16t^2 + 10 = 0$, which occurs for $t = \sqrt{10/16} \approx 0.79$ second. The range of the ball is approximately $132(.79) \approx 104.36$ feet.
- d. The maximum height occurs at time $T = 0$, and is 10 feet.

14.3.45

- a. $\mathbf{v}(t) = \int \langle 0, -32 \rangle dt = \langle 0, -32t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 250 \langle 1/2, \sqrt{3}/2 \rangle = \langle 125, 125\sqrt{3} \rangle$, we have $\mathbf{v}(t) = \langle 125, 125\sqrt{3} - 32t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 125, 125\sqrt{3} - 32t \rangle dt = \langle 125t, 125\sqrt{3}t - 16t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 20 \rangle$, we have $\mathbf{r}(t) = \langle 125t, 20 + 125\sqrt{3}t - 16t^2 \rangle$.

b.



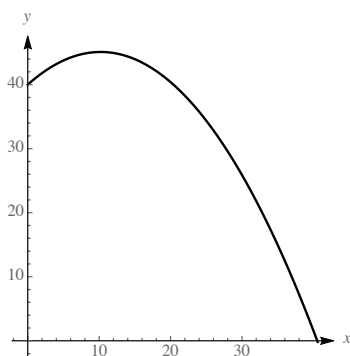
- c. The ball hits the ground when $20 + 125\sqrt{3}t - 16t^2 = 0$, which occurs for $t \approx 13.62$ seconds. The range of the ball is approximately $125 \cdot 13.62 \approx 1702$ feet.
- d. The maximum height occurs when $125\sqrt{3} - 32t = 0$, which is when $t \approx 6.77$ and it is about $20 + 125\sqrt{3}(6.77) - 16(6.77)^2 \approx 752.4$ feet.

14.3.46

- a. $\mathbf{v}(t) = \int \langle 0, -9.8 \rangle dt = \langle 0, -9.8t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 10\sqrt{2} \langle \sqrt{2}/2, \sqrt{2}/2 \rangle = \langle 10, 10 \rangle$, we have $\mathbf{v}(t) = \langle 10, 10 - 9.8t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 10, 10 - 9.8t \rangle dt = \langle 10t, 10t - 4.9t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 40 \rangle$, we have $\mathbf{r}(t) = \langle 10t, 40 + 10t - 4.9t^2 \rangle$.

b.



- c. The ball hits the ground when $40 + 10t - 4.9t^2 = 0$, which occurs when $t \approx 4.06$ seconds. The range of the ball is approximately $10 \cdot 4.06 = 40.6$ meters.
- d. The maximum height occurs at the time when $10 - 9.8t = 0$, which is when $t = 10/9.8 \approx 1.02$. The height at this time is about 45.1 meters.

14.3.47 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 0, 0, 10 \rangle dt = \langle 0, 0, 10t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 5, 0 \rangle$, we have $\mathbf{v}(t) = \langle 1, 5, 10t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 1, 5, 10t \rangle dt = \langle t, 5t, 5t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 5, 0 \rangle$, we have $\mathbf{r}(t) = \langle t, 5t + 5, 5t^2 \rangle$.

14.3.48 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 1, t, 4t \rangle dt = \langle t, t^2/2, 2t^2 \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 20, 0, 0 \rangle$, we have $\mathbf{v}(t) = \langle t + 20, t^2/2, 2t^2 \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle t + 20, t^2/2, 2t^2 \rangle dt = \langle t^2/2 + 20t, t^3/6, 2t^3/3 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, we have $\mathbf{r}(t) = \langle t^2/2 + 20t, t^3/6, 2t^3/3 \rangle$.

14.3.49 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle \sin t, \cos t, 1 \rangle dt = \langle -\cos t, \sin t, t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 2, 0 \rangle$, we have $\mathbf{v}(t) = \langle 1 - \cos t, \sin t + 2, t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 1 - \cos t, \sin t + 2, t \rangle dt = \langle t - \sin t, -\cos t + 2t, t^2/2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, we have $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t + 2t, t^2/2 \rangle$.

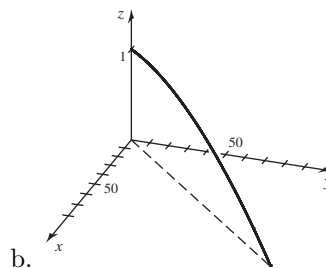
14.3.50 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle t, e^{-t}, 1 \rangle dt = \langle t^2/2, -e^{-t}, t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 0, 1 \rangle$, we have $\mathbf{v}(t) = \langle t^2/2, 1 - e^{-t}, t + 1 \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle t^2/2, 1 - e^{-t}, t + 1 \rangle dt = \langle t^3/6, t + e^{-t}, t^2/2 + t \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 4, 0, 0 \rangle$, we have $\mathbf{r}(t) = \langle t^3/6 + 4, t + e^{-t} - 1, t^2/2 + t \rangle$.

14.3.51

$$\begin{aligned} \text{a. } \mathbf{v}(t) &= \int \mathbf{a}(t) dt = \int \langle 0, 0, -9.8 \rangle dt = \\ &\langle 0, 0, -9.8t \rangle + \mathbf{C}. \quad \text{Because } \mathbf{v}(0) = \\ &\langle 200, 200, 0 \rangle, \quad \text{we have } \mathbf{v}(t) = \\ &\langle 200, 200, -9.8t \rangle. \end{aligned}$$

$$\begin{aligned} \text{Also, } \mathbf{r}(t) &= \int \mathbf{v}(t) dt = \\ &\int \langle 200, 200, -9.8t \rangle dt = \\ &\langle 200t, 200t, -4.9t^2 \rangle + \mathbf{D}, \quad \text{and be-} \\ &\text{cause } \mathbf{r}(0) = \langle 0, 0, 1 \rangle, \quad \text{we have} \\ &\mathbf{r}(t) = \langle 200t, 200t, -4.9t^2 + 1 \rangle. \end{aligned}$$



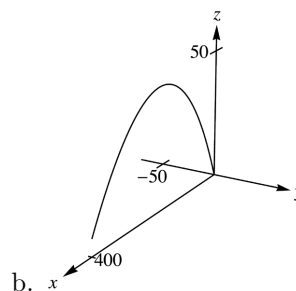
c. The bullet hits the ground when $-4.9t^2 + 1 = 0$, which occurs for $t \approx .452$ seconds. At this time, the bullet is approximately at the point $(200 \cdot 0.452, 200 \cdot 0.452, 0) \approx (90.35, 90.35, 0)$. So its range is approximately $\sqrt{90.35^2 + 90.35^2} \approx 127.8$ meters.

d. The maximum height of the bullet is its initial height of 1 meter.

14.3.52

$$\begin{aligned} \text{a. } \mathbf{v}(t) &= \int \mathbf{a}(t) dt = \int \langle 0, -.8, -9.8 \rangle dt = \\ &\langle 0, -.8t, -9.8t \rangle + \mathbf{C}. \quad \text{Because} \\ &\mathbf{v}(0) = \langle 50, 0, 30 \rangle, \quad \text{we have } \mathbf{v}(t) = \\ &\langle 50, -.8t, -9.8t + 30 \rangle. \end{aligned}$$

$$\begin{aligned} \text{Also, } \mathbf{r}(t) &= \int \mathbf{v}(t) dt = \\ &\int \langle 50, -.8t, -9.8t + 30 \rangle dt = \\ &\langle 50t, -.4t^2, -4.9t^2 + 30t \rangle + \mathbf{D}, \quad \text{and} \\ &\text{because } \mathbf{r}(0) = \langle 0, 0, 0 \rangle, \quad \text{we have} \\ &\mathbf{r}(t) = \langle 50t, -.4t^2, -4.9t^2 + 30t \rangle. \end{aligned}$$



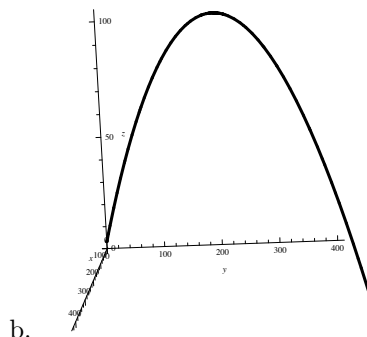
c. The ball hits the ground when $-4.9t^2 + 30t = 0$, which occurs for $t \approx 30/4.9 \approx 6.12$ seconds. At this time, the ball is at the point $((50)(6.12), -.4(6.12)^2, 0) \approx (306, -14.98, 0)$. So its range is approximately $\sqrt{306^2 + 14.98^2} \approx 306.4$ meters.

d. The maximum height of the ball occurs when $-9.8t + 30 = 0$, or when $t = 30/9.8 \approx 3.06$ seconds. At this time the ball's height is about $30(3.06) - 4.9(3.06)^2 \approx 45.92$ meters.

14.3.53

a. $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 10, 0, -32 \rangle dt = \langle 10t, 0, -32t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 60, 80, 80 \rangle$, we have $\mathbf{v}(t) = \langle 10t + 60, 80, -32t + 80 \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 10t + 60, 80, -32t + 80 \rangle dt = \langle 5t^2 + 60t, 80t, -16t^2 + 80t \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 3 \rangle$, we have $\mathbf{r}(t) = \langle 5t^2 + 60t, 80t, -16t^2 + 80t + 3 \rangle$.

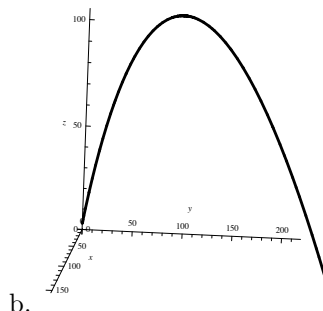


- c. The ball hits the ground when $-16t^2 + 80t + 3 = 0$, which occurs for $t \approx 5.04$ seconds. At this time, the ball is at the point $(5(5.04)^2 + 60(5.04), 80(5.04), 0) \approx (429.4, 403.2, 0)$. So its range is approximately $\sqrt{429.4^2 + 403.2^2} \approx 589$ feet.
- d. The maximum height of the ball occurs when $-32t + 80 = 0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2 + 80(2.5) + 3 = 103$ feet.

14.3.54

a. $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 0, 5, -32 \rangle dt = \langle 0, 5t, -32t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 30, 30, 80 \rangle$, we have $\mathbf{v}(t) = \langle 30, 5t + 30, -32t + 80 \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 30, 5t + 30, -32t + 80 \rangle dt = \langle 30t, 5t^2/2 + 30t, -16t^2 + 80t \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 3 \rangle$, we have $\mathbf{r}(t) = \langle 30t, 5t^2/2 + 30t, -16t^2 + 80t + 3 \rangle$.

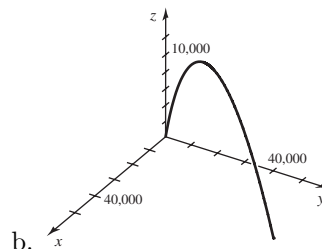


- c. The ball hits the ground when $-16t^2 + 80t + 3 = 0$, which occurs for $t \approx 5.04$ seconds. At this time, the ball is at the point $(30(5.04), 5(5.04)^2/2 + 30(5.04), 0) \approx (151.2, 214.7, 0)$. So its range is approximately $\sqrt{151.2^2 + 214.7^2} \approx 263$ feet.
- d. The maximum height of the ball occurs when $-32t + 80 = 0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2 + 80(2.5) + 3 = 103$ feet.

14.3.55

a. $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 0, 2.5, -9.8 \rangle dt = \langle 0, 2.5t, -9.8t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 300, 400, 500 \rangle$, we have $\mathbf{v}(t) = \langle 300, 2.5t + 400, 500 - 9.8t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 300, 2.5t + 400, 500 - 9.8t \rangle dt = \langle 300t, 1.25t^2 + 400t, 500t - 4.9t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 10 \rangle$, we have $\mathbf{r}(t) = \langle 300t, 1.25t^2 + 400t, 10 + 500t - 4.9t^2 \rangle$.

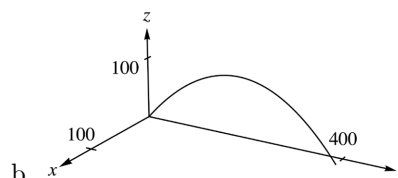


- c. The rocket hits the ground when $-4.9t^2 + 500t + 10 = 0$, which occurs for $t \approx 102.1$ seconds. At this time, the rocket is at the point $(30630, 53870.5, 0)$. So its range is approximately $\sqrt{30630^2 + 53870.5^2} \approx 61969.6$ meters.
- d. The maximum height of the rocket occurs when $-9.8t + 500 = 0$, or when $t = 500/9.8 \approx 51.02$ seconds. At this time the rocket's height is about $10 + 500(51.02) - 4.9(51.02)^2 \approx 12,765$ meters.

14.3.56

a. $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 1.2, 0, -32 \rangle dt = \langle 1.2t, 0, -32t \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 80, 80 \rangle$, we have $\mathbf{v}(t) = \langle 1.2t, 80, 80 - 32t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 1.2t, 80, 80 - 32t \rangle dt = \langle .6t^2, 80t, 80t - 16t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, we have $\mathbf{r}(t) = \langle .6t^2, 80t, 80t - 16t^2 \rangle$.



- c. The ball hits the ground when $-16t^2 + 80t = 0$, which occurs for $t = 80/16 = 5$ seconds. At this time, the ball is at the point $(15, 400, 0)$. So its range is approximately $\sqrt{15^2 + 400^2} \approx 400.28$ feet.
- d. The maximum height of the ball occurs when $-32t + 80 = 0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2 + 80 \cdot 2.5 = 100$ feet.

14.3.57

- a. False. For example, if $\mathbf{v}(t) = \langle \cos t, \sin t \rangle$, then its speed is constantly 1 even though its components aren't constant.
- b. True. They both generate $\{(x, y) \mid x^2 + y^2 = 1\}$.
- c. False. For example, $\langle t, t, t \rangle$ has variable magnitude but constant direction.
- d. True. If $\mathbf{a}(t) = \langle 0, 0, 0 \rangle$, then $\mathbf{v}(t) = \langle 0, 0, 0 \rangle + \mathbf{C}$ for a constant vector $\mathbf{C}$.
- e. False. Recall that for two-dimensional motion the range is given by $\frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}$, so doubling the speed should quadruple the range.

f. True. The time of flight is given by $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g}$, so doubling the speed doubles the time of flight.

g. True. For example, if $\mathbf{v}(t) = \langle e^t, e^t, e^t \rangle$, then $\mathbf{a}(t) = \langle e^t, e^t, e^t \rangle$ as well.

14.3.58 The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2 \cdot 20}{32} = 1.25$ seconds.

The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{400}{32} = 12.5$ feet.

The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{400}{64} = 6.25$ feet.

14.3.59 Note that $\langle u_0, v_0 \rangle = 75\langle\sqrt{3}, 1\rangle$

The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2 \cdot 75}{9.8} \approx 15.3$ seconds.

The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{11250\sqrt{3}}{9.8} \approx 1988.3$ meters.

The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{75^2}{19.6} \approx 287$ meters.

14.3.60 The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2 \cdot 80}{9.8} \approx 16.33$ seconds.

The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{6400}{9.8} \approx 653$ meters.

The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{80^2}{19.6} \approx 326.5$ meters.

14.3.61 Note that $\langle u_0, v_0 \rangle = 200\langle 1, \sqrt{3} \rangle$ The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2 \cdot 200\sqrt{3}}{32} = 21.65$ seconds.

The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{80,000\sqrt{3}}{32} = 4330.13$ feet.

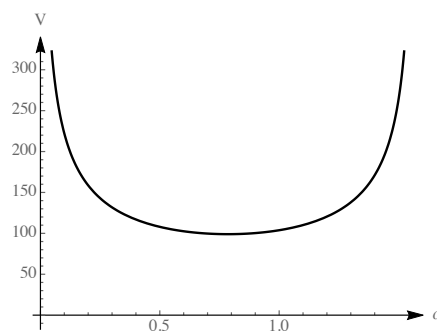
The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{(200\sqrt{3})^2}{64} = 1875$ feet.

14.3.62 Each of these values on the moon would be 6 times the corresponding value on the earth, because the factor of g in the denominator would result in an extra factor of 6 if g were replaced by $g/6$.

14.3.63 We desire $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = 300$ meters, so we require $\sin 2\alpha = 300 \cdot \frac{9.8}{60^2} \approx .81666$. So $2\alpha = \sin^{-1}(.81666)$, and $\alpha \approx 27.4$ degrees or $\alpha \approx 62.62$ degrees.

14.3.64

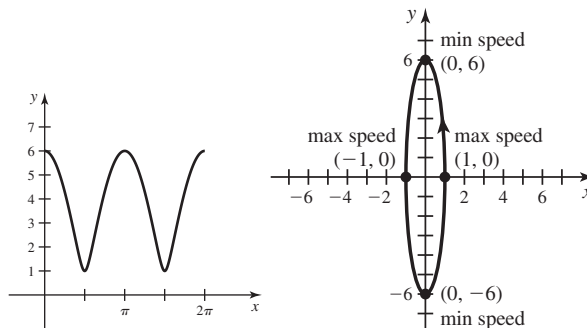
Let V stand for the initial speed. The range
 a. is given by $\frac{V^2 \sin 2\alpha}{g}$, so we require $\frac{9800}{\sin 2\alpha} = V^2$, so $V = \sqrt{9800 \csc 2\alpha}$.



- b. The speed is minimized when $\frac{dV}{d\alpha} = \frac{-9800 \csc 2\alpha \cot 2\alpha}{\sqrt{9800 \csc 2\alpha}} = 0$, which occurs when $\cos 2\alpha = 0$, or $\alpha = \pi/4$. At this value of α , the value of V is about 99 meters per second.

14.3.65

- a. The velocity is $\mathbf{v}(t) = \langle -a \sin t, b \cos t \rangle$ and the speed is $\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$.
- b.



- c. Yes, as the diagram indicates.
- d. Assume $a > b > 0$. Then the maximum speed occurs at $\pi/2$ and is equal to a , while the minimum speed occurs at π and is equal to b . So the ratio is $\frac{a}{b}$. In the case $b > a > 0$, the ratio is $\frac{b}{a}$.

14.3.66 Let the angle be α . Then $\mathbf{v}(t) = \langle 150 \cos \alpha, 150 \sin \alpha - 32t \rangle$, and

$$\mathbf{r}(t) = \langle 150t \cos \alpha, 150t \sin \alpha - 16t^2 \rangle.$$

Because we require the ball to land in the hole, we need the point $(390, 40)$ to be on the curve. So $150t \cos \alpha = 390$ and $150t \sin \alpha - 16t^2 = 40$. Thus $t = \frac{390}{150 \cos \alpha}$, and therefore $150 \cdot \frac{390 \cdot \sin \alpha}{150 \cos \alpha} - 16 \left(\frac{390}{150 \cos \alpha} \right)^2 = 40$.

This can be written $390 \tan \alpha - \frac{2704}{25} \sec^2 \alpha - 40 = 0$, or

$$390 \tan \alpha - \frac{2704}{25} (1 + \tan^2 \alpha) - 40 = -\frac{2704}{25} \tan^2 \alpha + 390 \tan \alpha - \frac{3704}{25} = 0.$$

By the quadratic formula, we have

$$\tan \alpha = \frac{1}{208} (375 - \sqrt{81361}) \quad \text{and} \quad \tan \alpha = \frac{1}{208} (375 + \sqrt{81361}).$$

Applying the inverse tangent and then writing the answer in degrees, we obtain $\alpha = 72.51$ degrees and $\alpha = 23.34$ degrees

14.3.67 Let the angle be α . Then $\mathbf{v}(t) = \langle 120 \cos \alpha, 120 \sin \alpha - 32t \rangle$, and

$$\mathbf{r}(t) = \langle 120t \cos \alpha, 120t \sin \alpha - 16t^2 \rangle.$$

Because we require the ball to land in the hole, we need the point $(420, -50)$ to be on the curve. So $120t \cos \alpha = 420$ and $120t \sin \alpha - 16t^2 = -50$. Thus $t = \frac{420}{120 \cos \alpha}$, and therefore $120 \cdot \frac{420 \cdot \sin \alpha}{120 \cos \alpha} - 16 \left(\frac{420}{120 \cos \alpha} \right)^2 = -50$. This can be written $420 \tan \alpha - 196 \sec^2 \alpha + 50 = -196 \tan^2 \alpha + 420 \tan \alpha - 146 = 0$.

By the quadratic formula, we have

$$\tan \alpha = \frac{1}{14} (15 - \sqrt{79}) \quad \text{and} \quad \tan \alpha = \frac{1}{14} (15 + \sqrt{79}).$$

Applying the inverse tangent and then writing the answer in degrees, we obtain $\alpha = 59.63$ degrees and $\alpha = 23.58$ degrees.

14.3.68 Let s be the initial speed of the ball. Note that $\mathbf{v}(t) = \langle s\sqrt{2}/2, s\sqrt{2}/2 - 32t \rangle$ and

$$\mathbf{r}(t) = \langle st\sqrt{2}/2, st\sqrt{2}/2 - 16t^2 \rangle.$$

Because we want the second coordinate to be 40 when the first coordinate is 390, we have $st\sqrt{2}/2 = 390$ and $st\sqrt{2}/2 - 16t^2 = 40$. Solving the first equation for t yields $t = \frac{780}{\sqrt{2}s}$. Putting this value into the second equation yields $s \cdot \frac{780}{\sqrt{2}s} \cdot \frac{\sqrt{2}}{2} - 16 \left(\frac{780}{\sqrt{2}s} \right)^2 = 40$. Solving this last equation for s yields $s = \frac{\sqrt{8} \cdot 780}{\sqrt{350}} \approx 117.9$.

14.3.69 Let s be the initial speed of the ball. Note that $\mathbf{v}(t) = \langle s\sqrt{3}/2, s/2 - 32t \rangle$ and

$$\mathbf{r}(t) = \langle st\sqrt{3}/2, st/2 - 16t^2 \rangle.$$

Because we want the second coordinate to be -50 when the first coordinate is 420, we have $st\sqrt{3}/2 = 420$ and $st/2 - 16t^2 = -50$. Solving the first equation for t yields $t = \frac{840}{\sqrt{3}s}$. Putting this value into the second equation yields $s \cdot \frac{840}{\sqrt{3}s} \cdot \frac{1}{2} - 16 \left(\frac{840}{\sqrt{3}s} \right)^2 = -50$. Solving this last equation for s yields $s \approx 113.4$.

14.3.70

- a. $\mathbf{v}(t) = \langle 40, -9.8t \rangle$ and $\mathbf{r}(t) = \langle 40t, 8 - 4.9t^2 \rangle$. Let $x = 40t$ and $y = 8 - 4.9t^2$. Then the equation of trajectory is $y = 8 - 4.9 \left(\frac{x}{40} \right)^2$. The equation of the outrun surface is $y = -\frac{1}{\sqrt{3}}x$. These curves intersect when $8 - 4.9 \left(\frac{x}{40} \right)^2 = -\frac{1}{\sqrt{3}}x$, which when solved for x yields 201.487, and then $y = -116.329$. The distance from the origin to the landing points is $\sqrt{201.487^2 + (-116.329)^2} \approx 232.657$ meters.
- b. In this scenario, $\mathbf{v}(t) = \langle 40 - .15t, -9.8t \rangle$ and $\mathbf{r}(t) = \langle 40t - .075t^2, 8 - 4.9t^2 \rangle$. Let $x = 40t - .075t^2$ and $y = 8 - 4.9t^2$. Then $\frac{8-y}{4.9} = t^2$, so $x = 40\sqrt{\frac{8-y}{4.9}} - .075\left(\frac{8-y}{4.9}\right)$. Because we also have $x = -\sqrt{3}y$, we are looking for the solution to the equation $-\sqrt{3}y = 40\sqrt{\frac{8-y}{4.9}} - .075\left(\frac{8-y}{4.9}\right)$. This results in $y = -114.29$ and so $x = 197.95$. Then the length of the jump is $\sqrt{(-114.29)^2 + 197.95^2} \approx 228.575$ meters.
- c. $\mathbf{v} = \langle 40 \cos \theta, 40 \sin \theta - 9.8t \rangle$, and $\mathbf{r}(t) = \langle 40t \cos \theta, 8 + 40t \sin \theta - 4.9t^2 \rangle$. Let $x = 40t \cos \theta$ and $y = 8 + 40t \sin \theta - 4.9t^2$. Then the equation of trajectory is $y = 8 + x \tan \theta - 4.9 \left(\frac{x}{40 \cos \theta} \right)^2$. The equation of the outrun surface is $y = -\frac{1}{\sqrt{3}}x$. These intersect when $8 + x \tan \theta - 4.9 \left(\frac{x}{40 \cos \theta} \right)^2 = -\frac{1}{\sqrt{3}}x$. Solving for x in terms of θ (using a computer algebra system) gives

$$x = \frac{0.5 \left(1.04785\sqrt{\tan^2 \theta + 1.05164 \tan \theta + 0.392835} + \tan \theta + 0.57735 \right)}{0.0030625 \tan^2 \theta + 0.0030625}.$$

This is maximized for $\theta \approx 29.41$ degrees.

14.3.71

- a. $\mathbf{v} = \langle 130, 0, -3 - 32t \rangle$ and $\mathbf{r}(t) = \langle 130t, 0, 6 - 3t - 16t^2 \rangle$. When $x = 60$, $t = 6/13$, so $z = 6 - 3(6/13) - 16(6/13)^2 \approx 1.207$ feet. The flight lasts $t = 6/13$ seconds.
- b. Suppose that the initial velocity is $\langle 130, 0, b \rangle$, so that $\mathbf{v}(t) = \langle 130t, 0, 6 + bt - 16t^2 \rangle$. So $z = 3 = 6 + b(6/13) - 16(6/13)^2$, which when solved for b gives $b = .8846$.

c. In this scenario, we have $\mathbf{v}(t) = \langle 130, 8t, -3 - 32t \rangle$ and $\mathbf{r}(t) = \langle 130t, 4t^2, 6 - 3t - 16t^2 \rangle$. As before, $x = 60$ when $t = 6/13$, and at that time $y = 4(6/13)^2 \approx 0.8521$ feet.

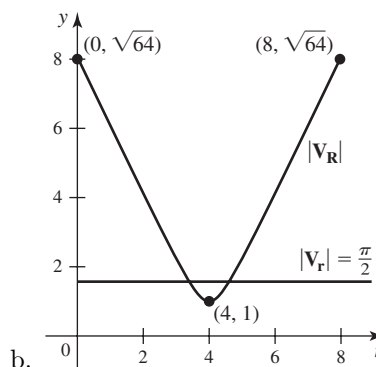
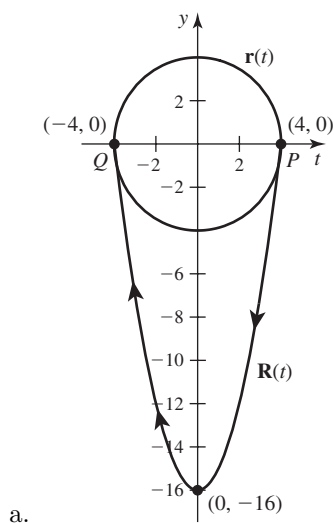
d. It moves more in the second half, because of the factor of t^2 . This makes life more difficult for the batter!

e. In this case $\mathbf{v}(t) = \langle 130, ct, -3 - 32t \rangle$ and $\mathbf{r}(t) = \langle 130t, -3 + ct^2/2, 6 - 3t - 16t^2 \rangle$. Again, we have $t = 6/13$, and so we require $-3 + c \left(\frac{6}{13} \right)^2 \cdot \frac{1}{2} = 0$, so $c \approx 28.17$.

14.3.72 We have $x = u_0 t + x_0$, and $y = -\frac{gt^2}{2} + v_0 t + y_0$. Eliminating t gives $y = -\frac{g}{2}((x - x_0)/u_0)^2 + v_0((x - x_0)/u_0) + y_0$, which is a segment of a parabola. If $y(T) = 0$, then we have $-\frac{g}{2}T^2 + v_0 T + y_0 = 0$, so $T^2 - \frac{2v_0}{g}T - \frac{2y_0}{g} = 0$, so by the quadratic formula we have $T = \frac{v_0}{g} + \sqrt{\left(\frac{v_0}{g}\right)^2 + \frac{2y_0}{g}} = \frac{v_0 + \sqrt{v_0^2 + 2gy_0}}{g}$.

14.3.73 We have $\mathbf{v}(t) = \langle v_0 \cos \alpha, v_0 \sin \alpha - gt \rangle$, and $\mathbf{r}(t) = \left\langle v_0 t \cos \alpha, y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2 \right\rangle$. Suppose that object hits the ground at $(a, 0)$. Then $a = v_0 T \cos \alpha$ and $0 = y_0 + v_0 T \sin \alpha - \frac{1}{2}gT^2$ where T is the time of the flight. So by the quadratic formula, $T = \frac{v_0 \sin \alpha + \sqrt{v_0^2 \sin^2 \alpha + 2gy_0}}{g}$. Thus $a = v_0 T \cos \alpha = v_0 (\cos \alpha) \left(\frac{v_0 \sin \alpha + \sqrt{v_0^2 \sin^2 \alpha + 2gy_0}}{g} \right)$ is the range. Because the maximum height when $y_0 = 0$ is $\frac{v_0^2 \sin^2 \alpha}{2g}$, the maximum height in this scenario is $y_0 + \frac{v_0^2 \sin^2 \alpha}{2g}$.

14.3.74



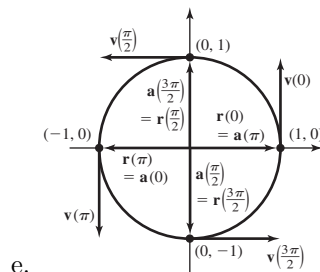
c. Both travelers arrive at $t = 8$.

14.3.75

a. The object traverses the circle once over the interval $[0, 2\pi/\omega]$.

- b. The velocity is $\mathbf{v}(t) = \langle -A\omega \sin(\omega t), A\omega \cos(\omega t) \rangle$. The velocity is not constant in direction, but it is constant in speed, because the speed is $|A\omega|$.

- c. The acceleration is $\mathbf{a}(t) = -A\omega^2 \langle \cos \omega t, \sin \omega t \rangle$.
- d. The position and velocity are orthogonal. The position and acceleration are in opposite directions.



14.3.76

- a. Note that $\mathbf{r}(t) = \langle (-7/5)t + 1, (6/5)t + 2, (6/5)t + 4 \rangle$ has $\mathbf{r}(0) = \langle 1, 2, 4 \rangle$ and $\mathbf{r}(5) = \langle -6, 8, 10 \rangle$.
- b. Consider $\mathbf{r}(t) = e^t \left\langle -\frac{7}{11}, \frac{6}{11}, \frac{6}{11} \right\rangle + \left\langle \frac{13}{6}, 1, 3 \right\rangle$ for $\ln(11/6) \leq t \leq \ln(77/6)$. Note that

$$\mathbf{r}(\ln(11/6)) = \frac{11}{6} \langle -7/11, 6/11, 6/11 \rangle + \langle 13/6, 1, 3 \rangle = \langle -7/6, 1, 1 \rangle + \langle 13/6, 1, 3 \rangle = \langle 1, 2, 4 \rangle$$

and

$$\mathbf{r}(\ln(77/6)) = \frac{77}{6} \langle -7/11, 6/11, 6/11 \rangle + \langle 13/6, 1, 3 \rangle = \langle -49/6, 7, 7 \rangle + \langle 13/6, 1, 3 \rangle = \langle -6, 8, 10 \rangle.$$

$$\text{Also } |\mathbf{r}'(t)| = |\langle e^t(-7/11), e^t(6/11), e^t(6/11) \rangle| = e^t \sqrt{\frac{49 + 36 + 36}{11^2}} = e^t.$$

14.3.77

- a. Consider $\mathbf{r}(t) = \langle 5 \sin(\pi t/6), 5 \cos(\pi t/6) \rangle$ for $0 \leq t \leq 12$. Note that the speed is the constant $5\pi/6$, and that $\mathbf{r}(0) = (0, 5) = \mathbf{r}(12)$.
- b. Consider $\mathbf{r}(t) = \langle 5 \sin((1 - e^{-t})/5), 5 \cos((1 - e^{-t})/5) \rangle$ for $-\ln(10\pi + 1) \leq t \leq 0$. Note that

$$\mathbf{r}(-\ln(10\pi + 1)) = (0, 5) = \mathbf{r}(0).$$

$$\text{Also note that the speed is } |\mathbf{r}'(t)| = |e^{-t} \langle \cos((1 - e^{-t})/5), -\sin((1 - e^{-t})/5) \rangle| = e^{-t}.$$

14.3.78

- a. Let $\mathbf{r}(t) = \langle \cos 6t, \sin 6t, 8t \rangle$. Then $\mathbf{r}$ moves along the circular helix, but the speed is

$$|\langle -6 \sin 6t, 6 \cos 6t, 8 \rangle| = \sqrt{36 \sin^2 6t + 36 \cos^2 6t + 64} = 10.$$

- b. Let $\mathbf{r}(t) = \langle \cos(t^2/(2\sqrt{2})), \sin(t^2/(2\sqrt{2})), t/\sqrt{2} \rangle$. Then $\mathbf{r}$ moves along the circular helix, but the speed is

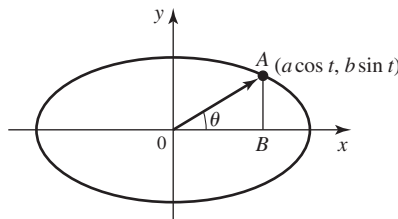
$$\left| \left\langle -\frac{t}{\sqrt{2}} \sin(t^2/(2\sqrt{2})), \frac{t}{\sqrt{2}} \cos(t^2/(2\sqrt{2})), t/\sqrt{2} \right\rangle \right| = \sqrt{\frac{t^2}{2} \sin^2(t^2/(2\sqrt{2})) + \frac{t^2}{2} \cos^2(t^2/(2\sqrt{2})) + \frac{t^2}{2}} = t$$

for $t \geq 0$.

14.3.79 Note that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, and $z = cy$, so this curve is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $z = cy$, which results in an ellipse in that plane.

14.3.80

- a. In right triangle ABO , we have $\tan \theta =$
 $\frac{AB}{OB} = \frac{b \sin t}{a \cos t} = \frac{b}{a} \tan t.$



b. $\theta = \tan^{-1}((b/a) \tan t)$, so $\theta'(t) = \frac{1}{1 + ((b/a) \tan t)^2} \cdot \frac{b}{a} \sec^2 t = \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t}.$

c. $\frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt} = \frac{1}{2} |\mathbf{r}(\theta(t))|^2 \cdot \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{1}{2} ab.$

- d. Because $\frac{dA}{dt}$ is a constant, the object sweeps out equal areas in equal times as it moves about the ellipse.

14.3.81 $\mathbf{r}(t)$ can be written $\langle R \cos \phi \cos t, R \sin \phi \cos t, R \sin t \rangle$ where R is the radius of the sphere and ϕ is a real constant. Note that

$$\mathbf{v}(t) = \langle -R \cos \phi \sin t, -R \sin \phi \sin t, R \cos t \rangle$$

and $\mathbf{a}(t) = \langle -R \cos \phi \cos t, -R \sin \phi \cos t, -R \sin t \rangle$, and that $\mathbf{r}(t) \cdot \mathbf{a}(t) = -R^2 \cos^2 \phi \cos^2 t - R^2 \sin^2 \phi \cos^2 t - R^2 \sin^2 t = -R^2(\cos^2 \phi + \sin^2 \phi) \cos^2 t - R^2 \sin^2 t = -R^2(\cos^2 t + \sin^2 t) = -R^2 = -|\mathbf{v}(t)|^2.$

14.3.82

- a. $|\mathbf{r}(t)|^2 = (a \cos t + b \sin t)^2 + (c \cos t + d \sin t)^2 = (a^2 + c^2) \cos^2 t + (2ab + 2cd) \sin t \cos t + (b^2 + d^2) \sin^2 t.$
 In order for the path to be a circle, it would be sufficient that $a^2 + c^2 = b^2 + d^2$ and that $ab + cd = 0.$
- b. In order for the path to be an ellipse, it would be sufficient that $ab + cd = 0.$

14.3.83

- a. If $t_1 > t_0$ are two values of t , we have $\mathbf{r}(t_1) - \mathbf{r}(t_0) = (f(t_1) - f(t_0)) \langle a, b, c \rangle$, which is always a vector in the same direction, regardless of the values of t_1 and $t_0.$
- b. $\mathbf{r}'(t) = f'(t) \langle a, b, c \rangle$ is a multiple of $\langle a, b, c \rangle$, so the tangent vector is always a multiple of the vector $\langle a, b, c \rangle$, so the motion of the object doesn't vary in direction, although it might vary in speed.

14.4 Length of Curves

14.4.1 $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt = \int_a^b \sqrt{1 + 4} dt = \sqrt{5}(b - a).$

14.4.2 To compute the length of the curve, compute $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt.$

14.4.3 The arc length is $L = \int_a^b |\mathbf{r}'(t)| dt = \int_a^b |\mathbf{v}(t)| dt.$

14.4.4 It travels a distance equal to the length of the path it traces in space: $\int_a^b |\mathbf{r}'(t)| dt = \int_a^b |\mathbf{v}(t)| dt.$

$$14.4.5 \quad L = \int_0^\pi | \langle -20 \sin(2t), 20 \cos(2t) \rangle | dt = 20\pi.$$

$$14.4.6 \quad \text{This is given by } \int_0^2 \sqrt{\langle 1, -8, 4 \rangle \cdot \langle 1, -8, 4 \rangle} dt = \int_0^2 \sqrt{1 + 64 + 16} dt = 18.$$

14.4.7 If the parameter t used to describe a trajectory also measures the arc length s of the curve that is generated, then we say that the curve is parametrized by its arc length. This occurs when $|\mathbf{v}(t)| = 1$.

14.4.8 Note that $|\mathbf{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$, so it is parametrized by arc length.

$$\text{The arc length is } s = \int_0^t |\mathbf{v}(t)| dt = \int_0^t 1 dt = t.$$

$$14.4.9 \quad L = \int_0^1 \sqrt{36t^2 + 64t^2} dt = \int_0^1 10t dt = 5t^2 \Big|_0^1 = 5.$$

$$14.4.10 \quad L = \int_0^1 \sqrt{9 + 16 + 1} dt = \sqrt{26}t \Big|_0^1 = \sqrt{26}.$$

$$14.4.11 \quad L = \int_0^\pi \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} dt = \int_0^\pi 3 dt = 3\pi.$$

$$14.4.12 \quad L = \int_0^{2\pi/3} \sqrt{(-12 \sin(3t))^2 + (12 \cos(3t))^2} dt = 12(2\pi/3) = 8\pi.$$

$$14.4.13 \quad \text{Note that } (\cos t + t \sin t)' = -\sin t + \sin t + t \cos t = t \cos t, \text{ and } (\sin t - t \cos t)' = \cos t - (\cos t - t \sin t) = t \sin t. \\ L = \int_0^{\pi/2} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt = \int_0^{\pi/2} t dt = \frac{t^2}{2} \Big|_0^{\pi/2} = \frac{\pi^2}{8}.$$

14.4.14 Note that $(\cos t + \sin t)' = -\sin t + \cos t$, and $(\cos t - \sin t)' = -\sin t - \cos t$.

$$L = \int_0^{2\pi} \sqrt{(-\sin t + \cos t)^2 + (-\sin t - \cos t)^2} dt = \int_0^{2\pi} \sqrt{2 - 2 \cos t \sin t + 2 \cos t \sin t} dt = \sqrt{2}(2\pi).$$

$$14.4.15 \quad L = \int_1^6 \sqrt{3^2 + (-4)^2 + 3^2} dt = \sqrt{34}(6 - 1) = 5\sqrt{34}.$$

$$14.4.16 \quad L = \int_0^{6\pi} \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} dt = 5(6\pi) = 30\pi.$$

$$14.4.17 \quad L = \int_0^{4\pi} \sqrt{1 + 64 \cos^2 t + 64 \sin^2 t} dt = \sqrt{65}(4\pi).$$

$$14.4.18 \quad L = \int_0^2 \sqrt{t^2 + (2t + 1)} dt = \int_0^2 (t + 1) dt = \left(\frac{t^2}{2} + t \right) \Big|_0^2 = 4.$$

$$14.4.19 \quad L = \int_0^{\ln 2} \sqrt{4e^{4t} + 16e^{4t} + 16e^{4t}} dt = \int_0^{\ln 2} 6e^{2t} dt = 3e^{2t} \Big|_0^{\ln 2} = 12 - 3 = 9.$$

$$14.4.20 \quad L = \int_0^4 \sqrt{4t^2 + 9t^4} dt = \int_0^4 t\sqrt{4 + 9t^2} dt. \text{ Let } u = 4 + 9t^2 \text{ so that } du = 18t dt. \text{ Then } L = \\ \frac{1}{18} \int_4^{148} \sqrt{u} du = \frac{1}{27} u^{3/2} \Big|_4^{148} = \frac{8}{27} (37\sqrt{37} - 1).$$

$$14.4.21 \quad L = \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt = \int_0^{\pi/2} 3 \sin t \cos t dt = \frac{3}{2} \sin^2 t \Big|_0^{\pi/2} = \frac{3}{2}.$$

$$14.4.22 \quad L = \int_0^{2\pi} \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} dt = 5(2\pi) = 10\pi.$$

$$14.4.23 \quad \text{The speed is } \sqrt{36t^4 + 9t^4 + 225t^4} = \sqrt{270}t^2. \text{ The length is thus } L = \int_0^4 \sqrt{270}t^2 dt = \sqrt{270} \frac{t^3}{3} \Big|_0^4 = \sqrt{30}(64 - 0) = 64\sqrt{30}.$$

$$14.4.24 \quad \text{The speed is } \sqrt{100t^2 \sin^2 t^2 + 100t^2 \cos^2 t^2 + 24^2 t^2} dt = \sqrt{100t^2 + 576t^2} = 26t. \text{ The length is thus } L = \int_0^2 26t dt = 13t^2 \Big|_0^2 = 52.$$

$$14.4.25 \quad \text{The speed is } 2\sqrt{(13\cos 2t)^2 + (-12\sin 2t)^2 + (-5\sin 2t)^2} = 2 \cdot \sqrt{169} = 26. \text{ The length is thus } L = \int_0^\pi 26 dt = 26\pi.$$

$$14.4.26 \quad \text{The speed is } e^t \sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2 + 1^2} = e^t \sqrt{3}. \text{ The length is thus } L = \int_0^{\ln 2} \sqrt{3} e^t dt = \sqrt{3} e^t \Big|_0^{\ln 2} = \sqrt{3}(2 - 1) = \sqrt{3}.$$

$$14.4.27 \quad \text{The length of the orbit is approximately } 4 \int_0^{\pi/2} \sqrt{\sin^2 t + 0.999 \cos^2 t} dt \text{ which can be verified with a computer algebra system to be about 6.28. Then the speed is approximately } \frac{6.28 \cdot 93,000,000}{365.25 \cdot 24} \approx 66,626 \text{ miles per hour.}$$

$$14.4.28 \quad \text{The length of the orbit is approximately}$$

$$4 \int_0^{\pi/2} 5.203 \sqrt{\sin^2 t + \left(\frac{1.517}{1.524}\right)^2 \cos^2 t} dt \approx 32.616.$$

$$\text{Then the speed is approximately } \frac{32.616 \cdot 93,000,000}{11.8618 \cdot 365.25 \cdot 24} \approx 29,171.7.$$

$$14.4.29 \quad L = \int_0^{2\pi} \sqrt{4\sin^2 t + 16\cos^2 t} dt = \int_0^{2\pi} \sqrt{4 + 12\cos^2 t} dt = 2 \int_0^{2\pi} \sqrt{1 + 3\cos^2 t} dt \approx 19.38.$$

$$14.4.30 \quad L = \int_0^{2\pi} \sqrt{4\sin^2 t + 16\cos^2 t + 36\sin^2 t} dt = \int_0^{2\pi} \sqrt{16\sin^2 t + 16\cos^2 t + 24\sin^2 t} dt = 2 \int_0^{2\pi} \sqrt{4 + 6\sin^2 t} dt \approx 32.85.$$

$$14.4.31 \quad L = \int_{-2}^2 \sqrt{1 + 64t^2} dt \approx 32.5.$$

$$14.4.32 \quad L = \int_0^{\ln 3} \sqrt{e^{2t} + 4e^{-2t} + 1} dt \approx 2.73.$$

$$14.4.33 \quad \text{Note that } |\mathbf{v}| = \sqrt{0 + \cos^2 t + \sin^2 t} = 1, \text{ so it does use arc length as its parameter.}$$

$$14.4.34 \quad \text{Note that } |\mathbf{v}| = \sqrt{1/3 + 1/3 + 1/3} = 1, \text{ so it does use arc length as its parameter.}$$

14.4.35 Note that $|\mathbf{v}| = \sqrt{1+4} \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle s/\sqrt{5}, 2s/\sqrt{5} \rangle$ for $0 \leq s \leq 3\sqrt{5}$. This has $|\mathbf{r}'(s)| = \frac{1}{\sqrt{5}}\sqrt{1+4} = 1$, so it does use arc length as its parameter.

14.4.36 Note that $|\mathbf{v}| = \sqrt{1+4+36} \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle (s/\sqrt{41}) + 1, (2s/\sqrt{41}) - 3, 6s/\sqrt{41} \rangle$ for $0 \leq s \leq 10\sqrt{41}$. This has $|\mathbf{r}'(s)| = \frac{1}{\sqrt{41}}\sqrt{1+4+36} = 1$, so it does use arc length as its parameter.

14.4.37 Note that $|\mathbf{v}| = \sqrt{4\sin^2 t + 4\cos^2 t} \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle 2\cos(s/2), 2\sin(s/2) \rangle$ for $0 \leq s \leq 4\pi$. This has $|\mathbf{r}'(s)| = \sqrt{\sin^2(s/2) + \cos^2(s/2)} = 1$, so it does use arc length as its parameter.

14.4.38 Note that $|\mathbf{v}| = \sqrt{17^2 \sin^2 t + 15^2 \cos^2 t + 8^2 \cos^2 t} = \sqrt{289} = 17 \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle 17\cos(s/17), 15\sin(s/17), 8\sin(s/17) \rangle$ for $0 \leq s \leq 17\pi$. This has

$$|\mathbf{r}'(s)| = \sqrt{\sin^2(s/17) + (15/17)^2 \cos^2(s/17) + (8/17)^2 \cos^2(s/17)} = 1,$$

so it does use arc length as its parameter.

14.4.39 Note that $|\mathbf{v}| = \sqrt{4t^2 \sin^2(t^2) + 4t^2 \cos^2(t^2)} = 2t \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle \cos s, \sin s \rangle$ for $0 \leq s \leq \pi$. This has $|\mathbf{r}'(s)| = \sqrt{\sin^2 s + \cos^2 s} = 1$, so it does use arc length as its parameter.

14.4.40 Note that $|\mathbf{v}| = \sqrt{4t^2 + 16t^2 + 64t^2} = \sqrt{84}t = 2\sqrt{21}t \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle (1/\sqrt{21})s, (2/\sqrt{21})s, (4/\sqrt{21})s \rangle$ for $\sqrt{21} \leq s \leq 16\sqrt{21}$. This has $|\mathbf{r}'(s)| = \frac{1}{\sqrt{21}}\sqrt{1+4+16} = 1$, so it does use arc length as its parameter.

14.4.41 Note that $|\mathbf{v}| = \sqrt{e^{2t} + e^{2t} + e^{2t}} = \sqrt{3}e^t \neq 1$, so it doesn't use arc length as its parameter.

Consider $\mathbf{r}(s) = \left\langle \frac{s}{\sqrt{3}} + 1, \frac{s}{\sqrt{3}} + 1, \frac{s}{\sqrt{3}} + 1 \right\rangle$ for $s \geq 0$. This has $|\mathbf{r}'(s)| = 1$, so it does use arc length as its parameter.

14.4.42 Note that $|\mathbf{v}| = \sqrt{(1/2)\sin^2 t + (1/2)\sin^2 t + \cos^2 t} = 1$, so it does use arc length as its parameter.

14.4.43

a. True. $L = \int_a^b S dt = S(b-a)$.

b. True. Both have length $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt = \int_a^b \sqrt{g'(t)^2 + f'(t)^2} dt$.

c. True. Both have length $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{(f'(u^2)(2u))^2 + (g'(u^2)(2u))^2} du$. The equality can be seen via the substitution $u^2 = t$.

d. False. It is not the case that $|\mathbf{r}'(t)| = 1$ for all t , because $|\mathbf{r}'(t)| = \sqrt{1+40t^2}$.

14.4.44

a. $\mathbf{r}(t) = \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0) \rangle$, where $0 \leq t \leq 1$.

b. $L = \int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} dt$. This is equal to

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

c. The distance formula also gives the length of this line segment to be

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

14.4.45

a. Let $x = a \cos t$, $y = b \sin t$ and $z = c \sin t$. Then $x^2 + y^2 + z^2 = a^2 \cos^2 t + b^2 \sin^2 t + c^2 \sin^2 t = a^2 \cos^2 t + (b^2 + c^2) \sin^2 t = a^2$, assuming $a^2 = b^2 + c^2$. So the curve lies on a sphere, but also note that $cy = bz$, so the curve also lies in the plane $cy - bz = 0$. So the curve is a circle centered at the origin.

b. The circle has arc length

$$L = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t + c^2 \cos^2 t} dt = \sqrt{a^2} \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t} dt = 2a\pi.$$

c. As in an exercise from the previous section, the curve describes a circle when $a^2 + c^2 + e^2 = b^2 + d^2 + f^2 = R^2$ and $ab + cd + ef = 0$. Note that $|\mathbf{r}(t)|^2 = (a \cos t + b \sin t)^2 + (c \cos t + d \sin t)^2 + (e \cos t + f \sin t)^2 = (a^2 + c^2 + e^2) \cos^2 t + (2ab + 2cd + 2ef) \sin t \cos t + (b^2 + d^2 + f^2) \sin^2 t$, so if the conditions are met, then the curve describes a circle of radius R and $L = \int_0^{2\pi} \sqrt{R^2} dt = 2\pi R$.

14.4.46 Assume $m \neq 0$. $\mathbf{r}'(t) = \langle mt^{m-1}, mt^{m-1}, (3m/2)t^{(3m/2-1)} \rangle$, so $|\mathbf{r}'(t)| = \frac{3|m|t^{m-1}}{2} \sqrt{\frac{4}{9} + \frac{4}{9} + t^m}$. So

$$L = \frac{3|m|}{2} \int_a^b t^{m-1} \sqrt{(8/9) + t^m} dt.$$

Let $u = \frac{8}{9} + t^m$, so that $\pm du = |m|t^{m-1} dt$. Then if $m > 0$ we have $L = \frac{3}{2} \int_{(8/9)+a^m}^{(8/9)+b^m} \sqrt{u} du = (8/9 + b^m)^{3/2} - (8/9 + a^m)^{3/2}$, and if $m < 0$, we have $L = \frac{3}{2} \int_{(8/9)+b^m}^{(8/9)+a^m} \sqrt{u} du = (8/9 + a^m)^{3/2} - (8/9 + b^m)^{3/2}$. Note that in the case $m = 0$, the curve is the constant $\mathbf{r}(t) = \langle 1, 1, 1 \rangle$, so $L = 0$.

14.4.47

a. $\mathbf{r}'(t) = h'(t) \langle A, B \rangle$, so $|\mathbf{r}'(t)| = |h'(t)| \sqrt{A^2 + B^2}$. Thus, $L = \sqrt{A^2 + B^2} \int_a^b |h'(t)| dt$

b. $L = \sqrt{2^2 + 5^2} \int_0^4 3t^2 dt = \sqrt{29} t^3 \Big|_0^4 = 64\sqrt{29}$.

c. $L = \sqrt{4^2 + 10^2} \int_1^8 |-1/t^2| dt = \sqrt{116} \frac{1}{t} \Big|_8^1 = \frac{7\sqrt{29}}{4}$.

14.4.48 With $A = 4$, $a = 1$, and $k = 35$, we have

$$\mathbf{r}(t) = \langle (4 + \cos 35t) \cos t, (4 + \cos 35t) \sin t, \sin 35t \rangle = \langle 4 \cos t + \cos t \cos 35t, 4 \sin t + \sin t \cos 35t, \sin 35t \rangle.$$

Then $\mathbf{r}'(t) = \langle -4 \sin t - 35 \cos t \sin 35t - \cos 35t \sin t, 4 \cos t - 35 \sin t \sin 35t + \cos 35t \cos t, 35 \cos 35t \rangle$. Using a computer algebra system, we find that $\int_0^{2\pi} |\mathbf{r}'(t)| dt \approx 221.4$ in.

14.4.49

- a. Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Then $y(t) = -4.9t^2 + 25t$ is 0 for $t > 0$ when $-4.9t + 25 = 0$, or $t = 25/4.9 \approx 5.102$ seconds.
- b. $L \approx \int_0^{5.102} \sqrt{400 + (25 - 9.8t)^2} dt$.
- c. Let $u = -9.8t + 25$ so that $du = -9.8dt$. Then $L \approx \frac{1}{-9.8} \int_{25}^{-24.9996} \sqrt{400 + u^2} du \approx 124.43$ meters.
- d. $x = u_0(5.102) = 20(5.102) = 102.04$ meters.

14.4.50

- a. Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Because $x^2 + y^2 = 1$, the path is the unit circle. The particle is at $(0, 1)$ at $t = 0$, and returns there at $t = \sqrt{2\pi}$.
- b. The length of the path is $2\pi = \int_0^{\sqrt{2\pi}} \sqrt{(2t \cos t^2)^2 + (2t \sin t^2)^2} dt = \int_0^{\sqrt{2\pi}} 2t dt$.
- c. This particle traces out the same circle as that one, but it does it faster, completing the circle in $\sqrt{2\pi}$ time units rather than 2π time units. Note that the speed of this particle is $2t$ rather than the constant 1 which is the speed of the other particle.
- d. This also traces out the same circle, over the time interval $[0, \sqrt[3]{2\pi}]$.
- e. It is also 2π because the arc is the same circle.
- f. The second runner would win the race. The runners occupy the same position at $t = 1$ (namely the point $(\sin(1), \cos(1))$), so that is when one passes the other.

14.4.51 Recall that a curve is parametrized by arc length exactly when $|\mathbf{v}| = 1$. We have $\mathbf{v} = \langle a, b, c \rangle$, so $|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$, which is equal to one if and only if $a^2 + b^2 + c^2 = 1$.

14.4.52 Suppose $a^2 = b^2 + c^2$. Recall that a curve is parametrized by arc length exactly when $|\mathbf{v}| = 1$. We have $\mathbf{v} = \langle -a \sin t, b \cos t, c \cos t \rangle$, so

$$|\mathbf{v}| = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t + c^2 \cos^2 t} = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = \sqrt{a^2},$$

which is equal to one if and only if $a^2 = b^2 + c^2 = 1$.

14.4.53
$$\int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{(cf'(t))^2 + (cg'(t))^2} dt = |c| \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt = |c| L.$$

14.4.54

- a. Let t_0 be a number on $[a, b]$, so that $\mathbf{r}(t_0) = \langle f(t_0), g(t_0), h(t_0) \rangle$ is a point on $\mathbf{r}$. Then $u^{-1}(t_0) = u_0$ is a number between $u^{-1}(a)$ and $u^{-1}(b)$. Note that $\mathbf{R}(u_0) = \langle f(u(u^{-1}(t_0))), g(u(u^{-1}(t_0))), h(u(u^{-1}(t_0))) \rangle = \langle f(t_0), g(t_0), h(t_0) \rangle = \mathbf{r}(t_0)$. So for every point on $\mathbf{r}$ there is a corresponding point on $\mathbf{R}$.

Now suppose that z_0 is a number between $u^{-1}(a)$ and $u^{-1}(b)$, so that $\mathbf{R}(z_0) = \langle f(u(z_0)), g(u(z_0)), h(u(z_0)) \rangle$ is a point on $\mathbf{R}$. Then $u(z_0) = s_0$ is a number between a and b , and $\mathbf{r}(s_0) = \langle f(s_0), g(s_0), h(s_0) \rangle = \langle f(u(z_0)), g(u(z_0)), h(u(z_0)) \rangle = \mathbf{R}(z_0)$. So for every point on $\mathbf{R}$ there is a corresponding point on $\mathbf{r}$.

Thus the two curves represent the same sets of points.

- b. Assume $u'(t) > 0$. The case $u'(t) < 0$ is similar. The length of $\mathbf{R}$ is

$$L = \int_{u^{-1}(a)}^{u^{-1}(b)} \sqrt{(f'(u)u'(t))^2 + (g'(u)u'(t))^2 + (h'(u)u'(t))^2} dt.$$

Let $s = u(t)$ so that $ds = u'(t) dt$. Then we have $L = \int_a^b \sqrt{f'(s)^2 + g'(s)^2 + h'(s)^2} ds$, which is the length of $\mathbf{r}$.

14.5 Curvature and Normal Vectors

14.5.1 A straight line has zero curvature.

14.5.2 The curvature is a measure of the magnitude of the rate of change of the unit tangent vector with respect to arc length. It is a scalar function.

$$\mathbf{14.5.3} \quad \kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| \text{ or } \kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3}.$$

14.5.4 The principal unit normal vector of a curve is a vector function whose value for any point on the curve is the vector perpendicular to the tangent to the curve, having unit length, and pointing to the inside of the curve.

$$\mathbf{14.5.5} \quad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

$$\mathbf{14.5.6} \quad \mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T}, \text{ where } a_N = \kappa |\mathbf{v}|^2 = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|} \text{ and } a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

14.5.7 $\mathbf{B}$ is a length one vector mutually perpendicular to $\mathbf{T}$ and $\mathbf{N}$. The three vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right-handed coordinate system.

$$\mathbf{14.5.8} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

14.5.9 The torsion is the rate at which the curve moves out of the osculating plane.

$$\mathbf{14.5.10} \quad \tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2}.$$

$$\mathbf{14.5.11} \quad \mathbf{r}'(t) = \langle 2, 4, 6 \rangle, \text{ so } \mathbf{T} = \frac{1}{2\sqrt{14}} \langle 2, 4, 6 \rangle = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle.$$

$$\text{So } \frac{d\mathbf{T}}{dt} = \langle 0, 0, 0 \rangle, \text{ and } \kappa = 0.$$

$$\mathbf{14.5.12} \quad \mathbf{r}'(t) = \langle -2 \sin t, -2 \cos t \rangle, \text{ so } \mathbf{T}(t) = \langle -\sin t, -\cos t \rangle. \text{ So}$$

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{8} |\langle -2 \cos t, 2 \sin t, 0 \rangle \times \langle -2 \sin t, -2 \cos t, 0 \rangle| = \frac{1}{2}.$$

$$\mathbf{14.5.13} \quad \mathbf{r}'(t) = \langle 2, 4 \cos t, -4 \sin t \rangle, \text{ so } \mathbf{T}(t) = \frac{1}{\sqrt{5}} \langle 1, 2 \cos t, -2 \sin t \rangle. \text{ So}$$

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{2\sqrt{5}} \left| \frac{1}{\sqrt{5}} \langle 0, -2 \sin t, -2 \cos t \rangle \right| = \frac{1}{5}.$$

$$\mathbf{14.5.14} \quad \mathbf{r}'(t) = \langle -2t \sin t^2, 2t \cos t^2 \rangle, \text{ so } \mathbf{T}(t) = \langle -\sin t^2, \cos t^2 \rangle. \text{ So}$$

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{2t} |\langle -2t \cos(t^2), -2t \sin(t^2) \rangle| = 1.$$

$$\mathbf{14.5.15} \quad \mathbf{r}'(t) = \langle \sqrt{3} \cos t, \cos t, -2 \sin t \rangle, \text{ so } |\mathbf{v}(t)| = \sqrt{3 \cos^2 t + \cos^2 t + 4 \sin^2 t} = 2.$$

$$\text{Thus, } \mathbf{T}(t) = \frac{1}{2} \langle \sqrt{3} \cos t, \cos t, -2 \sin t \rangle. \text{ So}$$

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{2} \left| \frac{1}{2} \langle -\sqrt{3} \sin t, -\sin t, -2 \cos t \rangle \right| = \frac{1}{2} \cdot \frac{1}{2} \cdot 2 = \frac{1}{2}.$$

14.5.16 $\mathbf{r}'(t) = \langle 1, -\tan t \rangle$, so $|\mathbf{v}(t)| = \sqrt{1 + \tan^2 t} = \sec t$. So $\mathbf{T}(t) = \langle \cos t, -\sin t \rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \cos t |\langle -\sin t, -\cos t \rangle| = \cos t.$$

14.5.17 $\mathbf{r}'(t) = \langle 1, 4t \rangle$, so $|\mathbf{v}(t)| = \sqrt{16t^2 + 1}$. Thus, $\mathbf{T}(t) = \frac{1}{\sqrt{16t^2 + 1}} \langle 1, 4t \rangle$.

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{(16t^2 + 1)^{3/2}} |\langle 0, 4, 0 \rangle \times \langle 1, 4t, 0 \rangle| = \frac{1}{(16t^2 + 1)^{3/2}} |\langle 0, 0, -4 \rangle| = \frac{4}{(16t^2 + 1)^{3/2}}.$$

14.5.18 $\mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle$, so $|\mathbf{v}(t)| = 3\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} = 3\sqrt{\cos^2 t \sin^2 t} = 3|\cos t \sin t|$. Thus $\mathbf{T}(t) = \langle -\cos t, \sin t \rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{3|\sin t \cos t|} |\langle \sin t, \cos t \rangle| = \frac{1}{3} |\sec t \csc t|$$

14.5.19 $\mathbf{r}'(t) = \langle \cos(\pi t^2/2), \sin(\pi t^2/2) \rangle$, so $|\mathbf{v}(t)| = 1$. So $\mathbf{T}(t) = \mathbf{v}(t) = \langle \cos(\pi t^2/2), \sin(\pi t^2/2) \rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = 1 \cdot |\langle -\pi t \sin(\pi t^2/2), \pi t \cos(\pi t^2/2) \rangle| = \pi t.$$

14.5.20 $\mathbf{r}'(t) = \langle \cos(t^2), \sin(t^2) \rangle$, so $|\mathbf{v}(t)| = 1$. So $\mathbf{T}(t) = \mathbf{v}(t) = \langle \cos(t^2), \sin(t^2) \rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = 1 \cdot |\langle -2t \sin(t^2), 2t \cos(t^2) \rangle| = 2t.$$

14.5.21 $\mathbf{r}'(t) = \langle 3 \sin t, 3 \cos t \rangle$ and $\mathbf{r}''(t) = \langle 3 \cos t, -3 \sin t \rangle$. So

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{27} |\langle 3 \cos t, -3 \sin t, 0 \rangle \times \langle 3 \sin t, 3 \cos t, 0 \rangle| = \frac{1}{27} |\langle 0, 0, 9 \rangle| = \frac{1}{3}.$$

14.5.22 $\mathbf{r}'(t) = \langle 4, 3 \cos t, -3 \sin t \rangle$ and $\mathbf{r}''(t) = \langle 0, -3 \sin t, -3 \cos t \rangle$.

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{125} |\langle 0, -3 \sin t, -3 \cos t \rangle \times \langle 4, 3 \cos t, -3 \sin t \rangle| = \frac{1}{125} |\langle -9, 12 \cos t, 12 \sin t \rangle| = \frac{15}{125} = \frac{3}{25}.$$

14.5.23 $\mathbf{r}'(t) = \langle 2t, 1, 0 \rangle$ and $\mathbf{r}''(t) = \langle 2, 0, 0 \rangle$. So

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{(4t^2 + 1)^{3/2}} |\langle 2, 0, 0 \rangle \times \langle 2t, 1, 0 \rangle| = \frac{1}{(4t^2 + 1)^{3/2}} |\langle 0, 0, 2 \rangle| = \frac{2}{(4t^2 + 1)^{3/2}}.$$

14.5.24 $\mathbf{r}'(t) = \langle \sqrt{3} \cos t, \cos t, -2 \sin t \rangle$ and $\mathbf{r}''(t) = \langle -\sqrt{3} \sin t, -\sin t, -2 \cos t \rangle$. So

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{8} \left| \langle -\sqrt{3} \sin t, -\sin t, -2 \cos t \rangle \times \langle \sqrt{3} \cos t, \cos t, -2 \sin t \rangle \right| = \frac{1}{8} \left| \langle 2, -2\sqrt{3}, 0 \rangle \right| = \frac{4}{8} = \frac{1}{2}.$$

14.5.25 $\mathbf{r}'(t) = \langle -4 \sin t, \cos t, -2 \sin t \rangle$ and $\mathbf{r}''(t) = \langle -4 \cos t, -\sin t, -2 \cos t \rangle$.

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{|\langle 2, 0, -4 \rangle|}{\left(\sqrt{20 \sin^2(t) + \cos^2(t)} \right)^3} = \frac{2\sqrt{5}}{(20 \sin^2(t) + \cos^2(t))^{3/2}}.$$

14.5.26 $\mathbf{r}'(t) = \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \rangle$ and $\mathbf{r}''(t) = \langle -2e^t \sin t, 2e^t \cos t, e^t \rangle$. $|\mathbf{v}(t)| = \sqrt{3}e^t$. So

$$\begin{aligned}\kappa &= \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{3\sqrt{3}e^{3t}} |\langle -2e^t \sin t, 2e^t \cos t, e^t \rangle \times \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \rangle| \\ &= \frac{1}{3\sqrt{3}e^{3t}} |e^{2t} \langle \cos t - \sin t, \cos t + \sin t, -2 \rangle| = \frac{\sqrt{6}e^{2t}}{3\sqrt{3}e^{3t}} = \frac{\sqrt{2}}{3e^t}.\end{aligned}$$

14.5.27 $\mathbf{r}'(t) = \langle 2 \cos t, -2 \sin t \rangle$, so $\mathbf{T} = \langle \cos t, -\sin t \rangle$ and $|\mathbf{T}| = 1$. $\mathbf{N}(t) = \langle -\sin t, -\cos t \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = -\sin t \cos t + \sin t \cos t = 0$.

14.5.28 $\mathbf{r}'(t) = \langle 4 \cos t, 4 \sin t, 10 \rangle$, and $|\mathbf{r}'(t)| = 2\sqrt{29}$, so $\mathbf{T} = \frac{1}{\sqrt{29}} \langle 2 \cos t, -2 \sin t, 5 \rangle$ and $|\mathbf{T}| = 1$. $\mathbf{N}(t) = \langle -\sin t, -\cos t, 0 \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

14.5.29 $\mathbf{r}'(t) = \langle t, -3, 0 \rangle$, and $|\mathbf{r}'(t)| = \sqrt{t^2 + 9}$, so $\mathbf{T} = \frac{1}{\sqrt{t^2 + 9}} \langle t, -3, 0 \rangle$ and $|\mathbf{T}| = 1$. We have $\mathbf{T}'(t) = \left\langle \frac{9}{(\sqrt{t^2 + 9})^3}, \frac{3t}{(\sqrt{t^2 + 9})^3}, 0 \right\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{t^2 + 9}} \langle 3, t, 0 \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

14.5.30 $\mathbf{r}'(t) = \langle t, t^2 \rangle$, and $|\mathbf{r}'(t)| = |t| \sqrt{t^2 + 1}$, so $\mathbf{T} = \frac{1}{\sqrt{t^2 + 1}} \langle 1, t \rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \left\langle -\frac{t}{(\sqrt{t^2 + 1})^3}, \frac{1}{(\sqrt{t^2 + 1})^3} \right\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{t^2 + 1}} \langle -t, 1 \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

14.5.31 $\mathbf{r}'(t) = \langle -2t \sin t^2, 2t \cos t^2 \rangle$, and $|\mathbf{r}'(t)| = 2t$, so $\mathbf{T} = \langle -\sin t^2, \cos t^2 \rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle -2t \cos t^2, -2t \sin t^2 \rangle$, so $\mathbf{N}(t) = \langle -\cos t^2, -\sin t^2 \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

14.5.32 $\mathbf{r}'(t) = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t \rangle$, and $|\mathbf{r}'(t)| = 3 |\sin t \cos t|$, so $\mathbf{T} = \langle -\cos t, \sin t \rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle \sin t, \cos t \rangle$, so $\mathbf{N}(t) = \langle \sin t, \cos t \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

14.5.33 $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, and $|\mathbf{r}'(t)| = \sqrt{4t^2 + 1}$, so $\mathbf{T} = \frac{1}{\sqrt{4t^2 + 1}} \langle 2t, 1 \rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \left\langle \frac{2}{(\sqrt{4t^2 + 1})^3}, -\frac{4t}{(\sqrt{4t^2 + 1})^3} \right\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{4t^2 + 1}} \langle 1, -2t \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

14.5.34 $\mathbf{r}'(t) = \langle 1, -\tan t \rangle$, and $|\mathbf{r}'(t)| = \sec t$, so $\mathbf{T} = \langle \cos t, -\sin t \rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle -\sin t, -\cos t \rangle$, so $\mathbf{N}(t) = \langle -\sin t, -\cos t \rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

14.5.35 $\mathbf{r}'(t) = \langle 1, 4, -6 \rangle$ and $\mathbf{r}''(t) = \langle 0, 0, 0 \rangle$. So $\kappa = 0$ and thus $a_N = 0$. Also, $a_T = 0$. We have $\mathbf{a} = 0 \cdot \mathbf{T} + 0 \cdot \mathbf{N} = \langle 0, 0, 0 \rangle$.

14.5.36 $\mathbf{r}'(t) = \langle -10 \sin t, -10 \cos t \rangle$ and $\mathbf{r}''(t) = \langle -10 \cos t, 10 \sin t \rangle$. Note that $\mathbf{a} \cdot \mathbf{v} = 0$, so $a_T = 0$. $\mathbf{a} \times \mathbf{v} = \langle 0, 0, 100 \rangle$, so $a_N = \frac{100}{10} = 10$. We have $\mathbf{a} = \langle -10 \cos t, 10 \sin t \rangle = 0 \cdot \mathbf{T} + 10\mathbf{N}$.

14.5.37 In problem 26 above, we computed $|\mathbf{v}(t)| = \sqrt{3}e^t$ and $\kappa = \frac{\sqrt{2}}{3e^t}$. Also, $\mathbf{v}(t) \cdot \mathbf{a}(t) = 3e^{2t}$. Thus $a_T = \frac{3e^{2t}}{\sqrt{3}e^t} = \sqrt{3}e^t$ and $a_N = \frac{\sqrt{2}}{3e^t} \cdot 3e^{2t} = \sqrt{2}e^t$.

14.5.38 $\mathbf{r}'(t) = \langle 1, 2t \rangle$ and $\mathbf{r}''(t) = \langle 0, 2 \rangle$. Note that $\mathbf{a} \cdot \mathbf{v} = 4t$, so $a_T = \frac{4t}{\sqrt{4t^2 + 1}}$. $\mathbf{a} \times \mathbf{v} = \langle 0, 0, -2 \rangle$, so $a_N = \frac{2}{\sqrt{4t^2 + 1}}$. We have $\mathbf{a} = \langle 0, 2 \rangle = \frac{4t}{\sqrt{4t^2 + 1}} \cdot \mathbf{T} + \frac{2}{\sqrt{4t^2 + 1}} \cdot \mathbf{N}$.

14.5.39 $\mathbf{r}'(t) = \langle 3t^2, 2t \rangle$ and $\mathbf{r}''(t) = \langle 6t, 2 \rangle$. Note that $\mathbf{a} \cdot \mathbf{v} = 18t^3 + 4t$, so $a_T = \frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2}} = \frac{18t^2 + 4}{\sqrt{9t^2 + 4}}$. $\mathbf{a} \times \mathbf{v} = \langle 0, 0, 6t^2 \rangle$, so $a_N = \frac{6t^2}{t\sqrt{9t^2 + 4}} = \frac{6t}{\sqrt{9t^2 + 4}}$. We have $\mathbf{a} = \langle 6t, 2 \rangle = \frac{18t^2 + 4}{\sqrt{9t^2 + 4}} \cdot \mathbf{T} + \frac{6t}{\sqrt{9t^2 + 4}} \cdot \mathbf{N}$.

14.5.40 $\mathbf{r}'(t) = \langle -20 \sin t, 20 \cos t, 30 \rangle$ and $\mathbf{r}''(t) = \langle -20 \cos t, -20 \sin t, 0 \rangle$. Note that $\mathbf{a} \cdot \mathbf{v} = 0$, so $a_T = 0$.
 $\mathbf{a} \times \mathbf{v} = \langle -600 \sin t, 600 \cos t, -400 \rangle$, so $a_N = \frac{200\sqrt{13}}{10\sqrt{13}} = 20$. We have $\mathbf{a} = \langle -20 \cos t, -20 \sin t, 0 \rangle = 0 \cdot \mathbf{T} + 20\mathbf{N}$.

14.5.41 $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \langle \cos t, -\sin t, 0 \rangle \times \langle -\sin t, -\cos t, 0 \rangle = \langle 0, 0, -1 \rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

14.5.42 $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left\langle \frac{2 \cos(t)}{\sqrt{29}}, \frac{-2 \sin(t)}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle \times \langle -\sin t, -\cos t, 0 \rangle = \frac{1}{\sqrt{29}} \langle 5 \cos t, -5 \sin t, -2 \rangle$. Also,
 $\tau = \frac{-d\mathbf{B}/dt}{ds/dt} \cdot \mathbf{N} = \left\langle \frac{5 \sin(t)}{58}, \frac{5 \cos(t)}{58}, 0 \right\rangle \cdot \langle -\sin t, -\cos t, 0 \rangle = -\frac{5}{58}$.

14.5.43 $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left\langle \frac{t}{\sqrt{t^2+9}}, -\frac{3}{\sqrt{t^2+9}}, 0 \right\rangle \times \left\langle \frac{3}{\sqrt{t^2+9}}, \frac{t}{\sqrt{t^2+9}}, 0 \right\rangle = \langle 0, 0, 1 \rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

14.5.44 $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left\langle \frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}}, 0 \right\rangle \times \left\langle -\frac{t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}}, 0 \right\rangle = \langle 0, 0, 1 \rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

14.5.45 $\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, -1 \rangle$, so $\mathbf{T} = \langle (-2/\sqrt{5}) \sin t, (2/\sqrt{5}) \cos t, -1/\sqrt{5} \rangle$.

$\mathbf{r}''(t) = \langle -2 \cos t, -2 \sin t, 0 \rangle$, so $\mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle$. Thus, $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}} \langle -\sin t, \cos t, 2 \rangle$. Also,
 $\tau = \frac{-d\mathbf{B}/dt}{ds/dt} \cdot \mathbf{N} = -\frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} (\langle -\cos t, -\sin t, 0 \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle) = -\frac{1}{5}$.

14.5.46 $\mathbf{r}'(t) = \langle 1, \sinh t, -\cosh t \rangle$, so $\mathbf{r}''(t) = \langle 0, \cosh t, -\sinh t \rangle$. $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 1, \sinh t, \cosh t \rangle$. Note that
 $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{1 + \sinh^2 t + \cosh^2 t} = \sqrt{2} \cosh t$. Therefore, $\mathbf{B} = \frac{1}{\sqrt{2}} \langle \operatorname{sech} t, \tanh t, 1 \rangle$.

Now $\mathbf{r}'''(t) = \langle 0, \sinh t, -\cosh t \rangle$ so $(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = \langle 1, \sinh t, \cosh t \rangle \cdot \langle 0, \sinh t, -\cosh t \rangle = -1$.
Thus $\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = -\frac{1}{2 \cosh^2 t} = -\frac{1}{2} \operatorname{sech}^2 t$.

14.5.47 $\mathbf{r}'(t) = \langle 12, -5 \sin t, 5 \cos t \rangle$, so $\mathbf{r}''(t) = \langle 0, -5 \cos t, -5 \sin t \rangle$. Thus,

$$\mathbf{T} = \langle 12/13, (-5/13) \sin t, (5/13) \cos t \rangle$$

and

$$\mathbf{N} = \langle 0, -\cos t, -\sin t \rangle.$$

So

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \langle (5/13), (12/13) \sin t, (-12/13) \cos t \rangle.$$

$$\text{Also, } \tau = -\frac{d\mathbf{B}/dt}{ds/dt} \cdot \mathbf{N} = -\frac{1}{13} \cdot \frac{1}{13} (\langle 0, 12 \cos t, 12 \sin t \rangle \cdot \langle 0, -\cos t, -\sin t \rangle) = -\frac{1}{169} \cdot (-12) = \frac{12}{169}.$$

14.5.48 $\mathbf{r}'(t) = \langle t \sin t, t \cos t, 1 \rangle$, and $\mathbf{r}''(t) = \langle \sin t + t \cos t, \cos t - t \sin t, 0 \rangle$. Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle t \sin t - \cos t, \sin t + t \cos t, -t^2 \rangle.$$

We then have $\mathbf{B} = \frac{1}{\sqrt{1+t^2+t^4}} \langle t \sin t - \cos t, \sin t + t \cos t, -t^2 \rangle$. Note that

$$\mathbf{r}'''(t) = \langle 2 \cos t - t \sin t, -2 \sin t - t \cos t, 0 \rangle.$$

Then we have $\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = -\frac{(2+t^2)}{1+t^2+t^4}$.

14.5.49

- a. False. For example, consider $\mathbf{r}(t) = \langle \cos t, \sin t, 1 \rangle$. Then $\mathbf{T} = \langle -\sin t, \cos t, 0 \rangle$ and also $\mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle$. Note that $\mathbf{T}$ and $\mathbf{N}$ lie in the xy -plane, but $\mathbf{r}$ doesn't.
- b. False. $\mathbf{T}$ does depend on the orientation, but $\mathbf{N}$ doesn't. Reversing the orientation changes $\mathbf{T}$ to $-\mathbf{T}$, but leaves $\mathbf{N}$ alone.
- c. False. Note that $|\mathbf{T}|$ is independent of orientation, so $\left| \frac{d\mathbf{T}}{ds} \right|$ is too.
- d. True. As we have already seen for the circle, $\mathbf{v} \cdot \mathbf{a} = 0$. Thus $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} = 0 \cdot \mathbf{T} + \kappa |\mathbf{v}|^2 \mathbf{N} = \frac{1}{R} \mathbf{N}$.
- e. False. For example, if the car's motion is given by $\mathbf{r}(t) = \langle 60 \cos t, 60 \sin t \rangle$, then the speed is a constant 60, but $\mathbf{a} = \langle -60 \cos t, -60 \sin t \rangle \neq \langle 0, 0 \rangle$.
- f. False. If it lies in the xy -plane, it will have zero torsion.
- g. False. If it lies in the xy -plane, it might have very large curvature but zero torsion.

14.5.50 Let $\mathbf{r}(t) = \langle t, f(t), 0 \rangle$. Then $\mathbf{r}'(t) = \langle 1, f'(t), 0 \rangle$ and $\mathbf{r}''(t) = \langle 0, f''(t), 0 \rangle$. Note that $|\mathbf{v}(t)| = \sqrt{1 + f'(t)^2}$, and $\mathbf{a} \times \mathbf{v} = \langle 0, 0, -f''(t) \rangle$, so $\kappa = \frac{|f''(t)|}{(1 + f'(t)^2)^{3/2}}$.

14.5.51 $f'(x) = 2x$ and $f''(x) = 2$, so

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

14.5.52 $f'(x) = -\frac{x}{\sqrt{a^2 - x^2}}$ and $f''(x) = -\frac{a^2}{(a^2 - x^2)^{3/2}}$, so

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{\left| -\frac{a^2}{(a^2 - x^2)^{3/2}} \right|}{\left(1 + \left(-\frac{x}{\sqrt{a^2 - x^2}} \right)^2 \right)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}.$$

14.5.53 $f'(x) = 1/x$ and $f''(x) = -1/x^2$, so

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{1/x^2}{(1 + (1/x)^2)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}.$$

14.5.54 $f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$, so

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{\sec^2 x}{(1 + \tan^2 x)^{3/2}} = \frac{\sec^2 x}{\sec^3 x} = \cos x.$$

14.5.55 $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$ and $\mathbf{r}''(t) = \langle f''(t), g''(t) \rangle$. So

$$\mathbf{a} \times \mathbf{v} = \langle f''(t), g''(t), 0 \rangle \times \langle f'(t), g'(t), 0 \rangle = \langle 0, 0, g'(t)f''(t) - f'(t)g''(t) \rangle.$$

Because $|\mathbf{v}| = \sqrt{(f'(t))^2 + (g'(t))^2}$, we have

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}}.$$

14.5.56 $f'(t) = a \cos t$ and $f''(t) = -a \sin t$, while $g'(t) = -a \sin t$ and $g''(t) = -a \cos t$. So

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}} = \frac{|-a^2 \cos^2 t - a^2 \sin^2 t|}{(a^2 \cos^2 t + a^2 \sin^2 t)^{3/2}} = \left| \frac{a^2}{a^3} \right| = \frac{1}{|a|}.$$

14.5.57 $f'(t) = a \cos t$ and $f''(t) = -a \sin t$, while $g'(t) = -b \sin t$ and $g''(t) = -b \cos t$. So

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}} = \frac{|-ab \cos^2 t - ab \sin^2 t|}{(a^2 \cos^2 t + b^2 \sin^2 t)^{3/2}} = \frac{|ab|}{(a^2 \cos^2 t + b^2 \sin^2 t)^{3/2}}.$$

14.5.58 $f'(t) = -3a \cos^2 t \sin t$ and $f''(t) = 3a \cos t(3 \sin^2 t - 1)$, while $g'(t) = 3a \sin^2 t \cos t$ and $g''(t) = 3a \sin t(3 \cos^2 t - 1)$. So

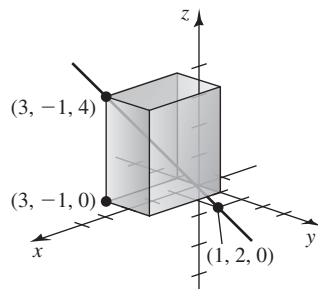
$$\kappa = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}} = \frac{|9a^2 \cos^2 t \sin^2 t(3 \cos^2 t - 1 + 3 \sin^2 t - 1)|}{27a^3 |\cos^3 t \sin^3 t|} = \frac{1}{3a |\cos t \sin t|}.$$

14.5.59 $f'(t) = 1$ and $f''(t) = 0$, while $g'(t) = 2at$ and $g''(t) = 2a$. So

$$\kappa = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}} = \frac{|2a - 0|}{(1 + 4a^2t^2)^{3/2}} = \frac{2|a|}{(1 + 4a^2t^2)^{3/2}}.$$

14.5.60

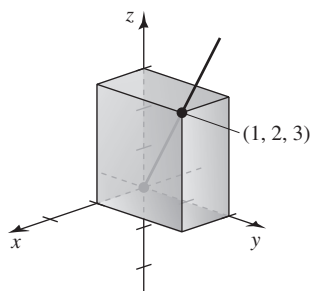
a.



- b. For line A: $\mathbf{v}(t) = \langle 2, -3, 4 \rangle$ and $\mathbf{a}(t) = \langle 0, 0, 0 \rangle$.
For line B: $\mathbf{v}(t) = \langle 6, -9, 12 \rangle$ and $\mathbf{a}(t) = \langle 0, 0, 0 \rangle$.
B has 3 times the velocity of A, but the same acceleration.
- c. For both lines, we have $a_N = 0$ and $a_T = 0$. $\mathbf{a} = \langle 0, 0, 0 \rangle = 0 \cdot \mathbf{T} = 0 \cdot \mathbf{N}$.

14.5.61

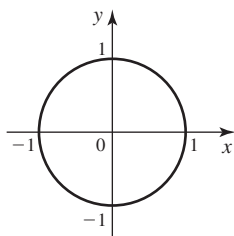
a.



- b. For line A: $\mathbf{v}(t) = \langle 1, 2, 3 \rangle$ and $\mathbf{a}(t) = \langle 0, 0, 0 \rangle$.
For line B: $\mathbf{v}(t) = \langle 2t, 4t, 6t \rangle$ and $\mathbf{a}(t) = \langle 2, 4, 6 \rangle$.
A has constant velocity and zero acceleration, while B has linearly increasing velocity and constant acceleration.
- c. For A, we have $a_T = a_N = 0$. For B, we have $a_T = \frac{56t}{\sqrt{56t}} = 2\sqrt{14}$ and $a_N = 0$ (because $\mathbf{v}$ and $\mathbf{a}$ are in the same direction).

14.5.62

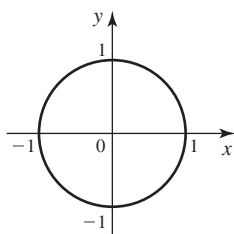
a.



- b. For curve A: $\mathbf{v}(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t \rangle$.
For curve B: $\mathbf{v}(t) = \langle -3 \sin 3t, 3 \cos 3t \rangle$ and $\mathbf{a}(t) = \langle -9 \cos 3t, -9 \sin 3t \rangle$.
A has constant velocity 1 while B has constant velocity 3. The magnitude of the acceleration on B is 9 times that on A.
- c. For A, we have $\mathbf{a} = \mathbf{N}$, so $a_T = 0$ and $a_N = 1$.
For B, we have $\mathbf{a} = 9\mathbf{N}$, so $a_T = 0$ and $a_N = 9$.

14.5.63

a.



b. For curve A: $\mathbf{v}(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t \rangle$.

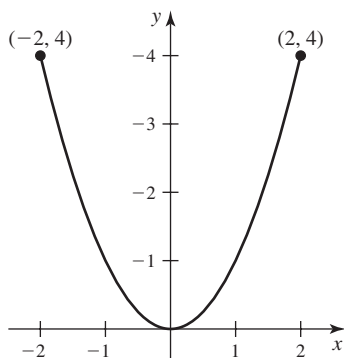
For curve B: $\mathbf{v}(t) = 2t \langle -\sin t^2, \cos t^2 \rangle$ and $\mathbf{a}(t) = \langle -4t^2 \cos t^2 - 2 \sin t^2, -4t^2 \sin t^2 + 2 \cos t^2 \rangle$.

A has constant velocity 1 while B does not have constant velocity.

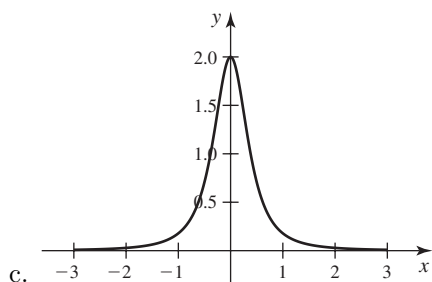
c. For A, we have $\mathbf{a} = \mathbf{N}$, so $a_T = 0$ and $a_N = 1$.
For B, we have $a_T = 2$ and $a_N = 4t^2$.

14.5.64

a.



$$\text{b. } \kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}.$$

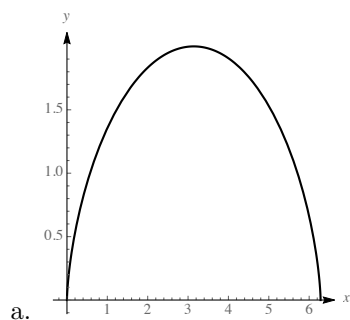


$$\text{d. } \kappa'(x) = -\frac{24x}{(1 + 4x^2)^{5/2}}. \quad \kappa \text{ has a maximum at } 0.$$

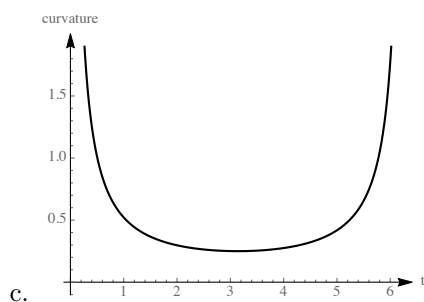
$$\kappa''(x) = \frac{24(16x^2 - 1)}{(1 + 4x^2)^{7/2}}. \quad \kappa \text{ has inflection points at } x = \pm 1/4.$$

e. Symmetry of $y = x^2$ implies symmetry in κ , which does occur. The parabola appears to have greater curvature near 0 and less near the endpoints, which is consistent with the graph of κ .

14.5.65



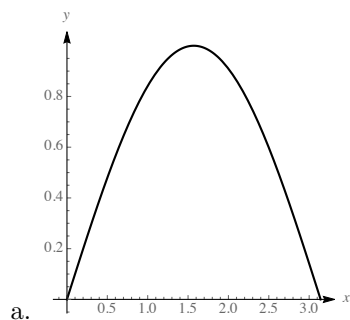
$$\begin{aligned} \text{b. } \kappa &= \frac{|f'g'' - f''g'|}{(f'(x)^2 + g'(x)^2)^{3/2}} = \\ &= \frac{|(1 - \cos t) \cos t - \sin t \sin t|}{((1 - \cos t)^2 + \sin^2 t)^{3/2}} = \\ &= \frac{1 - \cos t}{2\sqrt{2}(1 - \cos t)^{3/2}} = \frac{1}{2\sqrt{2}\sqrt{1 - \cos t}}. \end{aligned}$$



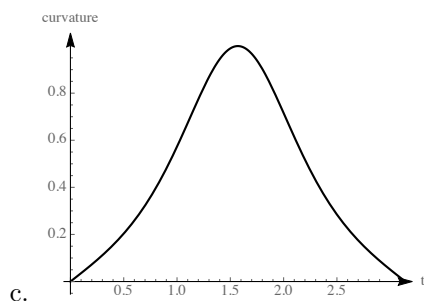
$$\begin{aligned} \text{d. } \kappa'(t) &= -\frac{\sin t}{4\sqrt{2}(1 - \cos t)^{3/2}}. \kappa \text{ has a minimum at} \\ &\pi. \kappa''(t) = \frac{\sin^2\left(\frac{t}{2}\right)(\cos t + 3)}{4\sqrt{2}(1 - \cos(t))^{5/2}}. \kappa \text{ has no inflec-} \\ &\text{tion points on the given interval.} \end{aligned}$$

- e. Symmetry of the given curve about π (on the interval $(0, 2\pi)$) implies symmetry in κ , which does occur. The curve is flatter near π and more curved near 0 and 2π .

14.5.66



$$\text{b. } \kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{\sin x}{(1 + \cos^2 x)^{3/2}}.$$



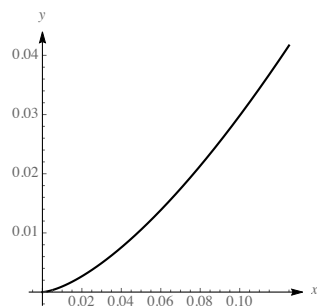
$$\text{d. } \kappa'(x) = -\frac{\cos(x)(\cos(2x) - 3)}{(\cos^2(x) + 1)^{5/2}}. \kappa \text{ has a maximum at } \pi/2.$$

$$\kappa''(x) = -\frac{2\sqrt{2}(-4\sin(x) - 19\sin(3x) + \sin(5x))}{(\cos(2x) + 3)^{7/2}}.$$

κ has inflection points at approximately 1.122 and $\pi - 1.122$.

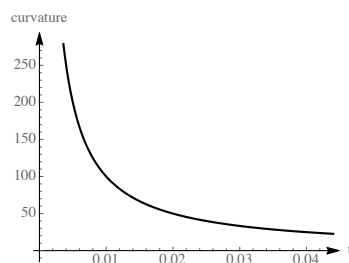
- e. Symmetry of the given curve about $\pi/2$ (on the interval $(0, \pi)$) implies symmetry in κ , which does occur. The curve is flatter near 0 and π and more curved near $\pi/2$.

14.5.67



a.

$$\text{b. } \kappa = \frac{|f'g'' - f''g'|}{(f'(x)^2 + g'(x)^2)^{3/2}} = \frac{|t \cdot 2t - 1 \cdot t^2|}{(t^2 + t^4)^{3/2}} = \frac{1}{t(1+t^2)^{3/2}}.$$



c.

$$\text{d. } \kappa'(t) = \frac{-4t^2 - 1}{t^2(t^2 + 1)^{5/2}}. \text{ } \kappa \text{ has no extrema. } \kappa''(t) = \frac{20t^4 + 7t^2 + 2}{t^3(t^2 + 1)^{7/2}}. \text{ } \kappa \text{ has no inflection points.}$$

e. The curve gets flatter as $t \rightarrow \infty$.

$$\text{14.5.68 } y'(x) = 1/x \text{ and } y''(x) = -1/x^2. \text{ So } \kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{1/x^2}{(1 + (1/x^2))^3/2} = \frac{x}{(x^2 + 1)^{3/2}}.$$

$\kappa'(x) = \frac{1 - 2x^2}{(x^2 + 1)^{5/2}}$, which is 0 for $x > 0$ only for $x = \sqrt{2}/2$. The first derivative test shows that this is where the maximum curvature exists, and the value of the maximum curvature is $\kappa(\sqrt{2}/2) = 2\sqrt{3}/9$.

$$\text{14.5.69 } y'(x) = e^x \text{ and } y''(x) = e^x. \text{ So } \kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{e^x}{(1 + (e^{2x}))^{3/2}}.$$

$\kappa'(x) = \frac{e^x - 2e^{3x}}{(e^{2x} + 1)^{5/2}}$, which is 0 for $x = -\frac{\ln 2}{2}$. The first derivative test shows that this is where the maximum curvature exists, and the value of the maximum curvature is $\kappa((-\ln 2)/2) = 2\sqrt{3}/9$.

14.5.70 By example 3, the curvature of a circle centered at the origin is the reciprocal of the radius of the circle. Because curvature is independent of the coordinate system, the curvature of the circle of curvature is the reciprocal of its radius, but by definition, its curvature is κ , so its radius is $\frac{1}{\kappa}$.

14.5.71 $\mathbf{r}'(t) = \langle 1, 2t \rangle$ and $\mathbf{r}''(t) = \langle 0, 2 \rangle$. Thus $\kappa = \frac{2}{(1 + 4t^2)^{3/2}}$. At $t = 0$, we have $\kappa = 2$. So we are seeking a circle of radius $1/2$. The center of the osculating circle is the point along the normal line to the curve at $(0, 0)$ which is $1/2$ unit from $(0, 0)$ and is on the inside of the curve, so it is $(0, 1/2)$. The equation is $x^2 + (y - (1/2))^2 = \frac{1}{4}$.

14.5.72 $y' = 1/x$ and $y'' = -1/x^2$, so $\kappa(x) = \frac{|1/x^2|}{(1 + (1/x)^2)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}$. At $x = 1$ we have $\kappa = \frac{1}{2\sqrt{2}}$, so the radius of the osculating circle is $2\sqrt{2}$. The center of the osculating circle is the point along the normal line to the curve at $(1, 0)$ which is $2\sqrt{2}$ units from $(1, 0)$ and is on the inside of the curve, so it is $(3, -2)$. The equation of the osculating circle is $(x - 3)^2 + (y + 2)^2 = 8$.

14.5.73 $\mathbf{r}'(t) = \langle 1 - \cos t, \sin t \rangle$ and $\mathbf{r}''(t) = \langle \sin t, -\cos t \rangle$. Thus $\kappa(\pi) = \frac{(2-0)}{(2^2+0^2)^{3/2}} = \frac{2}{8} = \frac{1}{4}$, so the radius of the osculating circle is 4. The center of the osculating circle is the point along the normal line to the curve at $(\pi, 2)$ which is 4 units from $(\pi, 2)$ and is on the inside of the curve, so it is $(\pi, -2)$. The equation of the osculating circle is $(x - \pi)^2 + (y + 2)^2 = 16$.

14.5.74

a. There is an instantaneous change in curvature from 0 to $\frac{1}{a}$ at the origin.

b. $\mathbf{r}'(t) = \left\langle \cos \frac{t^2}{2}, \sin \frac{t^2}{2} \right\rangle$ by the Fundamental Theorem of Calculus. So $|\mathbf{v}(t)| = 1$. Then $\kappa = |T'(t)| = \left| \left\langle -t \sin \frac{t^2}{2}, t \cos \frac{t^2}{2} \right\rangle \right| = t$. The curvature of the easement curve is 0 at the origin and gradually increases.

c. We would need the radius to be $\frac{1}{\kappa} = \frac{1}{t} = \frac{1}{1.2} \approx 0.83$.

14.5.75 $y' = n \cos nx$ and $y'' = -n^2 \sin nx$, so $\kappa(\pi/2n) = \frac{|-n^2 \sin(\pi/2)|}{(1 + (n^2 \cos^2(\pi/2))^{3/2}} = \frac{n^2}{(1+0)^{3/2}} = n^2$. This increases as n increases.

14.5.76 Note that $\mathbf{r}'(t) = \langle 1, 2t \rangle$, so $\mathbf{T} = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$. So $\mathbf{T}'(t) = \left\langle -\frac{4t}{(4t^2+1)^{3/2}}, \frac{2}{(4t^2+1)^{3/2}} \right\rangle$, so $\mathbf{N} = \frac{1}{\sqrt{4t^2+1}} \langle -2t, 1 \rangle$. Then

$$\mathbf{a} = \frac{2}{\sqrt{1+4t^2}} \left(\left\langle -\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}} \right\rangle + 2t \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle \right) = \langle 0, 2 \rangle.$$

14.5.77

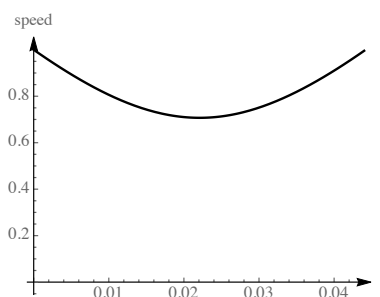
$\mathbf{r}'(t) = \langle V_0 \cos \alpha, V_0 \sin \alpha - gt \rangle$, so the speed is

$$\sqrt{V_0^2 \cos^2 \alpha + V_0^2 \sin^2 \alpha - 2gtV_0 \sin \alpha + g^2 t^2},$$

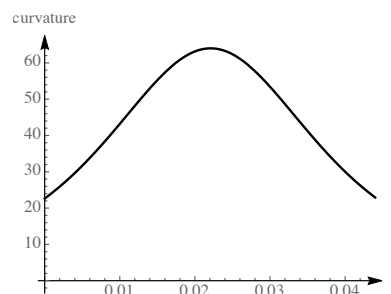
a. which can be written

$$\sqrt{V_0^2 - 2gtV_0 \sin \alpha + g^2 t^2}.$$

The graph shown is for $V_0 = 1$, $g = 32$, and $\alpha = 45$ degrees.



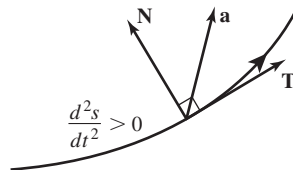
$\mathbf{a} = \langle 0, -g \rangle$. We have $\mathbf{a} \times \mathbf{v} = \langle 0, 0, gV_0 \cos \alpha \rangle$. So $\kappa = \frac{|gV_0 \cos \alpha|}{(V_0^2 - 2gtV_0 \sin \alpha + g^2 t^2)^{3/2}}$. The graph shown is for $V_0 = 1$, $g = 32$, and $\alpha = 45$ degrees.



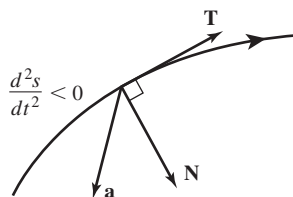
- c. The speed has a minimum and the curvature has a maximum at the halfway time of the flight, namely at $\frac{V_0 \sin \alpha}{g}$.

14.5.78

Suppose $\frac{d^2s}{dt^2} > 0$. Recall that $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$, and that $a_T = \frac{d^2s}{dt^2} > 0$. Also $a_N = \kappa |\mathbf{v}|^2 > 0$. So $\mathbf{a}$ is a linear combination of $\mathbf{T}$ and $\mathbf{N}$ with positive coefficients, so it lies between the two of them.



Suppose $\frac{d^2s}{dt^2} < 0$. Recall that $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$, and that $a_T = \frac{d^2s}{dt^2} < 0$. Also $a_N = \kappa |\mathbf{v}|^2 > 0$. So $\mathbf{a}$ is a linear combination of $\mathbf{T}$ and $\mathbf{N}$ with a negative coefficient on $\mathbf{T}$ and a positive coefficient on $\mathbf{N}$, so it does not lie between the two of them.



14.5.79 $\mathbf{r}'(t) = \langle pbt^{p-1}, pdt^{p-1}, pft^{p-1} \rangle = pt^{p-1} \langle b, d, f \rangle$. $\mathbf{r}''(t) = p(p-1)t^{p-2} \langle b, d, f \rangle$. Because for any t , $\mathbf{r}''(t)$ and $\mathbf{r}'(t)$ are multiples of $\langle b, d, f \rangle$, their cross product is $\langle 0, 0, 0 \rangle$. Thus $\kappa = \frac{0}{|\mathbf{v}|^3} = 0$. The given curve represents a straight line, which has zero curvature.

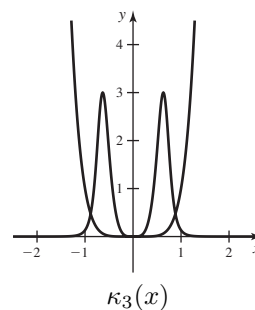
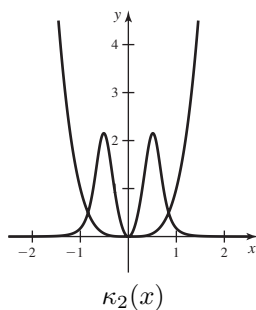
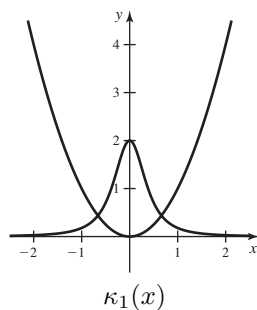
14.5.80 $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = \frac{d\mathbf{T}}{ds} |\mathbf{r}'(t)|$. So

$$\frac{\frac{d\mathbf{T}}{dt}}{\left| \frac{d\mathbf{T}}{dt} \right|} = \frac{\frac{d\mathbf{T}}{ds} |\mathbf{r}'(t)|}{\left| \frac{d\mathbf{T}}{ds} |\mathbf{r}'(t)| \right|} = \mathbf{N}.$$

14.5.81

- a. $f'_n(x) = 2nx^{2n-1}$ and $f''_n(x) = (2n)(2n-1)x^{2n-2}$. $\kappa = \frac{|2n(2n-1)x^{2n-2}|}{(1+4n^2x^{4n-2})^{3/2}}$. So $\kappa_1(x) = \frac{2}{(1+4x^2)^{3/2}}$, $\kappa_2(x) = \frac{12x^2}{(1+16x^6)^{3/2}}$, and $\kappa_3(x) = \frac{30x^4}{(1+36x^{10})^{3/2}}$.

b.



Note that the curves are symmetric about the y -axis.

- c. $\kappa'(x) = -\frac{4n(2n-1)x^{2n-1}(2n^2(4n-1)x^{4n} - (n-1)x^2)}{(4n^2x^{4n} + x^2)^2\sqrt{4n^2x^{4n-2} + 1}}$. By symmetry, we can concentrate on the critical points for $x > 0$. We have $\kappa'(x) = 0$ for $x > 0$ and $n > 1$, when

$$x = 2^{\frac{1}{2-4n}} \left(\frac{\sqrt{n^2(4n-1)}}{\sqrt{n-1}} \right)^{\frac{1}{1-2n}}.$$

For $n = 1$, the maximum occurs at 0. For $n = 2$, it occurs at $\frac{1}{\sqrt{2} \cdot 7^{1/6}}$. For $n = 3$, it occurs at $\frac{1}{3^{1/5}11^{1/10}}$.

- d. If the maximum curvature for f_n occurs at $\pm z_n$, then $\lim_{n \rightarrow \infty} z_n = 1$.

14.5.82

- a. Write $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$. Then $\mathbf{T} \times \mathbf{a} = \mathbf{T} \times (a_T \mathbf{T} + a_N \mathbf{N}) = a_T(\mathbf{T} \times \mathbf{T}) + a_N(\mathbf{T} \times \mathbf{N}) = 0 + a_N \mathbf{B}$. Then we have $\frac{\mathbf{v}}{|\mathbf{v}|} \times \mathbf{a} = \kappa |\mathbf{v}|^2 \mathbf{B}$, so $\mathbf{v} \times \mathbf{a} = \kappa |\mathbf{v}|^3 \mathbf{B}$.

- b. Taking norms of the last equation in part a, we have $|\mathbf{v} \times \mathbf{a}| = \kappa |\mathbf{v}|^3 \cdot 1$, so $\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$.

14.5.83 We have that $\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{\kappa |\mathbf{v}|^3}$ and $\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$, so $\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$.

14.5.84 $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{d\mathbf{B}/dt}{ds/dt} \cdot \mathbf{N} = -\frac{d\mathbf{B}/dt}{|\mathbf{v}|} \cdot \mathbf{N}$.

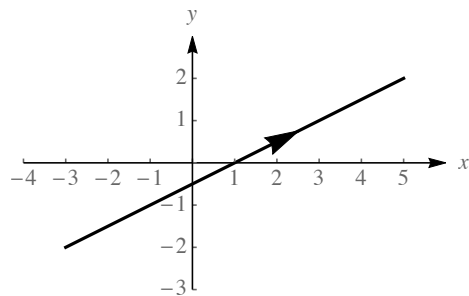
Chapter Fourteen Review

1

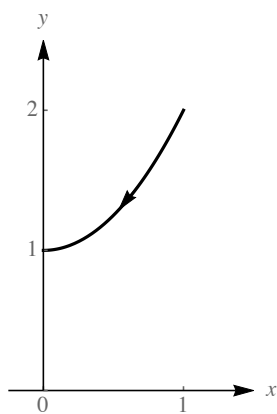
- False. $\mathbf{r}(0) = \langle 1, 1, 0 \rangle + \mathbf{C} = \langle 0, 0, 0 \rangle$, so $\mathbf{C} = \langle -1, -1, 0 \rangle$.
- True. The curvature of a circle is always the reciprocal of the radius.
- True. Note that $\frac{x^2}{9} + \frac{z^2}{36} = 1$.
- True. This follows because $\int \mathbf{r}'(t) dt = \int \langle 0, 0, 0 \rangle dt = \langle 0, 0, 0 \rangle + \mathbf{C} = \langle a, b, c \rangle$ for some real numbers a , b , and c .
- False. Its length is $\sqrt{169 \sin^2 t + 169 \cos^2 t} = 13 \neq 1$.
- False. For example, for the curve $\mathbf{r}(t) = \langle t^2, t \rangle$, we have $\mathbf{N} = \frac{1}{\sqrt{1+4t^2}} \langle 1, -2t \rangle$. So if, for example, $t = 2$, we have $\mathbf{r}(2) = \langle 4, 2 \rangle$ and $\mathbf{N} = \frac{1}{\sqrt{17}} \langle 1, -4 \rangle$, which aren't parallel.

2 This is a circle of radius one centered at $(0, 2, 0)$ and sitting in the plane $y = 2$.

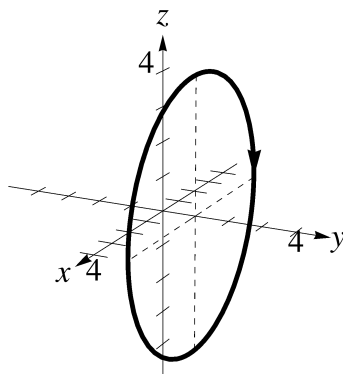
3 This is a straight line.



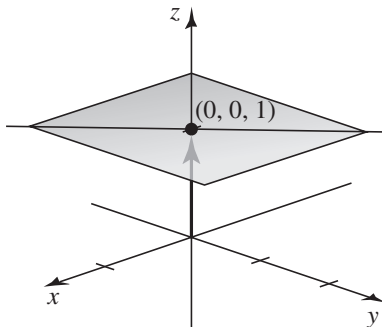
4 Note that if $x = \cos t$, then $y = 1 + x^2$. This is a portion of a parabola, from $(1, 2)$ to $(0, 1)$.



5 Note that $x^2 + z^2 = 16$. This is a circle of radius 4 centered at $(0, 1, 0)$ and sitting in the plane $y = 1$.



6 Note that if $\langle x, y, z \rangle = \langle e^t, 2e^t, 1 \rangle$, then $y = 2x$ and $z = 1$. So this is a line in the plane $z = 1$.



7 Note that $x^2 + y^2 + z^2 = 2$, so this curve lies on a sphere of radius 2. Also, every point satisfies $z = y$, so it is a tilted circle centered at the origin of radius $\sqrt{2}$, sitting in the plane $x = z$.

8 $\mathbf{r}(t) = (1-t)\langle 2, -3, 0 \rangle + t\langle 1, 4, 9 \rangle = \langle 2-t, -3+7t, 9t \rangle$, $0 \leq t \leq 1$.

9 A direction vector for the line is $\langle 1, 5, 2 \rangle \times \langle 1, 0, 3 \rangle = \langle 15, -1, -5 \rangle$. So the line is given by $\mathbf{r}(t) = \langle 4 + 15t, -2 - t, 3 - 5t \rangle$.

10 Let $x = 3\cos t + 2$ and $y = 1$ and $z = 3\sin t$. Then $(x-2)^2 + z^2 = 9$, as desired. So $\mathbf{r}(t) = \langle 3\cos t + 2, 1, 3\sin t \rangle$, $0 \leq t \leq 2\pi$.

11 Let $x = 2$, $y = 3\cos t$, and $z = 4\sin t$. Then $\frac{y^2}{9} + \frac{z^2}{16} = 1$, as desired. So $\mathbf{r}(t) = \langle 2, 3\cos t, 4\sin t \rangle$, $0 \leq t \leq 2\pi$.

12 Let $x = t$. Then we must have $y = x^2 = t^2$ and $z = x = t$. So $\mathbf{r}(t) = \langle t, t^2, t \rangle$.

13 Let $x = \cos t$ and $y = \sin t$. Then $x^2 + y^2 = 1$. Because $z = y$, we must also have $z = \sin t$. Thus $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t \rangle$, $0 \leq t \leq 2\pi$.

14 We must have $x^2 - 5y^2 = 10x^2 + 4y^2 - 36$, so $9x^2 + 9y^2 = 36$ and $x^2 + y^2 = 4$. Let $x = 2\cos t$ and $y = 2\sin t$. Then $z = 4\cos^2 t - 20\sin^2 t = 4\cos^2 t - 20(1 - \cos^2 t) = 24\cos^2 t - 20$. So $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 24\cos^2 t - 20 \rangle$, $0 \leq t \leq 2\pi$.

15 We must have $x^2 + 7y^2 + 2z^2 = 9$, so $x^2 + 9y^2 = 9$. Let $x = 3\cos t$ and $y = \sin t$ and $z = \sin t$. So $\mathbf{r}(t) = \langle 3\cos t, \sin t, \sin t \rangle$, $0 \leq t \leq 2\pi$.

16

a. $\lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 1, -3 \rangle$ and $\lim_{t \rightarrow \infty} \mathbf{r}(t)$ does not exist.

b. $\mathbf{r}'(t) = \langle 1, 2t \rangle$ so $\mathbf{r}'(0) = \langle 1, 0 \rangle$.

c. $\mathbf{r}''(t) = \langle 0, 2 \rangle$.

d. $\int \mathbf{r}(t) dt = \langle t^2/2 + t, t^3 - 3t \rangle + \langle C_1, C_2 \rangle$.

17

a. $\lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 1, 0 \rangle$ and $\lim_{t \rightarrow \infty} \mathbf{r}(t) = \langle 0, 1 \rangle$.

b. $\mathbf{r}'(t) = \left\langle \frac{-2}{(2t+1)^2}, \frac{1}{(1+t)^2} \right\rangle$ so $\mathbf{r}'(0) = \langle -2, 1 \rangle$.

c. $\mathbf{r}''(t) = \left\langle \frac{8}{(2t+1)^3}, \frac{-2}{(1+t)^3} \right\rangle$.

$$\text{d. } \int \mathbf{r}(t) dt = \left\langle \frac{1}{2} \ln |2t + 1|, t - \ln |t + 1| \right\rangle + \langle C_1, C_2 \rangle.$$

18

$$\text{a. } \lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 1, 0, 0 \rangle \text{ and } \lim_{t \rightarrow \infty} \mathbf{r}(t) = \left\langle 0, 0, \frac{\pi}{2} \right\rangle.$$

$$\text{b. } \mathbf{r}'(t) = \left\langle -2e^{-2t}, e^{-t} - te^{-t}, \frac{1}{1+t^2} \right\rangle \text{ so } \mathbf{r}'(0) = \langle -2, 1, 1 \rangle.$$

$$\text{c. } \mathbf{r}''(t) = \left\langle 4e^{-2t}, te^{-t} - 2e^{-t}, \frac{2t}{(1+t^2)^2} \right\rangle.$$

$$\text{d. } \int \mathbf{r}(t) dt = \left\langle -\frac{1}{2}e^{-2t}, (1+t)e^{-t}, t \tan^{-1}(t) - \frac{1}{2} \ln(t^2 + 1) \right\rangle + \langle C_1, C_2 \rangle.$$

19

$$\text{a. } \lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 0, 3, 0 \rangle \text{ and } \lim_{t \rightarrow \infty} \mathbf{r}(t) \text{ does not exist.}$$

$$\text{b. } \mathbf{r}'(t) = \langle 2 \cos 2t, -12 \sin 4t, 1 \rangle \text{ so } \mathbf{r}'(0) = \langle 2, 0, 1 \rangle.$$

$$\text{c. } \mathbf{r}''(t) = \langle -4 \sin 2t, -48 \cos 4t, 0 \rangle.$$

$$\text{d. } \int \mathbf{r}(t) dt = \left\langle -\frac{1}{2} \cos 2t, \frac{3}{4} \sin 4t, \frac{1}{2} t^2 \right\rangle + \langle C_1, C_2 \rangle.$$

$$\text{20 } \int_1^3 \langle 6t^2, 4t, 1/t \rangle dt = \langle 2t^3, 2t^2, \ln |t| \rangle \Big|_1^3 = \langle 54, 18, \ln 3 \rangle - \langle 2, 2, 0 \rangle = \langle 52, 16, \ln 3 \rangle.$$

$$\text{21 } \int_{-1}^1 \left\langle \sin \pi t, 1, \frac{2}{(1+t^2)} \right\rangle dt = \left\langle \frac{-\cos \pi t}{\pi}, t, 2 \tan^{-1} t \right\rangle \Big|_{-1}^1 = \langle 0, 2, \pi \rangle.$$

$$\text{22 } \mathbf{u}(0) \cdot \mathbf{v}'(0) + \mathbf{u}'(0) \cdot \mathbf{v}(0) = \langle 2, 7, 0 \rangle \cdot \langle 5, 0, 3 \rangle + \langle 3, 1, 2 \rangle \cdot \langle 3, -1, 0 \rangle = 10 + 0 + 0 + 9 - 1 + 0 = 18.$$

$$\text{23 } \mathbf{u}(0) \times \mathbf{v}'(0) + \mathbf{u}'(0) \times \mathbf{v}(0) = \langle 2, 7, 0 \rangle \times \langle 5, 0, 3 \rangle + \langle 3, 1, 2 \rangle \times \langle 3, -1, 0 \rangle = \langle 21, -6, -35 \rangle + \langle 2, 6, -6 \rangle = \langle 23, 0, -41 \rangle.$$

$$\text{24 } \mathbf{u}'(0) \cdot 5e^{5 \cdot 0} = \langle 3, 1, 2 \rangle \cdot 5 = \langle 15, 5, 10 \rangle.$$

$$\text{25 } \int \langle 1, \sin 2t, \sec^2 t \rangle dt = \left\langle t, -\frac{1}{2} \cos 2t, \tan t \right\rangle + \langle C_1, C_2, C_3 \rangle. \text{ Because } \mathbf{r}(0) = \langle 2, 2, 2 \rangle, \text{ we have } C_1 = 2, C_2 = \frac{5}{2}, \text{ and } C_3 = 2. \text{ So } \mathbf{r}(t) = \left\langle t + 2, -\frac{1}{2} \cos 2t + \frac{5}{2}, \tan t + 2 \right\rangle.$$

$$\text{26 } \int \langle e^t, 2e^{2t}, 6e^{3t} \rangle dt = \langle e^t, e^{2t}, 2e^{3t} \rangle + \langle C_1, C_2, C_3 \rangle. \text{ Because } \mathbf{r}(0) = \langle 1, 3, -1 \rangle, \text{ we have } C_1 = 0, C_2 = 2, \text{ and } C_3 = -3. \text{ So } \mathbf{r}(t) = \langle e^t, e^{2t} + 2, 2e^{3t} - 3 \rangle.$$

$$\text{27 } \int \left\langle \frac{4}{1+t^2}, 2t+1, 3t^2 \right\rangle dt = \langle 4 \tan^{-1} t, t^2 + t, t^3 \rangle + \langle C_1, C_2, C_3 \rangle. \text{ Because } \mathbf{r}(1) = \langle 0, 0, 0 \rangle, \text{ we have } C_1 = -\pi, C_2 = -2, \text{ and } C_3 = -1. \text{ So } \mathbf{r}(t) = \langle 4 \tan^{-1} t - \pi, t^2 + t - 2, t^3 - 1 \rangle.$$

$$\text{28 } \mathbf{r}'(t) = \langle 0, 6 \cos 2t, -6 \sin 2t \rangle. |\mathbf{r}'(t)| = 6. \text{ So } \mathbf{T}(t) = \langle 0, \cos 2t, -\sin 2t \rangle. \text{ At } t = \pi/4 \text{ we have } \mathbf{T}(\pi/4) = \langle 0, 0, -1 \rangle.$$

29 $\mathbf{r}'(t) = \langle 2e^t, 2e^{2t}, 1 \rangle$. $|\mathbf{r}'(t)| = \sqrt{(2e^{2t} + 1)^2} = 2e^{2t} + 1$. So $\mathbf{T}(t) = \left\langle \frac{2e^t}{2e^{2t} + 1}, \frac{2e^{2t}}{2e^{2t} + 1}, \frac{1}{2e^{2t} + 1} \right\rangle$. At $t = 0$ we have $\mathbf{T}(0) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$.

30

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 5t^2, 2t + 10 \rangle$, so the speed is $\sqrt{25t^4 + 4t^2 + 40t + 100}$.

b. $\mathbf{a}(t) = \langle 10t, 2 \rangle$.

31

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 4e^{4t}, 4e^{4t}, 2e^{4t} \rangle$, so the speed is $\sqrt{16e^{8t} + 16e^{8t} + 4e^{8t}} = \sqrt{36e^{8t}} = 6e^{4t}$.

b. $\mathbf{a}(t) = \langle 16e^{4t}, 16e^{4t}, 8e^{4t} \rangle$.

32 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle 1, 4 \rangle dt = \langle t, 4t \rangle + \langle C_1, C_2 \rangle$. Because $\mathbf{v}(0) = \langle 4, 3 \rangle$, we have $\mathbf{v}(t) = \langle t + 4, 4t + 3 \rangle$.
 $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle t + 4, 4t + 3 \rangle dt = \left\langle \frac{t^2}{2} + 4t, 2t^2 + 3t \right\rangle + \langle D_1, D_2 \rangle$. Because the initial position is $\langle 0, 2 \rangle$, we have $\mathbf{r}(t) = \left\langle \frac{t^2}{2} + 4t, 2t^2 + 3t + 2 \right\rangle$.

33 $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \langle \cos t, 2 \sin t \rangle dt = \langle \sin t, -2 \cos t \rangle + \langle C_1, C_2 \rangle$. Because $\mathbf{v}(0) = \langle 2, 1 \rangle$, we have $\mathbf{v}(t) = \langle 2 + \sin t, 3 - 2 \cos t \rangle$.

$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 2 + \sin t, 3 - 2 \cos t \rangle dt = \langle 2t - \cos t, 3t - 2 \sin t \rangle + \langle D_1, D_2 \rangle$. Because the initial position is $\langle 1, 2 \rangle$, we have $\mathbf{r}(t) = \langle 2t + 2 - \cos t, 3t + 2 - 2 \sin t \rangle$.

34 $\mathbf{r}'(t) = \langle 0, 8 \cos t, -\sin t \rangle$. The vectors $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal when $\mathbf{r} \cdot \mathbf{r}' = 0 + 64 \sin t \cos t - \sin t \cos t = 63 \sin t \cos t$ is zero. This occurs when either $\sin t = 0$ or $\cos t = 0$, so at $t = 0$, $t = \pi/2$, $t = 3\pi/2$ and $t = 2\pi$, which correspond to the points $(1, 0, 1)$, $(1, 8, 0)$, $(1, 0, -1)$ and $(1, -8, 0)$.

35

a. We have $\mathbf{a}(t) = \langle 0, -32 \rangle$ and $\mathbf{v}(0) = 80 \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \langle 40, 40\sqrt{3} \rangle$. So $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \langle 0, -32t \rangle + \langle 40, 40\sqrt{3} \rangle = \langle 40, -32t + 40\sqrt{3} \rangle$. Then

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t) dt = \int \langle 40, -32t + 40\sqrt{3} \rangle dt \\ &= \langle 40t + C_1, -16t^2 + 40\sqrt{3}t + C_2 \rangle = \langle 40t, -16t^2 + 40\sqrt{3}t + 3 \rangle \end{aligned}$$

because $\mathbf{r}(0) = \langle 0, 3 \rangle$.

b. The ball hits the ground when $-16t^2 + 40\sqrt{3}t + 3 = 0$, which occurs for $t \approx 4.37$ s. At this time the ball's x -position is about $40(4.37) \approx 174.9$ ft.

c. The ball's maximum height occurs when $-32t + 40\sqrt{3} = 0$, or $t = \frac{40\sqrt{3}}{32} \approx 2.17$ and is about $-16(2.17)^2 + 40\sqrt{3}(2.17) + 3 \approx 78$ ft.

36

- a. We have $\mathbf{a}(t) = \langle 0, -9.8 \rangle$ and $\mathbf{v}(0) = 6 \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \langle 3\sqrt{3}, 3 \rangle$. So $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \langle 0, -9.8t \rangle + \langle 3\sqrt{3}, 3 \rangle = \langle 3\sqrt{3}, -9.8t + 3 \rangle$. Then

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t) dt = \int \langle 3\sqrt{3}, -9.8t + 3 \rangle dt \\ &= \langle 3\sqrt{3}t + C_1, -4.9t^2 + 3t + C_2 \rangle = \langle 3\sqrt{3}t, -4.9t^2 + 3t + 2 \rangle \end{aligned}$$

because $\mathbf{r}(0) = \langle 0, 2 \rangle$.

- b. The rock hits the ground when $-4.9t^2 + 3t + 2 = 0$, which occurs for $t \approx 1.01$ s. At this time the rock's x -position is about $3\sqrt{3}(1.01) \approx 5.25$ ft.
- c. The rock's maximum height occurs when $-9.8t + 3 = 0$, or $t = \frac{3}{9.8} \approx 0.306$ and is about $-4.9(0.306)^2 + 3(0.306) + 2 \approx 2.46$ ft.

37

- a. We have $\mathbf{a}(t) = \langle 4, 0, -32 \rangle$ and $\mathbf{v}(0) = \langle 40, 20, 40 \rangle$. So $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \langle 4t, 0, -32t \rangle + \langle 40, 20, 40 \rangle = \langle 4t + 40, 20, 40 - 32t \rangle$. Then

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \langle 4t + 40, 20, 40 - 32t \rangle dt = \langle 2t^2 + 40t, 20t, -16t^2 + 40t + 2 \rangle$$

because $\mathbf{r}(0) = \langle 0, 0, 2 \rangle$.

- b. The ball hits the ground when $z = 0$, or $-16t^2 + 40t + 2 = 0$, or $t \approx 2.549$ s.
- c. The ball's position at $t = 2.549$ is approximately $\langle 115, 51, 0 \rangle$, so its distance from home plate is about $\sqrt{115^2 + 51^2} \approx 126$ ft.

38 We need $\frac{40^2 \sin 2\alpha}{9.8} = 100$, so $\sin 2\alpha = \frac{980}{1600}$. The two solutions to this equation are approximately $\alpha = 18.9^\circ, 71.1^\circ$.

39

- a. The trajectory is given by $\mathbf{r}(t) = \langle 50t, 50t - 16t^2 \rangle$. The projectile is at $y = 30$ when $-16t^2 + 50t - 30 = 0$, which occurs at $t = \frac{1}{16} (25 \pm \sqrt{145}) \approx .81$ and 2.32 . At these times, $x = 50t \approx 40.5$ and 116 . The first time represents when the projectile has not yet reached the cliff, while the second time represents when the projectile lands on the cliff, so the coordinates of the landing spot are approximately $(116, 30)$.
- b. The maximum height occurs where $y' = 0$, which occurs for $50 - 32t = 0$, or $t = 25/16$. The maximum height is $50 \cdot \frac{25}{16} - 16 \left(\frac{25}{16} \right)^2 = \frac{625}{16} \approx 39.06$ feet.
- c. As mentioned above, the flight ends at $t \approx 2.32$ seconds.
- d. The length of the trajectory is $\int_0^{2.32} \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^{2.32} \sqrt{2500 + (50 - 32t)^2} dt$.
- e. $L \approx 129$ feet.
- f. Suppose the launch angle is α . Then $\mathbf{r}(t) = \langle 50\sqrt{2}t \cos \alpha, 50\sqrt{2}t \sin \alpha - 16t^2 \rangle$. We want $y \geq 30$ when $x = 50$. We know that $x = 50$ when $t = \frac{\sec \alpha}{\sqrt{2}}$. At this time, we have $y = 50 \tan \alpha - 8 \sec^2 \alpha$. This expression is greater than or equal to 30 for approximately $41.5 \leq \alpha \leq 79.4$.

40 The initial velocity of the ball is given by $\langle s\sqrt{3}/2, s/2 \rangle$ where s is the initial speed of the ball. We have $\mathbf{r}(t) = \langle (s\sqrt{3}/2)t, -16t^2 + (s/2)t + 2 \rangle$. We know that $(s\sqrt{3}/2)t = 10$ when $-16t^2 + (s/2)t + 2 = 0$. Solving the first equation for t gives $t = 20/(s\sqrt{3})$. Putting this into the second equation gives $-16(20/(s\sqrt{3}))^2 + (s/2)(20/(s\sqrt{3})) = -2$. Solving for s gives $s \approx 16.6$ feet per second.

41 The square of the distance from $P(3, 1, 6)$ to the curve is $s(t) = (t - 3)^2 + t^4 + (3t - 6)^2$. Note that $s'(t) = 2(t - 3) + 4t^3 + 6(3t - 6) = 4t^3 + 20t - 42$. This is zero for $t \approx 1.47$. So the closest point is approximately $(1, 47, 3.15, 4.4)$.

42

- a. Let t_0 be the time until the ball is at the point $(18, 10)$. Using the theory from Section 14.3, the path of the ball is $\mathbf{r}(t) = \left\langle |v_0| \frac{\sqrt{2}}{2}t, -16t^2 + |v_0| \frac{\sqrt{2}}{2}t + 8 \right\rangle$. Thus $|v_0| \frac{\sqrt{2}}{2}t_0 = 18$. Thus $t_0 = \frac{18\sqrt{2}}{|v_0|}$. Because the second component of $\mathbf{r}$ is 10 at time t_0 , we have

$$-16 \left(\frac{18\sqrt{2}}{|v_0|} \right)^2 + 18 + 8 = 10.$$

Solving for $|v_0|$ gives $|v_0| = 18\sqrt{2}$ ft/s.

- b. $\mathbf{v}(0) = 18\sqrt{2} \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = \langle 18, 18 \rangle$.
- c. We have $\mathbf{r}(t) = \langle 18t, -16t^2 + 18t + 8 \rangle$.
- d. The front of the hoop is at the point $(17.25, 10)$. The distance is

$$s(t) = \sqrt{(18t - 17.25)^2 + (-16t^2 + 18t - 2)^2}.$$

- e. We will minimize the square of the distance: $(18t - 17.25)^2 + (-16t^2 + 18t - 2)^2$. The derivative of this is $36(18t - 17.25) + 2(-16t^2 + 18t - 2)(-32t + 18)$, which is 0 for $t \approx 0.973$. $s(0.973) \approx 5.42$ in.
- f. Yes, because the radius of the ball is less than 5.42 inches.

$$43 \quad L = \int_1^3 \sqrt{4t^2 + 8t + 4} \, dt = 2 \int_1^3 \sqrt{(t+1)^2} \, dt = 2 \int_1^3 (t+1) \, dt = 2 \left(\frac{t^2}{2} + t \right) \Big|_1^3 = 2(9/2 + 3 - (1/2 + 1)) = 9 + 6 - 1 - 2 = 12.$$

$$44 \quad L = \int_0^2 \sqrt{81t^7 + 9t^4} \, dt = \int_0^2 3t^2 \sqrt{9t^3 + 1} \, dt. \text{ Let } u = 9t^3 + 1 \text{ so that } du = 27t^2 \, dt. \text{ We have } L = \frac{1}{9} \int_1^{73} u^{1/2} \, du = \frac{2}{27} \left(u^{3/2} \right) \Big|_1^{73} = \frac{2}{27} (73\sqrt{73} - 1).$$

$$45 \quad L = \int_0^{\pi/2} \sqrt{\cos^2 t + (1 - \sin t)^2 + 16} \, dt = \int_0^{\pi/2} \sqrt{18 - 2 \sin t} \, dt \approx 6.42.$$

$$46 \quad L = \int_0^{\pi/4} \sqrt{1 + \tan^2 t + \sec^2 t} \, dt = \int_0^{\pi/4} \sqrt{2} \sec t \, dt = \sqrt{2} (\ln |\sec t + \tan t|) \Big|_0^{\pi/4} = \sqrt{2} (\ln(\sqrt{2} + 1) - \ln(1 + 0)) = \sqrt{2} \ln(\sqrt{2} + 1).$$

47

- a. $\mathbf{v}(t) = \int \langle 0, \sqrt{2}, 2t \rangle \, dt = \langle 0, \sqrt{2}t, t^2 \rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 0, 0 \rangle$, we have $\mathbf{C} = \langle 1, 0, 0 \rangle$, so $\mathbf{v}(t) = \langle 1, \sqrt{2}t, t^2 \rangle$.

$$b. \quad L = \int_0^3 \sqrt{1 + 2t^2 + t^4} \, dt = \int_0^3 (t^2 + 1) \, dt = \left(\frac{t^3}{3} + t \right) \Big|_0^3 = 9 + 3 = 12.$$

$$48 \quad |\mathbf{r}'(t)| = |\langle 4, -3 \rangle| = 5. \text{ Let } s = \int_1^t 5 \, du = 5t - 5. \text{ Solving for } t \text{ gives } t = \frac{s+5}{5}. \text{ Thus, } \mathbf{r}(s) = \langle 1 + 4(s+5)/5, -3(s+5)/5 \rangle = \left\langle 5 + \frac{4s}{5}, -3 - \frac{3s}{5} \right\rangle \text{ for } s \geq 0.$$

49 $|\mathbf{r}'(t)| = \left| \left\langle 2t, 2\sqrt{2}t^{1/2}, 2 \right\rangle \right| = \sqrt{4t^2 + 8t + 4} = 2\sqrt{(t+1)^2} = 2(t+1)$. Let $s = \int_0^t 2(u+1) du = (u^2 + 2u) \Big|_0^t = t^2 + 2t$. Then $s+1 = (t+1)^2$, so $t = \sqrt{s+1} - 1$. Thus,

$$\mathbf{r}(s) = \left\langle (\sqrt{s+1} - 1)^2, \frac{4\sqrt{2}}{3}(\sqrt{s+1} - 1)^{3/2}, 2(\sqrt{s+1} - 1) \right\rangle$$

for $s \geq 0$.

50

a. $\mathbf{r}'(t) = \langle -3 \sin t, 4 \cos t \rangle$, so $\mathbf{T} = \frac{1}{\sqrt{9+7\cos^2 t}} \langle -3 \sin t, 4 \cos t \rangle$. Because $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$ pointing toward the inside of the ellipse, it is $\mathbf{N} = \frac{1}{\sqrt{9+7\cos^2 t}} \langle -4 \cos t, -3 \sin t \rangle$. All of these are valid for $0 \leq t \leq 2\pi$.

b. $|\mathbf{r}'(t)| = \sqrt{9+7\cos^2 t}$. The derivative of this is $-\frac{7 \cos t \sin t}{\sqrt{9+7\cos^2 t}}$, which is 0 at $t = 0$, $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$. The speed is maximal at $t = 0$ and $t = \pi$ and minimal at $t = \pi/2$ and $t = 3\pi/2$.

c. $\kappa = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3} = \frac{|\langle 0, 0, -12 \rangle|}{(9+7\cos^2 t)^{3/2}} = \frac{12}{(9+7\cos^2 t)^{3/2}}$. The curvature is maximal at $t = \pi/2$ and $t = 3\pi/2$ where the denominator is minimized, and is minimal at $t = 0$ and $t = \pi$ where the denominator is maximized. Note that the velocity is maximized where the curvature is minimal, and vice versa.

d. In order for $\mathbf{r}$ and $\mathbf{N}$ to be parallel, we require $3 \cos t = m \cdot -\frac{4 \cos t}{\sqrt{9+7\cos^2 t}}$ and $4 \sin t = m \cdot -\frac{3 \sin t}{\sqrt{9+7\cos^2 t}}$ for some constant m . This only occurs when either $\sin t = 0$ or $\cos t = 0$. (If $\sin t = 0$, then $m = -3$ and if $\cos t = 0$ then $m = -4$.) So the corresponding points on $\mathbf{r}$ are $(3, 0)$, $(0, 4)$, $(-3, 0)$, and $(0, -4)$.

51

a. $\mathbf{r}'(t) = \langle -6 \sin t, 3 \cos t \rangle$, so $\mathbf{T} = \frac{1}{\sqrt{1+3\sin^2 t}} \langle -2 \sin t, \cos t \rangle$.

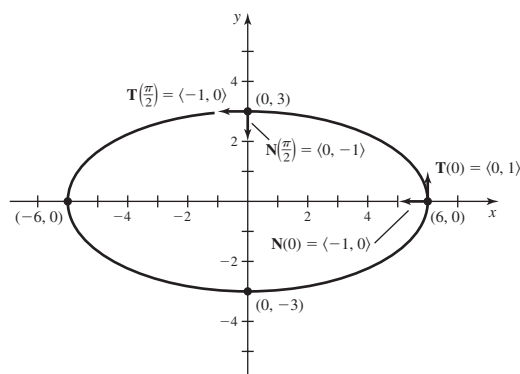
b. $\kappa = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3} = \frac{|\langle 0, 0, -18 \rangle|}{(3\sqrt{1+3\sin^2 t})^3} = \frac{2}{3(\sqrt{1+3\sin^2 t})^3}$.

c. Note that $\frac{1}{\sqrt{1+3\sin^2 t}} \langle -\cos t, -2 \sin t \rangle$ has length one, and is perpendicular to $\mathbf{T}$ (see part [d]), and points to the inside of the curve, so it is $\mathbf{N}$.

d. $\left| \frac{1}{\sqrt{1+3\sin^2 t}} \langle -\cos t, -2 \sin t \rangle \right| = \frac{\sqrt{1+3\sin^2 t}}{\sqrt{1+3\sin^2 t}} = 1$ and

$$\mathbf{T} \cdot \mathbf{N} = \frac{1}{\sqrt{1+3\sin^2 t}} (2 \sin t \cos t - 2 \sin t \cos t) = 0.$$

e.



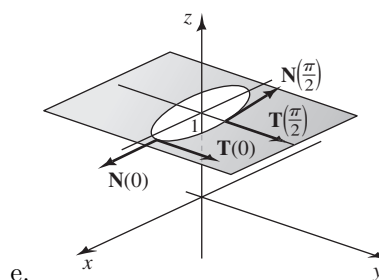
52

$$\text{a. } \mathbf{r}'(t) = \langle -\sin t, 2 \cos t, 0 \rangle, \quad \text{so } \mathbf{T} = \frac{1}{\sqrt{1+3\cos^2 t}} \langle -\sin t, 2 \cos t, 0 \rangle.$$

$$\text{b. } \kappa = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{(\sqrt{1+3\cos^2 t})^3} = \frac{|\langle 0, 0, -2 \rangle|}{(\sqrt{1+3\cos^2 t})^3} = \frac{2}{(\sqrt{1+3\cos^2 t})^3}.$$

c. Note that $\frac{1}{\sqrt{1+3\cos^2 t}} \langle -2 \cos t, -\sin t, 0 \rangle$ has length one, and is perpendicular to $\mathbf{T}$ (see part [d]), and points to the inside of the curve, so it is $\mathbf{N}$.

$$\text{d. } \left| \frac{1}{\sqrt{1+3\cos^2 t}} \langle -2 \cos t, -\sin t \rangle \right| = \frac{\sqrt{1+3\cos^2 t}}{\sqrt{1+3\cos^2 t}} = 1 \quad \text{and} \quad \mathbf{T} \cdot \mathbf{N} = \frac{1}{\sqrt{1+3\cos^2 t}} (2 \sin t \cos t - 2 \sin t \cos t) = 0.$$



e.

53

$$\text{a. } \mathbf{r}'(t) = \langle -\sin t, -2 \sin t, \sqrt{5} \cos t \rangle, \quad \text{so}$$

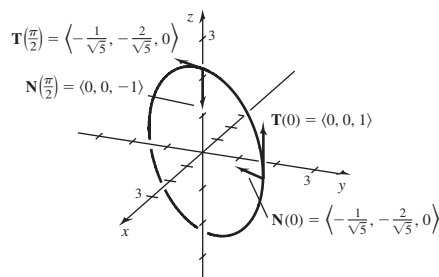
$$\mathbf{T} = \frac{1}{\sqrt{5}} \langle -\sin t, -2 \sin t, \sqrt{5} \cos t \rangle.$$

$$\text{b. } \kappa = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{(\sqrt{5})^3} = \frac{|\langle -2\sqrt{5}, \sqrt{5}, 0 \rangle|}{(\sqrt{5})^3} = \frac{1}{\sqrt{5}}.$$

$$\text{c. } \frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{5}} \langle -\cos t, -2 \cos t, -\sqrt{5} \sin t \rangle = \mathbf{N}.$$

$$\text{d. } \left| \frac{1}{\sqrt{5}} \langle -\cos t, -2 \cos t, -\sqrt{5} \sin t \rangle \right| = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

and $\mathbf{T} \cdot \mathbf{N} = \frac{1}{\sqrt{5}} (\sin t \cos t + 4 \sin t \cos t - 5 \sin t \cos t) = 0.$



e.

54

a. $\mathbf{r}'(t) = \langle 1, -2 \sin t, 2 \cos t \rangle$, so

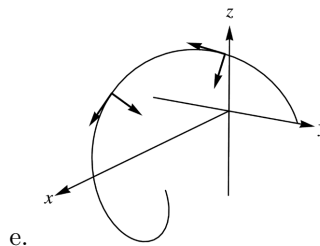
$$\mathbf{T} = \frac{1}{\sqrt{5}} \langle 1, -2 \sin t, 2 \cos t \rangle.$$

$$\text{b. } \kappa = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{(\sqrt{5})^3} = \frac{|\langle -4, -2 \sin(t), 2 \cos(t) \rangle|}{(\sqrt{5})^3} = \frac{2}{5}.$$

c. $\frac{dT}{dt} = \frac{1}{\sqrt{5}} \langle 0, -2 \cos t, -2 \sin t \rangle$, so

$$\mathbf{N} = \langle 0, -\cos t, -\sin t \rangle.$$

$$\text{d. } |\mathbf{N}| = \sqrt{\cos^2 t + \sin^2 t} = 1 \text{ and } \mathbf{T} \cdot \mathbf{N} = \frac{1}{\sqrt{5}}(0 + 2 \sin t \cos t - 2 \sin t \cos t) = 0.$$

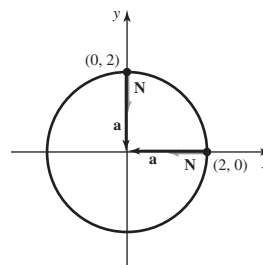


e.

55

a. $\mathbf{v}(t) = \langle -2 \sin t, 2 \cos t \rangle$, and $\mathbf{a}(t) = \langle -2 \cos t, -2 \sin t \rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a} \times \mathbf{v} = \langle -2 \cos t, -2 \sin t, 0 \rangle \times \langle -2 \sin t, 2 \cos t, 0 \rangle = \langle 0, 0, -4 \rangle$, so $a_N = \frac{4}{2} = 2$. We have $\mathbf{a} = \langle -2 \cos t, -2 \sin t \rangle = 2\mathbf{N} + 0 \cdot \mathbf{T}$.

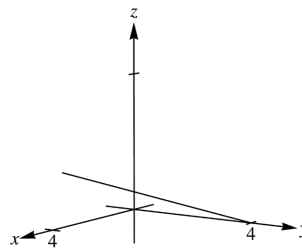
b. Note that at $t = 0$ we have $\mathbf{a} = \langle -2, 0 \rangle = 2 \langle -1, 0 \rangle = 2\mathbf{N}$, and at $t = \pi/2$ we have $\mathbf{a} = \langle 0, -2 \rangle = 2 \langle 0, -1 \rangle = 2\mathbf{N}$.



56

a. $\mathbf{v}(t) = \langle 3, -1, 1 \rangle$, and $\mathbf{a}(t) = \langle 0, 0, 0 \rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a} \times \mathbf{v} = \langle 0, 0, 0 \rangle$, so $a_N = 0$. We have $\mathbf{a} = \langle 0, 0, 0 \rangle = 0 \cdot \mathbf{N} + 0 \cdot \mathbf{T}$.

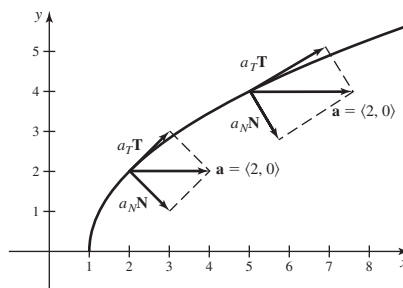
b. For all t , we have $\mathbf{a} = \langle 0, 0, 0 \rangle = 0 \cdot \mathbf{N} + 0 \cdot \mathbf{T}$.



57

- a. $\mathbf{v}(t) = \langle 2t, 2 \rangle$, and $\mathbf{a}(t) = \langle 2, 0 \rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 4t$, we have $a_T = \frac{4t}{2\sqrt{t^2+1}} = \frac{2t}{\sqrt{t^2+1}}$. Note that $\mathbf{a} \times \mathbf{v} = \langle 0, 0, 4 \rangle$, so $a_N = \frac{4}{2\sqrt{t^2+1}} = \frac{2}{\sqrt{t^2+1}}$. We have that $\mathbf{a} = \langle 2, 0 \rangle = \frac{2}{\sqrt{t^2+1}}\mathbf{N} + \frac{2t}{\sqrt{t^2+1}}\mathbf{T}$.

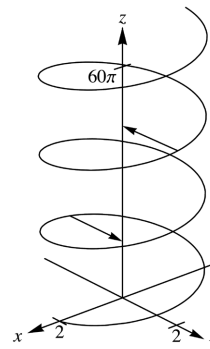
- b. At $t = 1$, we have $\mathbf{a} = \langle 2, 0 \rangle = \frac{2}{\sqrt{2}}\mathbf{T} + \frac{2}{\sqrt{2}}\mathbf{N} = \frac{2}{\sqrt{2}}\langle \sqrt{2}/2, \sqrt{2}/2 \rangle + \frac{2}{\sqrt{2}}\langle \sqrt{2}/2, -\sqrt{2}/2 \rangle$.
 At $t = 2$, we have $\mathbf{a} = \langle 2, 0 \rangle = \frac{4}{\sqrt{5}}\langle 2/\sqrt{5}, 1/\sqrt{5} \rangle + \frac{2}{\sqrt{5}}\langle 1/\sqrt{5}, -2/\sqrt{5} \rangle$.



58

- a. $\mathbf{v}(t) = \langle -2\sin t, 2\cos t, 10 \rangle$, and $\mathbf{a}(t) = \langle -2\cos t, -2\sin t, 0 \rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a} \times \mathbf{v} = \langle -20\sin(t), 20\cos(t), -4 \rangle$, so $a_N = \frac{4\sqrt{26}}{2\sqrt{26}} = 2$.

- b. At $t = \pi/2$, we have $\mathbf{a} = \langle 0, -2, 0 \rangle = 0 \cdot \mathbf{T} + 2\mathbf{N} = 2\langle 0, -1, 0 \rangle$.
 At $t = 3\pi/2$, we have $\mathbf{a} = \langle 0, 2, 0 \rangle = 0 \cdot \mathbf{T} + 2\mathbf{N} = 2\langle 0, 1, 0 \rangle$.



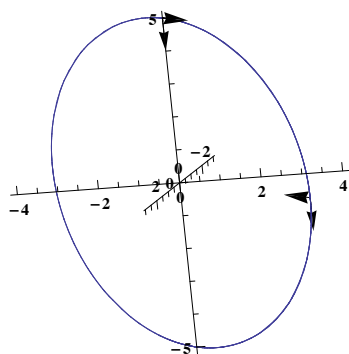
59 We have $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$, so $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle$. Thus, $\mathbf{T}(1) = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$.

Then $\mathbf{N}(1) = \left\langle -\frac{11}{\sqrt{266}}, -4\sqrt{\frac{2}{133}}, \frac{9}{\sqrt{266}} \right\rangle$, so $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left\langle \frac{3}{\sqrt{19}}, -\frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right\rangle$. Also,

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{3}{19}.$$

60

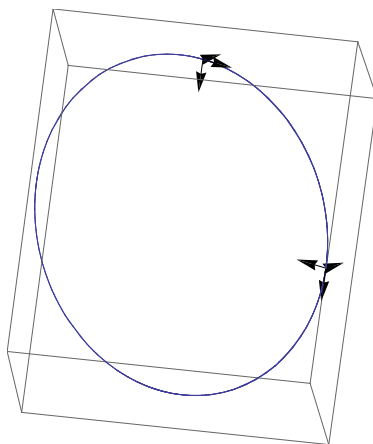
- a. $\mathbf{r}'(t) = \langle 3\cos t, 4\cos t, -5\sin t \rangle$, and $|\mathbf{r}'(t)| = 5$, so $\mathbf{T}(t) = \frac{1}{5}\langle 3\cos t, 4\cos t, -5\sin t \rangle$.
 b. $\mathbf{T}'(t) = \frac{1}{5}\langle -3\sin t, -4\sin t, -5\cos t \rangle$, which has length one, so $\mathbf{N}(t) = \frac{1}{5}\langle -3\sin t, -4\sin t, -5\cos t \rangle$.
 c. At $t = 0$ we have $\mathbf{T} = \langle 3/5, 4/5, 0 \rangle$ and $\mathbf{N} = \langle 0, 0, -1/5 \rangle$. At $t = \pi/2$ we have $\mathbf{T} = \langle 0, 0, -1 \rangle$ and $\mathbf{N} = \langle -3/5, -4/5, 0 \rangle$.



d. Yes.

e. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \left\langle -\frac{4}{5}, \frac{3}{5}, 0 \right\rangle$.

f.



g. One should check that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are all of unit length and are mutually orthogonal.

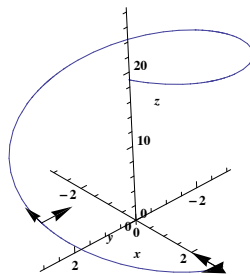
h. Because $\mathbf{B}$ is constant, $\tau = 0$.

61

a. $\mathbf{r}'(t) = \langle 3 \cos t, -3 \sin t, 4 \rangle$, and $|\mathbf{r}'(t)| = 5$, so $\mathbf{T}(t) = \frac{1}{5} \langle 3 \cos t, -3 \sin t, 4 \rangle$.

b. $\mathbf{T}'(t) = \frac{1}{5} \langle -3 \sin t, -3 \cos t, 0 \rangle$, so $\mathbf{N}(t) = \langle -\sin t, -\cos t, 0 \rangle$.

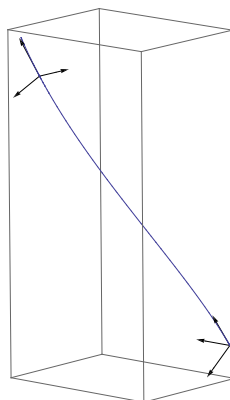
c. At $t = 0$ we have $\mathbf{T} = \langle 3/5, 0, 4/5 \rangle$ and $\mathbf{N} = \langle 0, -1, 0 \rangle$. At $t = \pi/2$ we have $\mathbf{T} = \langle 0, -3/5, 4/5 \rangle$ and $\mathbf{N} = \langle -1, 0, 0 \rangle$.



d. Yes.

e. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{5} \langle 4 \cos t, -4 \sin t, -3 \rangle$.

f.



g. One should check that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are all of unit length and are mutually orthogonal.

h. $\tau = -\frac{d\mathbf{B}/dt}{ds/dt} \cdot \mathbf{N} = \frac{1}{25} \langle -4 \sin t, -4 \cos t, 0 \rangle \cdot \langle -\sin t, -\cos t, 0 \rangle = -\frac{4}{25}$.

62

a. $\mathbf{v}(t) = \langle 2a_1t + b_1, 2a_2t + b_2, 2a_3t + b_3 \rangle$ and $\mathbf{a}(t) = \langle 2a_1, 2a_2, 2a_3 \rangle$. Thus $\mathbf{v} \times \mathbf{a} = 2 \langle a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2 \rangle$, which is constant. Thus $\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$ is a constant, so $\tau = 0$.

b. $\mathbf{a}'(t) = \langle 0, 0, 0 \rangle$, so $\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = 0$.

63

a. First consider the case where $a_3 = b_3 = c_3 = 0$. Let $t \neq s$ be in the interval I , and consider $\mathbf{r}(t) \times \mathbf{r}(s)$. We will show that this vector is always a multiple of the same constant vector. We have

$$\begin{aligned} \mathbf{r}(t) \times \mathbf{r}(s) &= \langle a_1f(t) + a_2g(t), b_1f(t) + b_2g(t), c_1f(t) + c_2g(t) \rangle \times \langle a_1f(s) + a_2g(s), b_1f(s) + b_2g(s), c_1f(s) + c_2g(s) \rangle \\ &= (f(t)g(s) - f(s)g(t)) \langle c_2b_1 - b_2c_1, a_2c_1 - a_1c_2, a_1b_2 - a_2b_1 \rangle, \end{aligned}$$

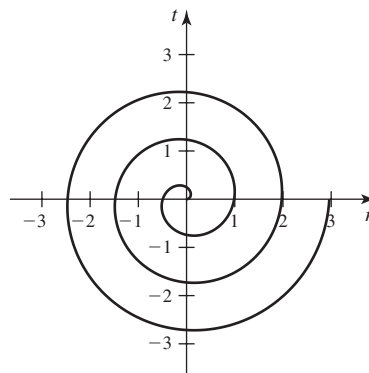
where this last computation admittedly requires patience. Because $\mathbf{r}(t) \times \mathbf{r}(s)$ is always orthogonal to the same vector, the vectors $\mathbf{r}(t)$ must all lie in the same plane.

Now consider the case where a_3 , b_3 , and c_3 are not necessarily 0, and consider $\mathbf{p}(t) = \mathbf{r}(t) - \langle a_3, b_3, c_3 \rangle$. Note that $\mathbf{p}(t)$ has the form required in the argument in the previous paragraph. Using the result above, the curve $\mathbf{p}(t)$ lies in a plane, which implies that $\mathbf{r}(t) = \mathbf{p}(t) + \langle a_3, b_3, c_3 \rangle$ lies in a plane as well, because we are just translating all the vectors $\mathbf{p}(t)$ by the same constant vector.

- b. If the curve lies in a plane, the $\mathbf{B}$ is always normal to the plane with length 1. Hence $\mathbf{B}$ is constant, so $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = 0$.

64

- a. The curve makes “one loop” for every 2π radians, so for $0 \leq \theta \leq 2\pi N$, it makes N loops. When $\theta = 2\pi N$, we have that the radius of the whole spiral is $R = tN$.



- b. $L = \int_0^{2\pi N} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \int_0^{2\pi N} \sqrt{\left(\frac{t\theta}{2\pi}\right)^2 + \left(\frac{t}{2\pi}\right)^2} d\theta = \frac{t}{2\pi} \int_0^{2\pi N} \sqrt{\theta^2 + 1} d\theta$.
- c. Note that $\lim_{\theta \rightarrow \infty} \frac{\theta^2 + 1}{\theta^2} = 1$, so when θ is large $\theta^2 \approx \theta^2 + 1$. So $L \approx \frac{t}{2\pi} \int_0^{2\pi N} \theta d\theta = \frac{t}{2\pi} \cdot (2\pi N)^2 \cdot \frac{1}{2} = t\pi N^2$. Because $N = \frac{R}{t}$, we have $L \approx \frac{\pi R^2}{t}$.
- d. Let $t\theta_1/(2\pi) = r$ and $t\theta_2/(2\pi) = R$. We have $\theta_1 = \frac{2\pi r}{t} = \frac{2\pi \cdot .025}{1.5 \cdot 10^{-6}} = \frac{\pi}{3} \cdot 10^5$, and $\theta_2 = \frac{2\pi R}{t} = \frac{2\pi \cdot 0.059}{1.5 \cdot 10^{-6}} \approx 2.36 \cdot \frac{\pi}{3} \cdot 10^5$.
- e. $L \approx \frac{1.5 \cdot 10^{-6}}{2\pi} \int_{(\pi/3) \cdot 10^5}^{(2.36\pi/3) \cdot 10^5} \theta d\theta \approx 5981.6$ meters. This is about 598,160 centimeters, and about 3.7 miles.

Chapter 15

Functions of Several Variables

15.1 Graphs and Level Curves

15.1.1 The independent variables are x and y and the dependent variable is z .

15.1.2 The domain of f is $\mathbb{R}^2$.

15.1.3 The domain of g is $\{(x, y) : x \neq 0 \text{ or } y \neq 0\}$.

15.1.4 The domain of h is $\{(x, y) : x - y \geq 0\}$.

15.1.5 We need three dimensions to plot points $(x, y, f(x, y))$.

15.1.6 Sketch the curves $f(x, y) = z_0$ in $\mathbb{R}^2$ for several values of z_0 .

15.1.7 $f(2, 1) = \sqrt{10 - 2 + 1} = \sqrt{9} = 3$. $f(-9, -3) = \sqrt{10 + 9 - 3} = \sqrt{16} = 4$.

15.1.8 $g(1, 5, 3) = \frac{1 + 5}{3} = 2$. $g(3, 7, 2) = \frac{3 + 7}{2} = 5$.

15.1.9

a. 1300 ft.

b. Katie did. She is 100 ft higher than Zeke.

15.1.10 Katie remains at a constant elevation of 1500 during her hike and there isn't enough information to determine Zeke's elevation during his hike, although he started and ended at an elevation of 1500 feet.

15.1.11 The level curves $x^2 + y^2 = z_0$ are circles centered at $(0, 0)$ in $\mathbb{R}^2$.

15.1.12 We need three dimensions to graph the level surfaces $f(x, y, z) = w_0$.

15.1.13 The function f has 6 independent variables, so $n = 6$.

15.1.14 We can sketch level surfaces in $\mathbb{R}^3$, or use colors to code the values of the function at points in $\mathbb{R}^3$.

15.1.15 The domain of f is $\mathbb{R}^2$.

15.1.16 The domain of f is $\mathbb{R}^2$.

15.1.17 The domain of f is $\{(x, y) : x^2 + y^2 \leq 25\}$, which is the set of all points on or within the circle of radius 5 centered at the origin.

15.1.18 The domain of f is $\{(x, y) : x^2 + y^2 > 25\}$, which is the set of all points outside the circle of radius 5 centered at the origin.

15.1.19 The domain of f is $\{(x, y) : y \neq 0\}$.

15.1.20 The domain of f is $\{(x, y) : x \neq \pm y\}$.

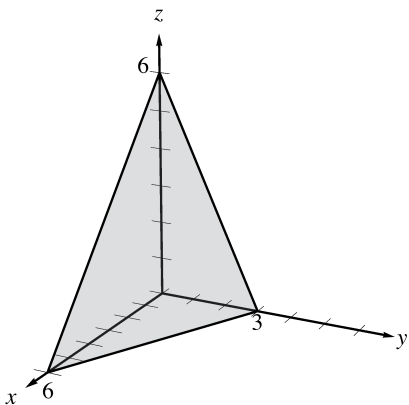
15.1.21 The domain of g is $\{(x, y) : y < x^2\}$.

15.1.22 The domain of f is $\{(x, y) : -1 \leq y - x^2 \leq 1\}$; which is the set of points lying between or on the parabolas $y = x^2 - 1$ and $y = x^2 + 1$.

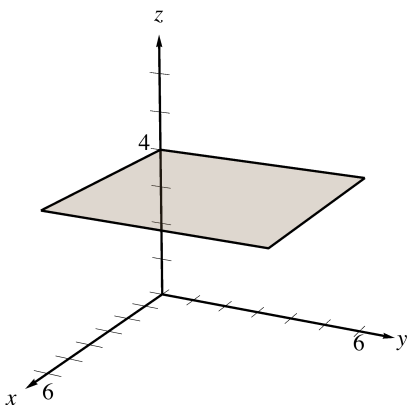
15.1.23 The domain of g is $\{(x, y) : xy \geq 0, (x, y) \neq (0, 0)\}$.

15.1.24 The domain of h is $\{(x, y) : x - 2y + 4 \geq 0\}$.

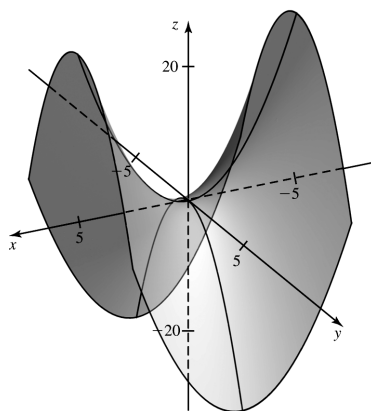
15.1.25 This surface is a plane; the function's domain is $\mathbb{R}^2$ and its range is $\mathbb{R}$.



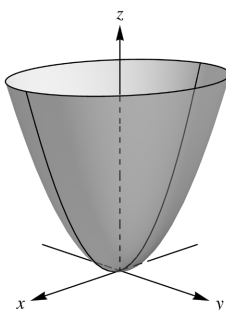
15.1.26 This surface is a plane; the function's domain is $\mathbb{R}^2$ and its range is $\{4\}$.



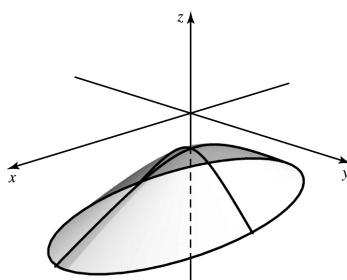
15.1.27 This surface is a hyperbolic paraboloid; the function's domain is $\mathbb{R}^2$ and its range is $\mathbb{R}$.



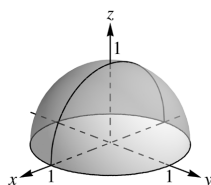
15.1.28 This surface is an elliptic paraboloid; the function's domain is $\mathbb{R}^2$ and its range is the interval $[0, \infty)$.



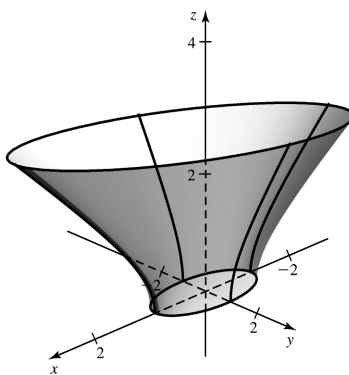
15.1.29 This surface is the lower part of a hyperboloid of two sheets; the function's domain is $\mathbb{R}^2$ and its range is the interval $(-\infty, -1]$.



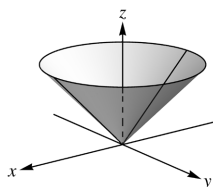
15.1.30 This surface is a hemisphere; the function's domain is $\{(x, y) : x^2 + y^2 \leq 1\}$ and its range is the interval $[0, 1]$.



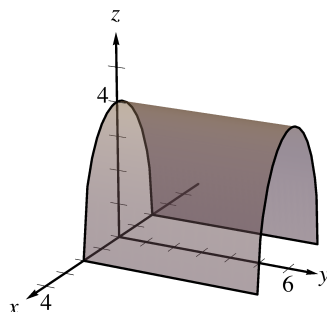
15.1.31 This surface is the upper part of a hyperboloid of one sheet; the function's domain is $\{(x, y) : x^2 + y^2 \geq 1\}$ and its range is the interval $[0, \infty)$.



15.1.32 This surface is the upper part of a circular cone; the function's domain is $\mathbb{R}^2$ and its range is the interval $[0, \infty)$.



15.1.33 This is the upper half of an elliptical cylinder. The domain is $\{(x, y) : -2 \leq x \leq 2, -\infty < y < \infty\}$, and the range is $[0, 4]$.



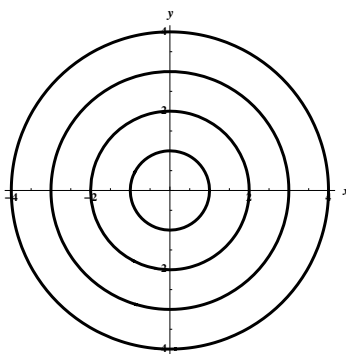
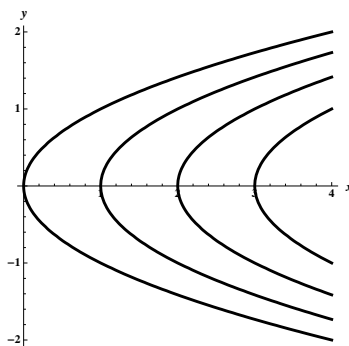
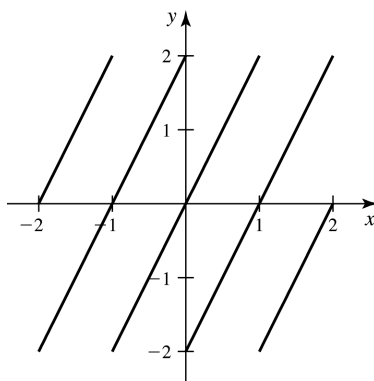
15.1.34

- B. Notice that the level curves for the function consist of lines parallel to the x -axis.
- E. Notice that the level curves for the function are hyperbolas.
- C. Notice that the level curves are oval and are elongated along the y -axis; therefore the level sets match (C).
- D. Notice that the level curves for the function are circles.
- A. Notice that the level curves are oval and are elongated along the x -axis ; therefore the level sets match (A).
- F. Notice that the level curves for the function are ellipses.

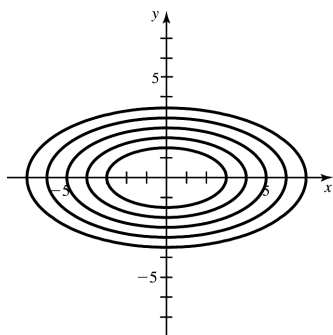
15.1.35

- A. Notice that the range of the function in (A) is $[-1, 1]$.

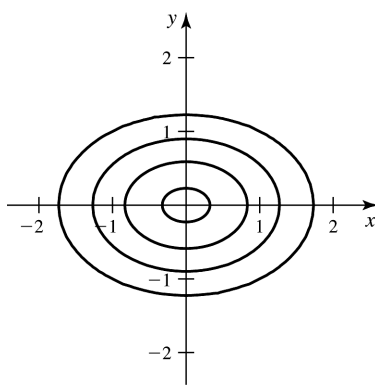
- b. D. Notice that the function in (D) becomes large and negative for (x, y) near $(0, 0)$.
- c. B. Notice that the function in (B) becomes large as you get close to $y = x$.
- d. C. Notice that the function in (C) is everywhere positive.

15.1.36**15.1.37****15.1.38**

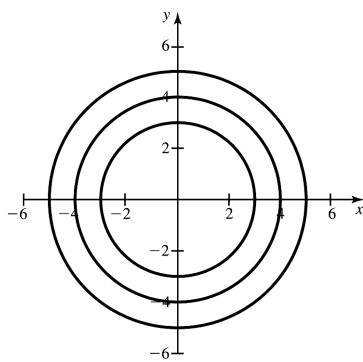
15.1.39



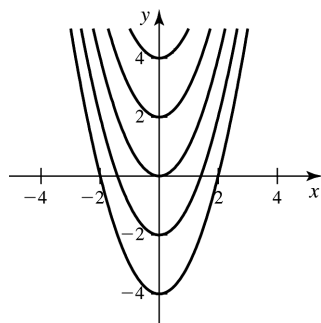
15.1.40



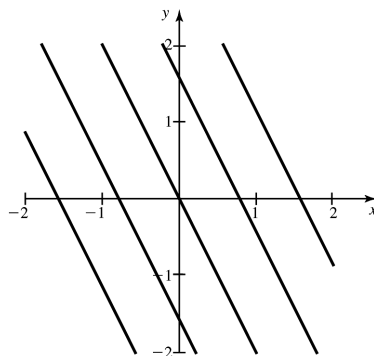
15.1.41



15.1.42

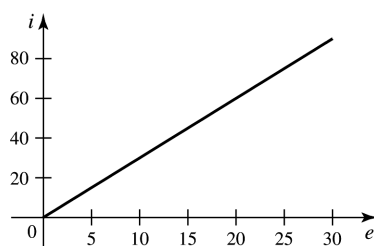


15.1.43



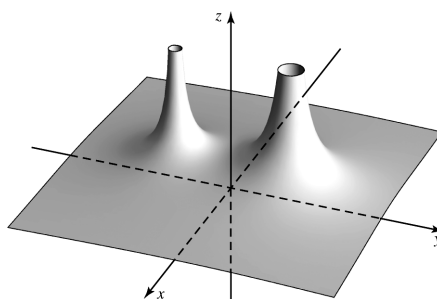
15.1.44

- a. His ERA was $A(24, 224) = \frac{9 \cdot 24}{224} = 0.9643$.
- b. His ERA is $A\left(4, \frac{1}{3}\right) = \frac{9 \cdot 4}{1/3} = 108$.
- c. The relationship is $e = \frac{i}{3}$, so a pitcher with an ERA of 3 gives up one run every three innings.



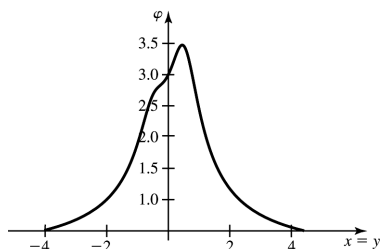
15.1.45

a.

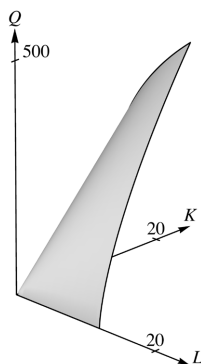


- b. The potential function is defined for all (x, y) in $\mathbb{R}^2$ except $(0, 1)$ and $(0, -1)$.
- c. We have $\varphi(2, 3) \approx 0.93 > \varphi(3, 2) \approx 0.87$.

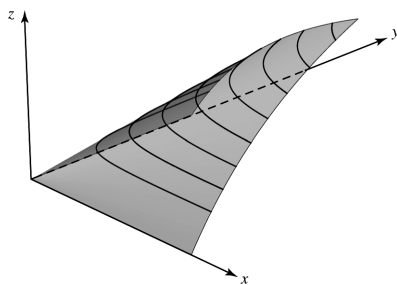
d.

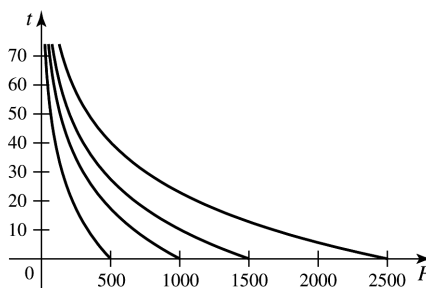
**15.1.46**

a.

b. We have $Q = cL^a K^b = 40 \cdot (10)^{1/3} K^{2/3}$.c. We have $Q = cL^a K^b = 40 \cdot (15)^{2/3} L^{1/3}$.**15.1.47**

a.

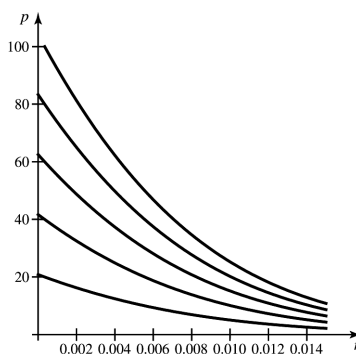
b. The maximum resistance is $R(10, 10) = 5$ ohms.c. This means $R(x, y) = R(y, x)$.**15.1.48**a. The set of points (P, t) with $B = 2000$ is given by $Pe^{0.04t} = 2000$; solving for P gives $P = 2000e^{-0.04t}$.b. The level curves are given by $P = Be^{0.04t}$ with $B = 500, 1000, 1500$ and 2000 .



- c. As t increases along a level curve, P decreases and vice versa.

15.1.49

- a. Solving for P in the equation $B(P, r, t) = 20000$ with $t = 20$ years gives $P = \frac{20000r}{(1+r)^{240} - 1}$.
- b. The level curves are given by $P = \frac{Br}{(1+r)^{240} - 1}$, with $B = 5000, 10000, 15000$ and 25000 .



15.1.50 The domain of f is $\mathbb{R}^3$.

15.1.51 The domain of g is $\{(x, y, z) : x \neq z\}$, which is all points in $\mathbb{R}^3$ not on the plane given by $x = z$.

15.1.52 The domain of p is $\{(x, y, z) : x^2 + y^2 + z^2 \geq 9\}$, which is all points in $\mathbb{R}^3$ on or outside the sphere of radius 3 centered at the origin.

15.1.53 The domain of f is $\{(x, y, z) : y \geq z\}$, which is all points in $\mathbb{R}^3$ on or below the plane given by $z = y$.

15.1.54 The domain of Q is $\mathbb{R}^3$.

15.1.55 The domain of F is $\{(x, y, z) : x^2 \leq y\}$, which is all points on the side of the vertical cylinder $y = x^2$ that contains the positive y -axis.

15.1.56 The domain of f is $\{(w, x, y, z) : w^2 + x^2 + y^2 + z^2 \leq 1\}$, which is all points in $\mathbb{R}^4$ on or inside the hypersphere of radius 1 centered at the origin.

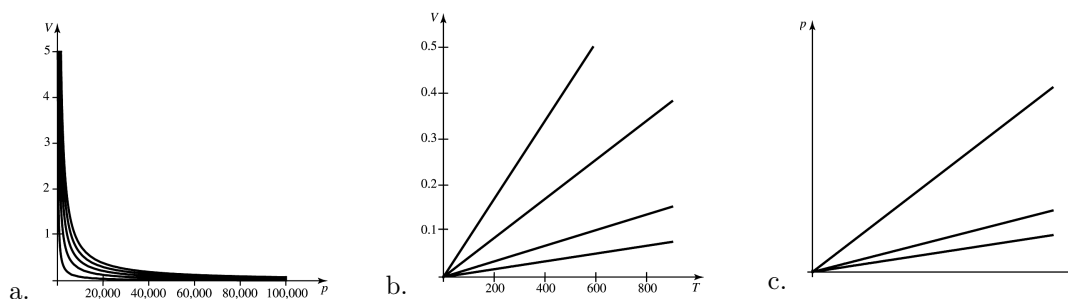
15.1.57

- a. False. This function has domain $\mathbb{R}^2$.
- b. False. The domain of a function of 4 variables is a region in $\mathbb{R}^4$.
- c. True. The level curves for the function defined by $z = 2x - 3y$ are lines of the form $2x - 3y = c$ for any constant c .

15.1.58

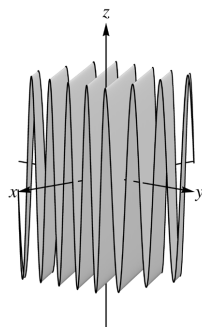
- a. His quarterback passer rating is $R = \frac{50 + 20 \cdot 71.43 + 80 \cdot 9.18 - 100 \cdot 0 + 100 \cdot 10.35}{24} \approx 135.33$.
- b. His quarterback passer rating is $R = \frac{50 + 20 \cdot 67.36 + 80 \cdot 6.48 - 100 \cdot 0.46 + 100 \cdot 8.23}{24} \approx 112.2$.
- c. If c , t and y are fixed then R is a decreasing linear function of t . This makes sense since a quarterback's rating should decrease if his interception percentage increases.

15.1.59



15.1.60

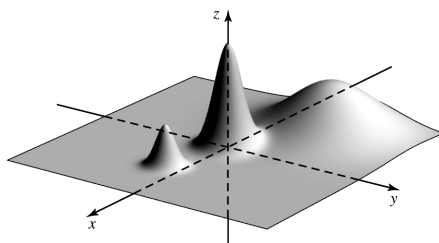
a.



- b. The domain of this function is $\mathbb{R}^2$.
- c. The maximum and minimum water heights are ± 10 .
- d. The maximum and minimum heights occur along the lines $2x - 3y = \frac{\pi}{2} + k\pi$, where k is any integer. A vector orthogonal to these lines is $\mathbf{v} = \langle 2, -3 \rangle$.

15.1.61

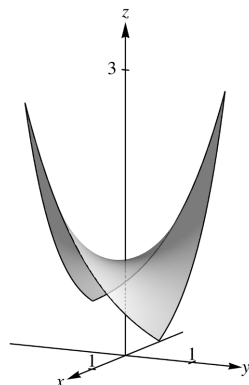
a.



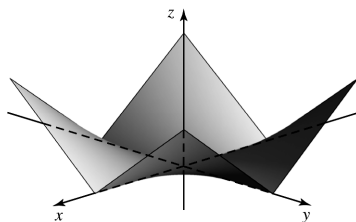
- b. The peaks occur near the points $(0, 0)$, $(-5, 3)$ and $(4, -1)$.
 c. We have $f(0, 0) \approx 10.17$, $f(-5, 3) \approx 5.00$, $f(4, -1) \approx 4.00$.

15.1.62

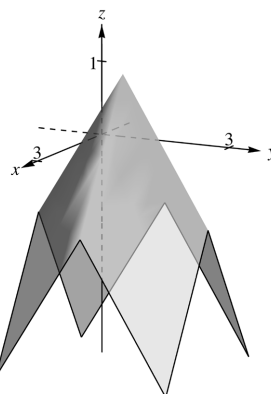
- a. The domain is $\mathbb{R}^2$ and the range is the interval $(0, \infty)$.
 b.

**15.1.63**

- a. The domain is $\mathbb{R}^2$ and the range is the interval $[0, \infty)$.
 b.

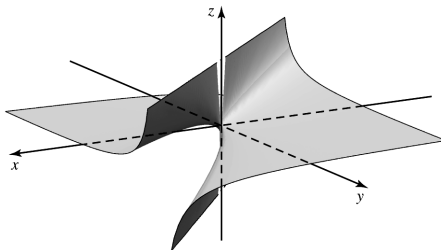
**15.1.64**

- a. The domain is $\mathbb{R}^2$ and the range is $\mathbb{R}$.
 b.

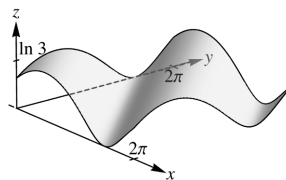


15.1.65

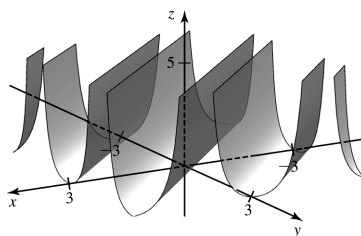
- a. The domain is $\{(x, y) : x \neq y\}$ and the range is $\mathbb{R}$.
 b.

**15.1.66**

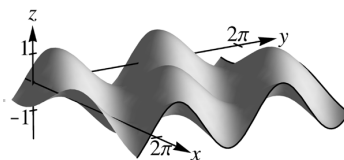
- a. The domain is $\mathbb{R}^2$ and the range is the interval $[0, \ln 3]$.
 b.

**15.1.67**

- a. The domain is $\{(x, y) : y \neq x + \frac{\pi}{2} + n\pi \text{ for any integer } n\}$ and the range is the interval $[0, \infty)$.
 b.

**15.1.68**

- a. The domain is $\mathbb{R}^2$ and the range is the interval $[-1, 1]$.
 b.



15.1.69 This function has a peak at the origin.

15.1.70 This function has a peak at the point $\left(\frac{1}{2}, -1\right)$.

15.1.71 This function has a depression at the point $(1, 0)$.

15.1.72 This function has a depression at the point $(1, 1)$.

15.1.73 The level curves are the lines given by $ax + by = d - cz_0$, where z_0 is a constant; these lines all have slope $-\frac{a}{b}$ (in the case $b = 0$ the lines are all vertical).

15.1.74 If $\frac{1}{x^2 + y^2 + z^2} = K$, then $x^2 + y^2 + z^2 = \frac{1}{K} = C$ for a constant C . Thus, the level surfaces are spheres centered at the origin.

15.1.75 If $x^2 + y^2 - z = C$, then $z = x^2 + y^2 - C$, so the level surfaces are paraboloids with vertices $(0, 0, -C)$.

15.1.76 If $x^2 - y^2 - z = C$, then $z = x^2 - y^2 - C$, so the level surfaces are hyperbolic paraboloids with a saddle point at $(0, 0, -C)$.

15.1.77 If $\sqrt{x^2 + 2z^2} = K$, then $x^2 + 2z^2 = C$ where $C = K^2$, so the level surfaces are elliptic cylinders parallel to the y -axis.

15.1.78 Factor the equation $x^2 - (y + z)x + yz = (x - y)(x - z)$; hence the domain of g is $\{(x, y, z) : x \neq y \text{ and } x \neq z\}$. The domain consists of all points not on the planes given by $x = y$ and $x = z$.

15.1.79 The domain of f is $\{(x, y) : x - 1 \leq y \leq x + 1\}$. This is the region between the two parallel lines given by $y = x - 1$ and $y = x + 1$.

15.1.80 The domain of f is $\{(x, y, z) : z > x^2 + y^2 - 2x - 3\}$, which is equivalent to $\{(x, y, z) : z > (x - 1)^2 + y^2 - 4\}$. This region consists of all points inside a circular paraboloid with vertex at $(1, 0, -4)$.

15.1.81 Factor the equation $z^2 - xz + yz - xy = (z - x)(z + y)$; hence the domain of h is $\{(x, y, z) : (z - x)(z + y) \geq 0\}$, which is equivalent to $D = \{(x, y, z) : (x \leq z \text{ and } y \geq -z) \text{ or } (x \geq z \text{ and } y \leq -z)\}$. The domain consists of all points above or below both the planes given by $z = x$ and $z = -y$ as well as the points on either one of these planes.

15.1.82

a. This “ball” is the solid octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$.

b. This “ball” is the solid cube with vertices (x, y, z) where $x, y, z = \pm 1$.

15.2 Limits and Continuity

15.2.1 The values of $|f(x, y) - L|$ can be made arbitrarily small if (x, y) is sufficiently close to (a, b) .

15.2.2 If $f(x, y)$ has a different limit as (x, y) approaches (a, b) along two different paths, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

15.2.3 If $f(x, y)$ is a polynomial, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$; in other words, the limit can be found by plugging in $x = a$, $y = b$ in $f(x, y)$.

15.2.4 We evaluate $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ along all paths that approach (a, b) from within the domain of f .

15.2.5 If the limits along different paths do not agree, then the limit does not exist.

15.2.6 Evaluating $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ along a finite number of paths does not establish that the limit is the same along *all* paths that approach (a,b) .

15.2.7 The function f must be defined at (a,b) , $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ must exist, and the limit must equal $f(a,b)$.

15.2.8 Yes, $(0,0)$ is a boundary point for R because every disc centered at $(0,0)$ contains points in R and points not in R (namely $(0,0)$). The set R is neither open nor closed because it contains some of its boundary points (namely points on the unit circle) but not all of them (namely $(0,0)$).

15.2.9 A rational function is continuous at all points where its denominator is nonzero.

15.2.10 Because xy^2z^3 is a polynomial, $\lim_{(x,y,z) \rightarrow (1,1,-1)} xy^2z^3 = 1 \cdot 1^2 \cdot (-1)^3 = -1$.

15.2.11 $\lim_{(x,y) \rightarrow (5,-5)} \frac{x^2 - y^2}{x + y} = \lim_{(x,y) \rightarrow (5,-5)} \frac{(x-y)(x+y)}{x+y} = \lim_{(x,y) \rightarrow (5,-5)} x - y = 5 - (-5) = 10$.

15.2.12 We have (along $y = 0$) $\lim_{(x,y) \rightarrow (1,0)} \frac{x^2 - 2x - y^2 + 1}{x^2 - 2x + y^2 + 1} = \lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x^2 - 2x + 1} = \lim_{x \rightarrow 1} 1 = 1$. But, along $x = 1$ we have $\lim_{(x,y) \rightarrow (1,0)} \frac{x^2 - 2x - y^2 + 1}{x^2 - 2x + y^2 + 1} = \lim_{y \rightarrow 0} \frac{-y^2}{y^2} = \lim_{y \rightarrow 0} -1 = -1$. Because we obtain different limits along different paths within the domain, $\lim_{(x,y) \rightarrow (1,0)} \frac{x^2 - 2x - y^2 + 1}{x^2 - 2x + y^2 + 1}$ does not exist.

15.2.13 $\lim_{(x,y) \rightarrow (2,9)} 101 = 101$.

15.2.14 $\lim_{(x,y) \rightarrow (1,-3)} (3x + 4y - 2) = 3 \cdot 1 + 4(-3) - 2 = -11$.

15.2.15 $\lim_{(x,y) \rightarrow (-3,3)} (4x^2 - y^2) = 4 \cdot (-3)^2 - (3)^2 = 27$.

15.2.16 $\lim_{(x,y) \rightarrow (2,-1)} (xy^8 - 3x^2y^3) = 2(-1)^8 - 3 \cdot 2^2(-1)^3 = 14$.

15.2.17 $\lim_{(x,y) \rightarrow (0,\pi)} \frac{\cos xy + \sin xy}{2y} = \frac{\cos 0 + \sin 0}{2\pi} = \frac{1}{2\pi}$.

15.2.18 $\lim_{(x,y) \rightarrow (e^2,4)} \ln \sqrt{xy} = \ln \sqrt{4e^2} = \ln 2e = 1 + \ln 2$.

15.2.19 $\lim_{(x,y) \rightarrow (2,0)} \frac{x^2 - 3xy^2}{x + y} = \frac{2^2 - 3 \cdot 2 \cdot 0^2}{2 + 0} = 2$.

15.2.20 $\lim_{(u,v) \rightarrow (1,-1)} \frac{10uv - 2v^2}{u^2 + v^2} = \frac{10 \cdot 1(-1) - 2(-1)^2}{1^2 + (-1)^2} = -6$.

15.2.21 $\lim_{(x,y) \rightarrow (6,2)} \frac{x^2 - 3xy}{x - 3y} = \lim_{(x,y) \rightarrow (6,2)} \frac{x(x - 3y)}{x - 3y} = \lim_{(x,y) \rightarrow (6,2)} x = 6$.

15.2.22 $\lim_{(x,y) \rightarrow (1,-2)} \frac{y^2 + 2xy}{y + 2x} = \lim_{(x,y) \rightarrow (1,-2)} \frac{y(y + 2x)}{y + 2x} = \lim_{(x,y) \rightarrow (1,-2)} y = -2$.

15.2.23 $\lim_{(x,y) \rightarrow (3,1)} \frac{x^2 - 7xy + 12y^2}{x - 3y} = \lim_{(x,y) \rightarrow (3,1)} \frac{(x - 3y)(x - 4y)}{x - 3y} = \lim_{(x,y) \rightarrow (3,1)} (x - 4y) = 3 - 4 = -1$.

$$15.2.24 \quad \lim_{(x,y) \rightarrow (-1,1)} \frac{2x^2 - xy - 3y^2}{x + y} = \lim_{(x,y) \rightarrow (-1,1)} \frac{(x+y)(2x-3y)}{x+y} = \lim_{(x,y) \rightarrow (-1,1)} (2x-3y) = -5.$$

$$15.2.25 \quad \lim_{(x,y) \rightarrow (2,2)} \frac{y^2 - 4}{xy - 2x} = \lim_{(x,y) \rightarrow (2,2)} \frac{(y+2)(y-2)}{x(y-2)} = \lim_{(x,y) \rightarrow (2,2)} \frac{y+2}{x} = \frac{2+2}{2} = 2.$$

15.2.26

$$\begin{aligned} \lim_{(x,y) \rightarrow (4,5)} \frac{\sqrt{x+y} - 3}{x+y-9} &= \lim_{(x,y) \rightarrow (4,5)} \frac{(\sqrt{x+y} - 3)(\sqrt{x+y} + 3)}{(x+y-9)(\sqrt{x+y} + 3)} \\ &= \lim_{(x,y) \rightarrow (4,5)} \frac{x+y-9}{(x+y-9)(\sqrt{x+y} + 3)} \\ &= \lim_{(x,y) \rightarrow (4,5)} \frac{1}{\sqrt{x+y} + 3} = \frac{1}{\sqrt{4+5} + 3} = \frac{1}{6}. \end{aligned}$$

15.2.27

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} \frac{\sqrt{y} - \sqrt{x+1}}{y-x-1} &= \lim_{(x,y) \rightarrow (1,2)} \frac{(\sqrt{y} - \sqrt{x+1})(\sqrt{y} + \sqrt{x+1})}{(y-x-1)(\sqrt{y} + \sqrt{x+1})} \\ &= \lim_{(x,y) \rightarrow (1,2)} \frac{y-x-1}{(y-x-1)(\sqrt{y} + \sqrt{x+1})} \\ &= \lim_{(x,y) \rightarrow (1,2)} \frac{1}{\sqrt{y} + \sqrt{x+1}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}. \end{aligned}$$

15.2.28

$$\begin{aligned} \lim_{(u,v) \rightarrow (8,8)} \frac{u^{1/3} - v^{1/3}}{u^{2/3} - v^{2/3}} &= \lim_{(u,v) \rightarrow (8,8)} \frac{u^{1/3} - v^{1/3}}{(u^{1/3} - v^{1/3})(u^{1/3} + v^{1/3})} \\ &= \lim_{(u,v) \rightarrow (8,8)} \frac{1}{u^{1/3} + v^{1/3}} = \frac{1}{2+2} = \frac{1}{4}. \end{aligned}$$

15.2.29 Observe that along the line $y = 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{x+2y}{x-2y} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$, whereas along the line $x = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x+2y}{x-2y} = \lim_{y \rightarrow 0} \frac{2y}{-2y} = -1.$$

15.2.30 Observe that along the line $y = x$, $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy}{3x^2 + y^2} = \lim_{x \rightarrow 0} \frac{4x^2}{4x^2} = 1$, whereas along the line

$$y = -x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{4xy}{3x^2 + y^2} = \lim_{x \rightarrow 0} -\frac{4x^2}{4x^2} = -1.$$

15.2.31 Observe that along the line $x = 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 - 2x^2}{y^4 + x^2} = \lim_{y \rightarrow 0} \frac{y^4}{y^4} = 1$, whereas along the line $y = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 - 2x^2}{y^4 + x^2} = \lim_{x \rightarrow 0} -\frac{2x^2}{x^2} = -2.$$

15.2.32 Observe that along the line $y = 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^2}{x^3 + y^2} = \lim_{x \rightarrow 0} \frac{x^3}{x^3} = 1$, whereas along the line $x = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^2}{x^3 + y^2} = \lim_{y \rightarrow 0} -\frac{y^2}{y^2} = -1.$$

15.2.33 Observe that along the line $y = x$, $\lim_{(x,y) \rightarrow (0,0)} \frac{y^3 + x^3}{xy^2} = \lim_{x \rightarrow 0} \frac{2x^3}{x^3} = 2$, whereas along the line

$$y = -x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{y^3 + x^3}{xy^2} = \lim_{x \rightarrow 0} \frac{0}{-x^3} = 0.$$

15.2.34 Observe that along the line $y = 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{\sqrt{x^2 - y^2}} = \lim_{x \rightarrow 0} \frac{0}{|x|} = 0$, whereas along the ray $x = 2y$, $y > 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{\sqrt{x^2 - y^2}} = \lim_{x \rightarrow 0} \frac{y}{\sqrt{3}y} = \frac{1}{\sqrt{3}}$.

15.2.35 The function f is continuous on $\mathbb{R}^2$.

15.2.36 The function f is continuous on $\mathbb{R}^2$ (the denominator of this rational function is always positive).

15.2.37 The function p is continuous at all points except the origin, where it is undefined.

15.2.38 The function S is continuous at all points except where $x^2 = y^2$, which is along the lines $x = y$ and $x = -y$.

15.2.39 The function f is continuous on $\mathbb{R}^2$ except where $x = 0$.

15.2.40 The function f is continuous on $\mathbb{R}^2$ except where $x = 0$ or $y = \pm 1$.

15.2.41 The function f is continuous on $\mathbb{R}^2$ except the origin. Note that along the line $y = x$, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + x^2} = \frac{1}{2} \neq 0$.

15.2.42 The function f is continuous on $\mathbb{R}^2$ except the origin. Note that along the line $x = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 - 2x^2}{y^4 + x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{y^4} = 1 \neq 0.$$

15.2.43 The function f is continuous on $\mathbb{R}^2$.

15.2.44 The function f is continuous on $\mathbb{R}^2$.

15.2.45 The function f is continuous on $\mathbb{R}^2$.

15.2.46 The function g is continuous on its domain, which is $D = \{(x, y) : x > y\}$.

15.2.47 The function h is continuous on $\mathbb{R}^2$.

15.2.48 The function p is continuous on $\mathbb{R}^2$.

15.2.49 The function f is continuous on its domain, which is $D = \{(x, y) : (x, y) \neq (0, 0)\}$.

15.2.50 The function f is continuous on its domain, which is $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

15.2.51 The function g is continuous on $\mathbb{R}^2$.

15.2.52 The function h is continuous on its domain, which is $D = \{(x, y) : x \geq y\}$.

15.2.53 The function f is continuous on $\mathbb{R}^2$.

15.2.54 The function f is continuous on $\mathbb{R}^2$.

15.2.55 $\lim_{(x,y,z) \rightarrow (1, \ln 2, 3)} ze^{xy} = 3e^{1 \cdot \ln 2} = 6$.

15.2.56 $\lim_{(x,y,z) \rightarrow (0, 1, 0)} \ln(1 + y)e^{xz} = (\ln 2)e^0 = \ln 2$.

15.2.57 $\lim_{(x,y,z) \rightarrow (1, 1, 1)} \frac{yz - xy - xz - x^2}{yz + xy + xz - y^2} = \frac{1 - 1 - 1 - 1}{1 + 1 + 1 - 1} = -1$.

$$15.2.58 \quad \lim_{(x,y,z) \rightarrow (1,1,1)} \frac{x - \sqrt{xz} - \sqrt{xy} + \sqrt{yz}}{x - \sqrt{xz} + \sqrt{xy} - \sqrt{yz}} = \lim_{(x,y,z) \rightarrow (1,1,1)} \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} - \sqrt{z})}{(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{z})} =$$

$$\lim_{(x,y,z) \rightarrow (1,1,1)} \frac{(\sqrt{x} - \sqrt{y})}{(\sqrt{x} + \sqrt{y})} = \frac{1-1}{1+1} = 0.$$

$$15.2.59 \quad \lim_{(x,y,z) \rightarrow (1,1,1)} \frac{x^2 + xy - xz - yz}{x - z} = \lim_{(x,y,z) \rightarrow (1,1,1)} \frac{(x-z)(x+y)}{x - z} = \lim_{(x,y,z) \rightarrow (1,1,1)} (x+y) = 2.$$

$$15.2.60 \quad \lim_{(x,y,z) \rightarrow (1,-1,1)} \frac{xz + 5x + yz + 5y}{x + y} = \lim_{(x,y,z) \rightarrow (1,-1,1)} \frac{(x+y)(z+5)}{x + y} = \lim_{(x,y,z) \rightarrow (1,-1,1)} (z+5) = 6.$$

15.2.61

- False. The limit may be different or not exist along other paths approaching $(0, 0)$.
- False. We may have $f(a, b)$ undefined, or $f(a, b) \neq L$.
- True. The limit must exist for f to be continuous at (a, b) .
- False. For example, take $P = (0, 0)$ and the domain of f to be $\{(x, y) : (x, y) \neq (0, 0)\}$.

$$15.2.62 \quad \text{Observe that along the line } y = 0, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{y^2}{x^8 + y^2} = \lim_{x \rightarrow 0} \frac{0}{x^8} = 0, \text{ whereas along the curve } y = x^4,$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^2}{x^8 + y^2} = \lim_{x \rightarrow 0} \frac{x^8}{2x^8} = \frac{1}{2}, \text{ therefore this limit does not exist.}$$

$$15.2.63 \quad \lim_{(x,y) \rightarrow (0,1)} \frac{y \sin x}{x(y+1)} = \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left(\lim_{y \rightarrow 1} \frac{y}{y+1} \right) = 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

$$15.2.64 \quad \lim_{(x,y) \rightarrow (1,1)} \frac{x^2 + xy - 2y^2}{2x^2 - xy - y^2} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x+2y)(x-y)}{(2x+y)(x-y)} = \lim_{(x,y) \rightarrow (1,1)} \frac{x+2y}{2x+y} = \frac{1+2}{2+1} = 1.$$

$$15.2.65 \quad \lim_{(x,y) \rightarrow (1,0)} \frac{y \ln y}{x} = \left(\lim_{x \rightarrow 1} \frac{1}{x} \right) \left(\lim_{y \rightarrow 0} \ln y^y \right) = 1 \cdot \ln 1 = 0$$

$$15.2.66 \quad \text{Observe that along the line } y = x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{xy} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1, \text{ whereas along the line } y = -x,$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{xy} = \lim_{x \rightarrow 0} \frac{x^2}{-x^2} = -1, \text{ therefore this limit does not exist.}$$

$$15.2.67 \quad \text{Observe that along the line } y = x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{|x-y|}{|x+y|} = \lim_{x \rightarrow 0} \frac{0}{2|x|} = 0, \text{ whereas along the line } y = 2x,$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|x-y|}{|x+y|} = \lim_{x \rightarrow 0} \frac{|x|}{3|x|} = \frac{1}{3}, \text{ therefore this limit does not exist.}$$

$$15.2.68 \quad \lim_{(u,v) \rightarrow (-1,0)} \frac{uve^{-v}}{u^2 + v^2} = \frac{(-1) \cdot 0 \cdot e^0}{(-1)^2 + 0^2} = 0.$$

$$15.2.69 \quad \text{Observe that } \lim_{(x,y) \rightarrow (2,0)} \frac{1 - \cos y}{xy^2} = \left(\lim_{x \rightarrow 2} \frac{1}{x} \right) \left(\lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2} \right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \text{ where the } y\text{-limit is}$$

evaluated by two applications of L'Hôpital's rule.

$$15.2.70 \quad \text{Let } u = xy; \text{ then } u \rightarrow 0 \text{ as } (x, y) \rightarrow (4, 0), \text{ so } \lim_{(x,y) \rightarrow (4,0)} x^2 y \ln xy = \lim_{(x,y) \rightarrow (4,0)} x \cdot xy \ln xy =$$

$$4 \lim_{u \rightarrow 0} u \ln u = 4 \lim_{u \rightarrow 0} \ln u^u = 4 \cdot 0 = 0.$$

15.2.71 Let $u = xy$; then $u \rightarrow 0$ as $(x, y) \rightarrow (1, 0)$, so $\lim_{(x,y) \rightarrow (1,0)} \frac{\sin xy}{xy} = \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$.

15.2.72 Let $u = xy$; then $u \rightarrow 0$ as $(x, y) \rightarrow (0, \frac{\pi}{2})$, so $\lim_{(x,y) \rightarrow (0,\pi/2)} \frac{1 - \cos xy}{4x^2y^3} =$
 $\lim_{(x,y) \rightarrow (0,\pi/2)} \left(\frac{1}{4y} \cdot \frac{1 - \cos xy}{x^2y^2} \right) = \frac{1}{2\pi} \lim_{u \rightarrow 0} \frac{1 - \cos u}{u^2} = \frac{1}{4\pi}$. (Use L'Hôpital's rule twice to see that $\lim_{u \rightarrow 0} (1 - \cos u)/u^2 = 1/2$).

15.2.73 Let $u = xy$; then $u \rightarrow 0$ as $(x, y) \rightarrow (0, 2)$, so $\lim_{(x,y) \rightarrow (0,2)} (2xy)^{xy} = \lim_{u \rightarrow 0} 2^u u^u = 1$.

15.2.74 Let $u = x + y$. Then $\lim_{(x,y) \rightarrow (3,3)} \frac{x^2 + 2xy - 6x + y^2 - 6y}{x + y - 6} = \lim_{(x,y) \rightarrow (3,3)} \frac{x^2 + 2xy + y^2 - (6x_6y)}{x + y - 6} =$
 $\lim_{(x,y) \rightarrow (3,3)} \frac{(x+y)^2 - 6(x+y)}{x + y - 6} = \lim_{u \rightarrow 6} \frac{u^2 - 6u}{u - 6} = \lim_{u \rightarrow 6} u = 6$.

15.2.75 Let $u = x + y$. Then $\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + 2xy - x + y^2 - y - 6}{x + y - 3} = \lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + 2xy + y^2 - x - y - 6}{x + y - 3} =$
 $\lim_{(x,y) \rightarrow (1,2)} \frac{(x+y)^2 - (x+y) - 6}{x + y - 3} = \lim_{u \rightarrow 3} \frac{u^2 - u - 6}{u - 3} = \lim_{u \rightarrow 3} \frac{(u+2)(u-3)}{u-3} = \lim_{u \rightarrow 3} (u+2) = 5$.

15.2.76 Let $u = x^2 + y^2$. Then

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \tan^{-1} \left(\frac{(2 + (x+y)^2 + (x-y)^2)}{2e^{x^2+y^2}} \right) &= \lim_{(x,y) \rightarrow (0,0)} \tan^{-1} \left(\frac{2 + 2(x^2 + y^2)}{2e^{x^2+y^2}} \right) \\ &= \lim_{u \rightarrow 0^+} \tan^{-1} \left(\frac{1+u}{e^u} \right) = \frac{\pi}{4}. \end{aligned}$$

15.2.77 Let $u = x^2 + y^2 - 1$. Then as $x^2 + y^2 \rightarrow 1$, $u \rightarrow 0$. Because $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$, we must have $b = 1$ in order for f to be continuous everywhere.

15.2.78 Let $u = xy$. Note that as $xy \rightarrow 0$, $u \rightarrow 0$. Also, $\lim_{xy \rightarrow 0} \frac{1 + 2xy - \cos(xy)}{xy} = \lim_{u \rightarrow 0} \frac{1 + 2u - \cos u}{u} =$
 $\lim_{u \rightarrow 0} \frac{2 + \sin u}{1} = 2$ by L'Hôpital's rule. Thus, we must have $a = 2$ in order for f to be continuous everywhere.

15.2.79 $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^2 \theta \sin \theta}{r^2} = \lim_{r \rightarrow 0} r \cos^2 \theta \sin \theta = 0$.

15.2.80 $\lim_{(x,y) \rightarrow (0,0)} \frac{x-y}{\sqrt{x^2+y^2}} = \lim_{r \rightarrow 0} \frac{r \cos \theta - r \sin \theta}{r} = \lim_{r \rightarrow 0} (\cos \theta - \sin \theta)$, which does not exist.

15.2.81 $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2 + x^2y^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^2 + r^4 \cos^2 \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} (1 + r^2 \cos^2 \theta \sin^2 \theta) = 1$.

15.2.82 Let $z = x + y$, then

$$\frac{\sin x + \sin y}{x + y} = \frac{\sin x + \sin(z-x)}{z} = \frac{\sin x + \sin z \cos x - \cos z \sin x}{z} = \sin x \left(\frac{1 - \cos z}{z} \right) + \cos x \left(\frac{\sin z}{z} \right).$$

As $(x, y) \rightarrow (0, 0)$ we have $z \rightarrow 0$; $\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$ and $\lim_{z \rightarrow 0} \frac{1 - \cos z}{z} = 0$, so $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin x + \sin y}{x + y} = 0 \cdot 0 + 1 \cdot 1 = 1$.

15.2.83 The limit is 0 along the lines $x = 0$ or $y = 0$. However, along the line $x = y$ we have

$\lim_{(x,y) \rightarrow (0,0)} \frac{ax^m y^n}{bx^{m+n} + cy^{m+n}} = \lim_{x \rightarrow 0} \frac{ax^{m+n}}{bx^{n+m} + cx^{m+n}} = \frac{a}{b+c} \neq 0$ because $a \neq 0$. Therefore this limit does not exist.

15.2.84 The limit is 0 along the line $y = 0$. However, along the curve $y = x^2$ we have $\lim_{(x,y) \rightarrow (0,0)} \frac{ax^{2(p-n)}y^n}{bx^{2p} + cy^p}$
 $= \lim_{x \rightarrow 0} \frac{ay^p}{(b+c)y^p} = \frac{a}{b+c} \neq 0$ because $a \neq 0$. Therefore this limit does not exist.

15.2.85 Because $\lim_{(x,y) \rightarrow (0,0)} e^{-1/(x^2+y^2)} = 0$, we should define $f(0,0) = 0$.

15.2.86 For any $\epsilon > 0$, let $\delta = \epsilon$; then $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |y-b| < \epsilon$ because $|y-b| \leq \sqrt{(x-a)^2 + (y-b)^2}$.

15.2.87 For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{2}$. Then $|x-a|, |y-b| \leq \sqrt{(x-a)^2 + (y-b)^2}$, so
 $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ which implies that $|x+y-(a+b)| \leq |x-a| + |y-b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

15.2.88 Suppose $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$, $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = M$. Let $\epsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1 \implies |f(x,y) - L| < \frac{\epsilon}{2}$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_2 \implies |g(x,y) - M| < \frac{\epsilon}{2}$. Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) + g(x,y) - (L+M)| \leq |f(x,y) - L| + |g(x,y) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

15.2.89 Observe first that this is trivial when $c = 0$, so assume $c \neq 0$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \frac{\epsilon}{|c|}$. Therefore $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |cf(x,y) - cL| = |c||f(x,y) - L| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$.

15.3 Partial Derivatives

15.3.1 The slope parallel to the x -axis is $f_x(a,b)$, and the slope parallel to the y -axis is $f_y(a,b)$.

15.3.2

- $f_x(x,y) = 6x$ and $f_y(x,y) = 3y^2$.
- $f_x(1,3) = 6$ and $f_y(1,3) = 27$.
- $f_{xx}(x,y) = 6$, $f_{yy}(x,y) = 6y$, $f_{xy}(x,y) = f_{yx}(x,y) = 0$.

15.3.3

- Negative.
- Negative.
- Negative.
- Positive.

15.3.4 $f_x(x,y) = 12x^5 + 2y$, $f_y(x,y) = 8y^7 + 2x$.

15.3.5 $f_x(x,y) = 6xy$, $f_y(x,y) = 3x^2$.

15.3.6 We have $f_x(x, y) = 2xy^3$, $f_y(x, y) = 3x^2y^2$; therefore $f_{xx}(x, y) = 2y^3$, $f_{yy}(x, y) = 6x^2y$, $f_{xy}(x, y) = f_{yx}(x, y) = 6xy^2$.

15.3.7 Observe that $f_x(x, \cos y) = 6x^2$, so $f_{xy}(x, \cos y) = 0$; and $f_y(x, \cos y) = 6y$, so $f_{yx}(x, \cos y) = 0$.

15.3.8 Observe that $f_x(x, \cos y) = e^y$, so $f_{xy}(x, \cos y) = e^y$; and $f_y(x, \cos y) = xe^y$, so $f_{yx}(x, \cos y) = e^y$.

15.3.9 $f_x(x, \cos y, \cos z) = y + z$; $f_y(x, \cos y, \cos z) = x + z$; $f_z(x, \cos y, \cos z) = x + y$.

15.3.10 Note that $\frac{\partial V}{\partial r} = 2\pi rh > 0$, so the volume is an increasing function of the radius r if the height h is fixed.

$$\begin{aligned} \mathbf{15.3.11} \quad f_x &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{5(x+h)y - 5xy}{h} = \lim_{h \rightarrow 0} \frac{5hy}{h} = \lim_{h \rightarrow 0} 5y = 5y. \\ f_y &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{5x(y+h) - 5xy}{h} = \lim_{h \rightarrow 0} \frac{5xh}{h} = \lim_{h \rightarrow 0} 5x = 5x. \end{aligned}$$

$$\begin{aligned} \mathbf{15.3.12} \quad f_x &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{x+h+y^2+4 - (x+y^2+4)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1. \\ f_y &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{x+(y+h)^2+4 - (x+y^2+4)}{h} = \lim_{h \rightarrow 0} \frac{y^2+2hy+h^2-y^2}{h} = \\ &\lim_{h \rightarrow 0} 2y+h = 2y. \end{aligned}$$

$$\begin{aligned} \mathbf{15.3.13} \quad f_x &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{\frac{x+h}{y} - \frac{x}{y}}{h} = \lim_{h \rightarrow 0} \frac{h}{hy} = \lim_{h \rightarrow 0} \frac{1}{y} = \frac{1}{y}. \\ f_y &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{\frac{x}{y+h} - \frac{x}{y}}{h} = \lim_{h \rightarrow 0} \frac{xy - x(y+h)}{y(y+h)h} = \lim_{h \rightarrow 0} \frac{-x}{y(y+h)} = -\frac{x}{y^2}. \end{aligned}$$

15.3.14

$$\begin{aligned} f_x &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)y} - \sqrt{xy}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)y} - \sqrt{xy}}{h} \cdot \frac{\sqrt{(x+h)y} + \sqrt{xy}}{\sqrt{(x+h)y} + \sqrt{xy}} \\ &= \lim_{h \rightarrow 0} \frac{xy + hy - xy}{h(\sqrt{(x+h)y} + \sqrt{xy})} = \frac{y}{2\sqrt{xy}}. \end{aligned}$$

$$\begin{aligned} f_y &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x(y+h)} - \sqrt{xy}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x(y+h)} - \sqrt{xy}}{h} \cdot \frac{\sqrt{x(y+h)} + \sqrt{xy}}{\sqrt{x(y+h)} + \sqrt{xy}} \\ &= \lim_{h \rightarrow 0} \frac{xy + xh - xy}{h(\sqrt{x(y+h)} + \sqrt{xy})} = \frac{x}{2\sqrt{xy}}. \end{aligned}$$

15.3.15 $f_x(x, y) = e^y$, $f_y(x, y) = xe^y$.

15.3.16 $f_x(x, y) = 12x^2y^2 + 6xy^3 = 6xy^2(2x + y)$. $f_y(x, y) = 8x^3y + 9x^2y^2 = x^2y(8x + 9y)$.

15.3.17 $f_x(x, y) = 2xye^{x^2y}$, $f_y(x, y) = x^2e^{x^2y}$.

15.3.18 $f_x(x, y) = 5(3xy + 4y^2 + 1)^4(3y) = 15y(3xy + 4y^2 + 1)^4$. $f_y(x, y) = 5(3xy + 4y^2 + 1)^4(3x + 8y) = 5(3x + 8y)(3xy + 4y^2 + 1)^4$.

$$15.3.19 \quad f_w(w, z) = \frac{(w^2 + z^2) \cdot 1 - w \cdot 2w}{(w^2 + z^2)^2} = \frac{z^2 - w^2}{(w^2 + z^2)^2}, \quad f_z(w, z) = -w(w^2 + z^2)^{-2} \cdot 2z = -\frac{2wz}{(w^2 + z^2)^2}.$$

$$15.3.20 \quad f_s(s, t) = \frac{(s+t)(1) - (s-t)(1)}{(s+t)^2} = \frac{2t}{(s+t)^2}, \quad f_t(s, t) = \frac{(s+t)(-1) - (s-t)(1)}{(s+t)^2} = \frac{-2s}{(s+t)^2}.$$

15.3.21

$$f_x(x, y) = \cos(xy) + x(-\sin(xy))y = \cos(xy) - xy \sin(xy),$$

$$f_y(x, y) = x(-\sin(xy))x = -x^2 \sin(xy).$$

$$15.3.22 \quad f_x(x, y) = \frac{1}{(x^4/y^4) + 1} \cdot \frac{2x}{y^2} = \frac{2xy^2}{x^4 + y^4}, \quad f_y(x, y) = \frac{1}{(x^4/y^4) + 1} \cdot \frac{-2x^2}{y^3} = -\frac{2x^2y}{x^4 + y^4}.$$

$$15.3.23 \quad s_y(y, z) = z^2(\sec^2 yz)z = z^3 \sec^2 yz, \quad s_z(y, z) = 2z \tan yz + z^2(\sec^2 yz)y = 2z \tan yz + yz^2 \sec^2 yz.$$

$$15.3.24 \quad g_x(x, z) = 1 \cdot \ln(z^2 + x^2) + x \cdot \frac{2x}{z^2 + x^2} = \ln(z^2 + x^2) + \frac{2x^2}{z^2 + x^2}, \quad g_z(x, z) = x \cdot \frac{2z}{z^2 + x^2} = \frac{2xz}{z^2 + x^2}.$$

$$15.3.25 \quad G_s(s, t) = \frac{t}{2\sqrt{st}} \cdot \frac{1}{s+t} + \sqrt{st} \cdot -\frac{1}{(s+t)^2} = \frac{\sqrt{st}(s+t) - 2s\sqrt{st}}{2s(s+t)^2} = \frac{\sqrt{st}(t-s)}{2s(s+t)^2}, \quad G_t(s, t) = \frac{s}{2\sqrt{st}} \cdot \frac{1}{s+t} + \sqrt{st} \cdot -\frac{1}{(s+t)^2} = \frac{\sqrt{st}(s+t) - 2t\sqrt{st}}{2t(s+t)^2} = \frac{\sqrt{st}(s-t)}{2t(s+t)^2}.$$

15.3.26

$$F_p(p, q) = \frac{1}{2} (p^2 + pq + q^2)^{-1/2} (2p + q) = \frac{2p + q}{2\sqrt{p^2 + pq + q^2}},$$

$$F_q(p, q) = \frac{1}{2} (p^2 + pq + q^2)^{-1/2} (p + 2q) = \frac{p + 2q}{2\sqrt{p^2 + pq + q^2}}.$$

$$15.3.27 \quad f_x(x, y) = 2yx^{2y-1}, \quad f_y(x, y) = 2x^{2y} \ln x.$$

15.3.28

$$g_x(x, y) = 5 \cos^4(x^2 y^3) (-\sin(x^2 y^3)) (2xy^3) = -10xy^3 \sin(x^2 y^3) \cos^4(x^2 y^3)$$

$$g_y(x, y) = 5 \cos^4(x^2 y^3) (-\sin(x^2 y^3)) (3x^2 y^2) = -15x^2 y^2 \sin(x^2 y^3) \cos^4(x^2 y^3).$$

$$15.3.29 \quad h_x(x, y) = 1 - \frac{2x}{2\sqrt{x^2 - 4y}} = \frac{\sqrt{x^2 - 4y} - x}{\sqrt{x^2 - 4y}}, \quad h_y(x, y) = \frac{4}{2\sqrt{x^2 - 4y}} = \frac{2}{\sqrt{x^2 - 4y}}.$$

$$15.3.30 \quad h_u(u, v) = \frac{1}{2} \left(\frac{uv}{u-v} \right)^{-1/2} \left(\frac{(u-v)v - uv \cdot 1}{(u-v)^2} \right) = -\frac{1}{2} u^{-1/2} v^{3/2} (u-v)^{-3/2},$$

$$h_v(u, v) = \frac{1}{2} \left(\frac{uv}{u-v} \right)^{-1/2} \left(\frac{(u-v)u - uv \cdot (-1)}{(u-v)^2} \right) = \frac{1}{2} u^{3/2} v^{-1/2} (u-v)^{-3/2}$$

$$15.3.31 \quad \text{Using the Fundamental Theorem of Calculus, we have } f_x(x, y) = -e^{x^2}, \text{ and } f_y(x, y) = 3y^2 e^{y^6}.$$

$$15.3.32 \quad g_x(x, y) = \frac{y}{\sqrt{1-xy}} \cdot \frac{1}{2\sqrt{xy}} \cdot y = \frac{y^2}{2\sqrt{xy}\sqrt{1-xy}}, \quad g_y(x, y) = \sin^{-1} \sqrt{xy} + \frac{y}{\sqrt{1-xy}} \cdot \frac{1}{2\sqrt{xy}} \cdot x = \frac{xy + 2\sqrt{xy}\sqrt{1-xy} \sin^{-1} \sqrt{xy}}{2\sqrt{xy}\sqrt{1-xy}}.$$

$$15.3.33 \quad \text{We have } f_x(x, y) = -\frac{2x}{1 + (x^2 + y^2)^2} \text{ and } f_y(x, y) = -\frac{2y}{1 + (x^2 + y^2)^2}.$$

15.3.34 We have $f_x(x, y) = -\frac{ye^{-xy}}{1 + e^{-xy}}$ and $f_y(x, y) = -\frac{xe^{-xy}}{1 + e^{-xy}}$.

15.3.35 We have $h_x(x, y, z) = (1 + 2y)^x \ln(1 + 2y)$, $h_y(x, y, z) = x(1 + 2y)^{x-1}$, and $h_z(x, y, z) = (1 + x + 2y)^z \ln(1 + x + 2y)$.

15.3.36 We have $f_x(x, y) = f_y(x, y) = 2 \sin(2(x + y)) - 2 \cos(x + y) \sin(x + y)$.

15.3.37 By the Fundamental Theorem of Calculus, $f_x(x, y) = -\frac{\partial}{\partial x} \int_x^y h(s) ds = -h(x)$ and similarly $f_y(x, y) = h(y)$.

15.3.38 We have $f_x(x, y) = 2x \sin y$, $f_y(x, y) = x^2 \cos y$; therefore $f_{xx}(x, y) = 2 \sin y$, $f_{yy}(x, y) = -x^2 \sin y$, $f_{xy}(x, y) = f_{yx}(y, x) = 2x \cos y$.

15.3.39 We have $h_x(x, y) = 3x^2 + y^2$, $h_y(x, y) = 2xy$; therefore $h_{xx}(x, y) = 6x$, $h_{yy}(x, y) = 2x$, $h_{xy}(x, y) = h_{yx}(y, x) = 2y$.

15.3.40 We have $f_x(x, y) = 10x^4y^2 + 2xy$, $f_y(x, y) = 4x^5y + x^2$; therefore $f_{xx}(x, y) = 40x^3y^2 + 2y$, $f_{yy}(x, y) = 4x^5$, $f_{xy}(x, y) = f_{yx}(y, x) = 20x^4y + 2x$.

15.3.41 We have $f_x(x, y) = 4y^3 \cos 4x$, $f_y(x, y) = 3y^2 \sin 4x$; therefore $f_{xx}(x, y) = -16y^3 \sin 4x$, $f_{yy}(x, y) = 6y \sin 4x$, $f_{xy}(x, y) = f_{yx}(y, x) = 12y^2 \cos 4x$.

15.3.42

$$f_x(x, y) = 2 \sin(x^3y) \cos(x^3y)(3x^2y) = 6x^2y \sin(x^3y) \cos(x^3y) = 3x^2y \sin(2x^3y),$$

while

$$f_y(x, y) = 2 \sin(x^3y) \cos(x^3y)(x^3) = 2x^3 \sin(x^3y) \cos(x^3y) = x^3 \sin(2x^3y).$$

Then,

$$f_{xx}(x, y) = 6xy \sin(2x^3y) + 3x^2y \cos(2x^3y)(6x^2y) = 6xy(\sin(2x^3y) + 3x^3y \cos(2x^3y)),$$

and

$$f_{xy}(x, y) = 3x^2 \sin(2x^3y) + 3x^2y \cos(2x^3y)(2x^3) = 3x^2(\sin(2x^3y) + 2x^3y \cos(2x^3y)),$$

and

$$f_{yx}(x, y) = 3x^2 \sin(2x^3y) + x^3 \cos(2x^3y)(6x^2y) = 3x^2(\sin(2x^3y) + 2x^3y \cos(2x^3y)),$$

and

$$f_{yy}(x, y) = x^3 \cos(2x^3y)(2x^3) = 2x^6 \cos(2x^3y).$$

15.3.43 We have

$$p_u(u, v) = \frac{2u}{u^2 + v^2 + 4},$$

$$p_v(u, v) = \frac{2v}{u^2 + v^2 + 4}.$$

Therefore

$$p_{uu}(u, v) = \frac{(u^2 + v^2 + 4) \cdot 2 - 2u \cdot 2u}{(u^2 + v^2 + 4)^2} = \frac{-2u^2 + 2v^2 + 8}{(u^2 + v^2 + 4)^2},$$

$$p_{vv}(u, v) = \frac{(u^2 + v^2 + 4) \cdot 2 - 2v \cdot 2v}{(u^2 + v^2 + 4)^2} = \frac{2u^2 - 2v^2 + 8}{(u^2 + v^2 + 4)^2},$$

and

$$p_{uv}(u, v) = p_{vu}(u, v) = -2u(u^2 + v^2 + 4)^{-2} \cdot 2v = -\frac{4uv}{(u^2 + v^2 + 4)^2}.$$

15.3.44

$$Q_r(r, s) = \frac{3r^2 s e^{r^3 s}}{s} = 3r^2 e^{r^3 s},$$

while

$$Q_s(r, s) = \frac{s e^{r^3 s} r^3 - e^{r^3 s}}{s^2} = \frac{e^{r^3 s} (r^3 s - 1)}{s^2}.$$

So

$$Q_{rr}(r, s) = 6r e^{r^3 s} + 3r^2 e^{r^3 s} (3r^2 s) = 3r e^{r^3 s} (2 + 3r^3 s),$$

while

$$Q_{rs}(r, s) = Q_{sr}(r, s) = 3r^2 e^{r^3 s} (r^3) = 3r^5 e^{r^3 s},$$

and

$$Q_{ss}(r, s) = \frac{s^2 (r^3 e^{r^3 s} (r^3 s - 1) + e^{r^3 s} (r^3)) - e^{r^3 s} (r^3 s - 1)(2s)}{s^4} = \frac{e^{r^3 s} (r^6 s^2 - 2r^3 s + 2)}{s^3}.$$

15.3.45 We have $F_r(r, s) = e^s$, $F_s(r, s) = r e^s$; therefore $F_{rr}(r, s) = 0$, $F_{ss}(r, s) = r e^s$, $F_{rs}(r, s) = F_{sr}(r, s) = e^s$.

15.3.46 We have

$$H_x(x, y) = \frac{x}{\sqrt{4 + x^2 + y^2}},$$

$$H_y(x, y) = \frac{y}{\sqrt{4 + x^2 + y^2}}.$$

Therefore

$$H_{xx}(x, y) = \frac{\left(\sqrt{4 + x^2 + y^2}\right) \cdot 1 - x \frac{x}{\sqrt{4 + x^2 + y^2}}}{4 + x^2 + y^2} = \frac{4 + y^2}{(4 + x^2 + y^2)^{3/2}},$$

and

$$H_{yy}(x, y) = \frac{\left(\sqrt{4 + x^2 + y^2}\right) \cdot 1 - y \frac{y}{\sqrt{4 + x^2 + y^2}}}{4 + x^2 + y^2} = \frac{4 + x^2}{(4 + x^2 + y^2)^{3/2}},$$

and

$$H_{xy}(x, y) = H_{yx}(x, y) = -\frac{1}{2} x (4 + x^2 + y^2)^{-3/2} \cdot 2y = -\frac{xy}{(4 + x^2 + y^2)^{3/2}}.$$

15.3.47

$$f_x(x, y) = \frac{1}{1 + x^6 y^4} \cdot (3x^2 y^2) = \frac{3x^2 y^2}{1 + x^6 y^4},$$

and

$$f_y(x, y) = \frac{1}{1 + x^6 y^4} \cdot (2x^3 y) = \frac{2x^3 y}{1 + x^6 y^4}.$$

Then

$$f_{xx}(x, y) = \frac{(1 + x^6 y^4)(6xy^2) - (3x^2 y^2)(6x^5 y^4)}{(1 + x^6 y^4)^2} = \frac{6xy^2(1 - 2x^6 y^4)}{(1 + x^6 y^4)^2},$$

and

$$f_{xy}(x, y) = f_{yx}(x, y) = \frac{(1 + x^6 y^4)(6x^2 y) - (3x^2 y^2)(4x^6 y^3)}{(1 + x^6 y^4)^2} = \frac{6x^2 y(1 - x^6 y^4)}{(1 + x^6 y^4)^2},$$

and

$$f_{yy}(x, y) = \frac{(1 + x^6 y^4)(2x^3) - (2x^3 y)(4x^6 y^3)}{(1 + x^6 y^4)^2} = \frac{2x^3(1 - 3x^6 y^4)}{(1 + x^6 y^4)^2}.$$

15.3.48 Observe that $f_x(x, y) = 6xy^{-1} + 2x^{-2}y^2$, so $f_{xy}(x, y) = -6xy^{-2} + 4x^{-2}y$; and $f_y(x, y) = -3x^2y^{-2} - 4x^{-1}y$, so $f_{yx}(x, y) = -6xy^{-2} + 4x^{-2}y$.

15.3.49 Observe that $f_x(x, y) = e^{x+y}$, so $f_{xy}(x, y) = e^{x+y}$; and $f_y(x, y) = e^{x+y}$, so $f_{yx}(x, y) = e^{x+y}$.

15.3.50 Observe that $f_x(x, y) = \frac{1}{2}x^{-1/2}y^{1/2}$, so $f_{xy}(x, y) = \frac{1}{4}x^{-1/2}y^{-1/2} = \frac{1}{\sqrt{4xy}}$; and $f_y(x, y) = \frac{1}{2}x^{1/2}y^{-1/2}$, so $f_{yx}(x, y) = \frac{1}{4}x^{-1/2}y^{-1/2} = \frac{1}{\sqrt{4xy}}$.

15.3.51 Observe that $f_x(x, y) = -y \sin xy$, so $f_{xy}(x, y) = -\sin xy - xy \cos xy$; and $f_y(x, y) = -x \sin xy$, so $f_{yx}(x, y) = -\sin xy - xy \cos xy$.

15.3.52 Note that $f_x(x, y) = e^{\sin xy} y \cos xy$, so

$$f_{xy} = e^{\sin xy} x \cos xy (y \cos xy) + e^{\sin xy} (\cos xy - yx \sin xy) = e^{\sin xy} (xy \cos^2 xy + \cos xy - yx \sin xy).$$

Also $f_y(x, y) = e^{\sin xy} (x \cos xy)$. So

$$f_{yx}(x, y) = e^{\sin xy} y \cos xy (x \cos xy) + e^{\sin xy} (\cos xy - yx \sin xy) = e^{\sin xy} (xy \cos^2 xy + \cos xy - yx \sin xy).$$

15.3.53 Note that $f_x(x, y) = 4(2x - y^3)^3(2) = 8(2x - y^3)^3$, so $f_{xy} = 24(2x - y^3)^2(-3y^2) = -72y^2(2x - y^3)^2$. Also, $f_y(x, y) = 4(2x - y^3)^3(-3y^2) = -12y^2(2x - y^3)^3$. Then $f_{yx}(x, y) = -12y^2(3(2x - y^3)^2(2)) = -72y^2(2x - y^3)^2$.

15.3.54 $G_r(r, s, t) = \frac{s+t}{2\sqrt{rs+rt+st}}; G_s(r, s, t) = \frac{r+t}{2\sqrt{rs+rt+st}}; G_t(r, s, t) = \frac{r+s}{2\sqrt{rs+rt+st}}.$

15.3.55 $h_x(x, y, z) = h_y(x, y, z) = h_z(x, y, z) = -\sin(x + y + z).$

15.3.56 $g_x(x, y, z) = 4xy - 3z^4; f_y(x, y, z) = 2x^2 + 20yz^2; f_z(x, y, z) = -12xz^3 + 20y^2z.$

15.3.57 $F_u(u, v, w) = \frac{1}{v+w}; F_v(u, v, w) = -\frac{u}{(v+w)^2}; F_w(u, v, w) = -\frac{u}{(v+w)^2}.$

15.3.58 $Q_x(x, y, z) = yz \sec^2 xyz; Q_y(x, y, z) = xz \sec^2 xyz; Q_z(x, y, z) = xy \sec^2 xyz.$

15.3.59 $G_r(r, s, t) = \frac{s^3 t^5}{2\sqrt{rs^3 t^5}}; G_s(r, s, t) = \frac{3rs^2 t^5}{2\sqrt{rs^3 t^5}}; G_t(r, s, t) = \frac{5rs^3 t^4}{2\sqrt{rs^3 t^5}}.$

15.3.60 $g_w(w, x, y, z) = g_x(w, x, y, z) = -\sin(w+x)\sin(y-z); g_y(w, x, y, z) = \cos(w+x)\cos(y-z); g_z(w, x, y, z) = -\cos(w+x)\cos(y-z).$

15.3.61 $h_w(w, x, y, z) = \frac{z}{xy}; h_x(w, x, y, z) = -\frac{wz}{x^2 y}; h_y(w, x, y, z) = -\frac{wz}{xy^2}; h_z(w, x, y, z) = \frac{w}{xy}.$

15.3.62 $F_w(w, x, y, z) = \sqrt{x+2y+3z}; F_x(w, x, y, z) = \frac{w}{2\sqrt{x+2y+3z}}; F_y(w, x, y, z) = \frac{w}{\sqrt{x+2y+3z}}; F_z(w, x, y, z) = \frac{3w}{2\sqrt{x+2y+3z}}.$

15.3.63

a. $R'(t) = \frac{(ct+d)a - (at+b)c}{(ct+d)^2} = \frac{ad-bc}{(ct+d)^2}.$

b. $g_x(x, y, z) = -\frac{8z}{3(2x-y+z)^2}; g_y(x, y, z) = \frac{4z}{3(2x-y+z)^2}; g_z(x, y, z) = \frac{4(2x-y)}{3(2x-y+x)^2}.$

$$15.3.64 \quad f_x(2, 3) \approx \frac{f(2.1, 3) - f(2, 3)}{0.1} = \frac{4.347 - 4.243}{0.1} = 1.04.$$

$$15.3.65 \quad f_y(2, 3) \approx \frac{f(2, 3.1) - f(2, 3)}{0.1} = \frac{4.384 - 4.243}{0.1} = 1.41.$$

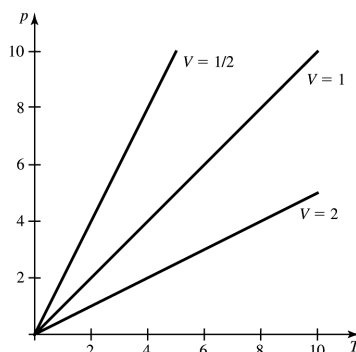
$$15.3.66 \quad f_x(2.2, 3.4) \approx \frac{f(2.3, 3.4) - f(2.2, 3.4)}{0.1} = \frac{5.156 - 5.043}{0.1} = 1.13. \text{ Answers may vary.}$$

$$15.3.67 \quad f_y(2.4, 3.3) \approx \frac{f(2.4, 3.4) - f(2.4, 3.3)}{0.1} = \frac{5.267 - 5.112}{0.1} = 1.55. \text{ Answers may vary.}$$

$$15.3.68 \quad f_x(0.42, 0.5) \approx \frac{1.75 - 1.5}{0.5 - 0.42} \approx 3.125; \quad f_y(0.42, 0.5) \approx \frac{1.75 - 1.5}{0.7 - 0.5} \approx 1.25. \text{ Answers may vary.}$$

15.3.69

- We have $V = \frac{kT}{P}$, so $\frac{\partial V}{\partial P} = -\frac{kT}{P^2}$. Because this partial derivative is negative, the volume decreases as the pressure increases at a fixed temperature.
- We have $\frac{\partial V}{\partial T} = \frac{k}{P}$. Because this partial derivative is positive, the volume increases as the temperature increases at a fixed pressure.
-



15.3.70

- We have $B_w = \frac{1}{h^2}$.
- The partial derivative $B_w > 0$ so B is an increasing function of w .
- We have $B_h = -\frac{2w}{h^3}$.
- The partial derivative $B_h < 0$ so B is a decreasing function of h .

15.3.71

- Solving for R gives $R = (R_1^{-1} + R_2^{-1})^{-1}$, so $\frac{\partial R}{\partial R_1} = -(R_1^{-1} + R_2^{-1})^{-2}(-R_1^{-2}) = \frac{R_2^2}{(R_1 + R_2)^2}$ and similarly $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2}$.
- We have $-R^{-2} \frac{\partial R}{\partial R_1} = -R_1^{-2} \implies \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}$ and similarly $\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2}$.
- Because $\frac{\partial R}{\partial R_1} > 0$, an increase in R_1 causes an increase in R .
- Because $\frac{\partial R}{\partial R_2} > 0$, a decrease in R_2 causes a decrease in R .

15.3.72

- We have $V_h = \frac{\pi}{3} (3r \cdot 2h - 3h^2) = \pi (2rh - h^2)$ and $V_r = \pi h^2$.
- Because $V_r = \pi h^2$, V_r is greater when $h = 0.8r$.
- Solve $V_r = \pi h^2 = 1$ to obtain $h = \pi^{-1/2}$.
- The maximum value of $2rh - h^2$ as a function of h occurs when $2r - 2h = 0$, which gives $h = r$.

15.3.73 We see that $u_t = -16e^{-4t} \cos 2x = u_{xx}$.

15.3.74 We see that $u_t = -10e^{-t} \sin x = u_{xx}$.

15.3.75 We see that $u_t = -a^2 A e^{-a^2 t} \cos ax = u_{xx}$.

15.3.76 We see that $u_t = -e^{-t} (2 \sin x + 3 \cos x) = u_{xx}$.

15.3.77

- Observe that as $f(x, y) = 0$ along either coordinate axis but on the line $y = x$, $f(x, y) = -\frac{x^2}{2x^2} = -\frac{1}{2}$, so $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist, and hence f is not continuous at $(0, 0)$.
- By Theorem 15.6, f is not differentiable at $(0, 0)$.
- Because f is identically 0 on the coordinate axes, $f_x(0, 0) = f_y(0, 0) = 0$.
- We have $f_x(x, y) = -\left(\frac{(x^2 + y^2)y - xy \cdot 2x}{(x^2 + y^2)^2}\right) = \frac{(x^2 - y^2)y}{(x^2 + y^2)^2}$. Along the line $x = 2y$, $f_x(x, y) = \frac{3y^3}{25y^4} = \frac{3}{25} \cdot \frac{1}{y}$, which does not converge to 0 as $y \rightarrow 0$. Hence f_x is not continuous at $(0, 0)$. A similar argument shows that f_y is also not continuous at $(0, 0)$.
- Theorem 15.5 does not apply because the partials f_x and f_y are not continuous at $(0, 0)$, and Theorem 15.6 does not apply because f is not differentiable at $(0, 0)$.

15.3.78

- Observe that as $f(x, y) = 0$ along either coordinate axis but on the curve $x = y^2$, $f(x, y) = \frac{2y^4}{2y^4} = 1$, so $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist, and hence f is not continuous at $(0, 0)$.
- By Theorem 15.6, f is not differentiable at $(0, 0)$.
- Because f is identically 0 on the coordinate axes, $f_x(0, 0) = f_y(0, 0) = 0$.
- We have $f_x(x, y) = \frac{(x^2 + y^4)2y^2 - 2xy^2(2x)}{(x^2 + y^4)^2} = \frac{2y^2(y^4 - x^2)}{(x^2 + y^4)^2}$. Along the curve $x = 2y^2$, $f_x(x, y) = -\frac{6y^6}{25y^8} = -\frac{6}{25} \cdot \frac{1}{y^2}$, which does not converge to 0 as $y \rightarrow 0$. Hence f_x is not continuous at $(0, 0)$.
We also have $f_y(x, y) = \frac{(x^2 + y^4)4xy - 2xy^2(4y^3)}{(x^2 + y^4)^2} = \frac{4xy(x^2 - y^4)}{(x^2 + y^4)^2}$. Along the curve $x = 2y^2$, $f_y(x, y) = -\frac{24y^7}{25y^8} = -\frac{24}{25} \cdot \frac{1}{y}$, which does not converge to 0 as $y \rightarrow 0$. Hence f_y is not continuous at $(0, 0)$.
- Theorem 15.5 does not apply because the partials f_x and f_y are not continuous at $(0, 0)$, and Theorem 15.6 does not apply because f is not differentiable at $(0, 0)$.

15.3.79

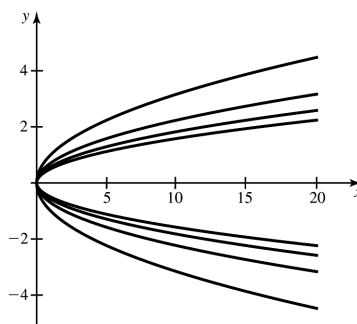
- a. False. $\frac{\partial}{\partial x} y^{10} = 0$ because x and y are independent variables.
- b. False. $\frac{\partial^2}{\partial x \partial y} (xy)^{1/2} = \frac{1}{2} \cdot x^{-1/2} \cdot \frac{1}{2} y^{-1/2} = \frac{1}{4\sqrt{xy}}$.
- c. True. If f has continuous partial derivatives of all orders, then the order of differentiation for mixed partials can be exchanged.

15.3.80

- a. There are 3 choices for each of the variables we differentiate with respect to, so there are 9 possible second partial derivatives: $w_{xx}, w_{yy}, w_{zz}, w_{xy}, w_{yx}, w_{xz}, w_{zx}, w_{yz}, w_{zy}$.
- b. This function has continuous partial derivatives of all orders, so $w_{xy} = w_{yx}, w_{xz} = w_{zx}$ and $w_{yz} = w_{zy}$.
- c. There are 4 choices for each of the variables we differentiate with respect to, so there are 16 possible second partial derivatives.

15.3.81

- a. We have $z_x = \frac{1}{y^2}$ and $z_y = -\frac{2x}{y^3}$.
- b.



- c. We observe that z increases at the same rate as x , which makes sense because $z_x = 1$ along this line.
- d. We observe that z increases when $y < 0$, is undefined when $y = 0$ and decreases when $y > 0$, which is consistent with $z_y = -\frac{2}{y^3}$ along this line.

15.3.82

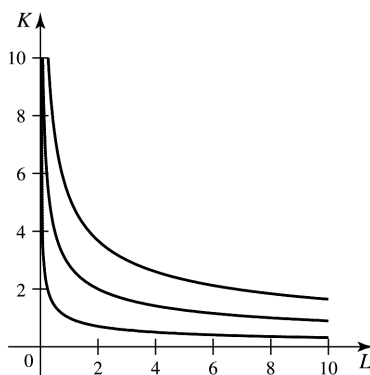
- a. $V_x = 2xh$, $V_h = x^2$.
- b. $\Delta V \approx 2xh\Delta x = 2 \cdot 0.5 \cdot 1.5 \cdot 0.01 = 0.015 \text{ m}^3$.
- c. $\Delta V = x^2\Delta h = (0.5)^2(-0.01) = -0.0025 \text{ m}^3$ (notice that because V is a linear function of h , the linear approximation is exact).
- d. Notice that for fixed height h , $\frac{\Delta V}{V} \approx \frac{2xh\Delta x}{x^2h} = 2\frac{\Delta x}{x}$. Therefore a 10% change in x will produce (approximately) a 20% change in V .
- e. Notice that for fixed base x , $\frac{\Delta V}{V} = \frac{\Delta h}{h}$, so a 10% change in h will produce a 10% change in V .

15.3.83

- a. We have $\varphi_x(x, y) = -\frac{2x}{(x^2 + (y-1)^2)^{3/2}} - \frac{x}{(x^2 + (y+1)^2)^{3/2}}$ and $\varphi_y(x, y) = -\frac{2(y-1)}{(x^2 + (y-1)^2)^{3/2}} - \frac{y+1}{(x^2 + (y+1)^2)^{3/2}}$.
- b. Observe that $|\varphi_x(x, y)| \leq \frac{2|x|}{|x|^{3/2}} + \frac{|x|}{|x|^{3/2}} = \frac{3}{|x|^{1/2}}$ and similarly $|\varphi_y(x, y)| \leq \frac{2|y-1|}{|y-1|^{3/2}} + \frac{|y+1|}{|y+1|^{3/2}} = \frac{2}{|y-1|^{1/2}} + \frac{1}{|y+1|^{1/2}}$, which both converge to 0 as $x, y \rightarrow \infty$.
- c. We see that $\varphi_x(0, y) = 0$ as long as $y \neq \pm 1$. This is consistent with the observation that along horizontal lines $y = y_0$ the potential function takes its maximum at $x = 0$.
- d. We see that $\varphi_y(x, 0) = \frac{1}{(x^2 + 1)^{3/2}}$. This implies that if we cross the x -axis at any point from below to above, the potential function is increasing.

15.3.84

- a. We have $Q_L = \frac{1}{3}L^{-2/3}K^{2/3}$ and $Q_K = \frac{2}{3}L^{1/3}K^{-1/3}$.
- b. We have $\Delta Q \approx Q_K(10, 20) \Delta K = \frac{2}{3}(10)^{1/3}(20)^{-1/3} \cdot 0.5 \approx 0.2646$.
- c. We have $\Delta Q \approx Q_L(10, 20) \Delta L = \frac{1}{3}(10)^{-2/3}(20)^{2/3} \cdot (-0.5) \approx -0.2646$.
- d.



- e. As we move along the vertical line $L = 2$ in the positive K -direction, Q increases, which is consistent with $Q_K > 0$.
- f. As we move along the horizontal line $K = 2$ in the positive L -direction, Q increases, which is consistent with $Q_L > 0$.

15.3.85 Observe that $f_x(x, y) = \frac{(cx + dy)a - (ax + by)c}{(cx + dy)^2} = \frac{y(ad - bc)}{(cx + dy)^2} = 0$; similarly, $f_y(x, y) = 0$.

Suppose $a \neq 0$ (other cases can be handled similarly). Then $ad = bc$ implies $d = \frac{bc}{a}$; this also shows that $c \neq 0$, otherwise both c and $d = 0$ and the function is undefined. Hence $ax + by = \frac{a}{c} \left(cx + \frac{bc}{a}y \right) = \frac{a}{c}(cx + dy)$, so $f(x, y) = \frac{a}{c}$ is a constant function which implies $f_x = f_y = 0$.

15.3.86

- a. The period T is found by solving $\frac{\pi t}{2} = 2\pi$, so $t = 4$.
- b. We have $u_t = 2 \sin(\pi x) \cos\left(\frac{\pi t}{2}\right) \cdot \frac{\pi}{2} = \pi \sin(\pi x) \cos\left(\frac{\pi t}{2}\right)$.
- c. For fixed t , the quantity u_t is largest when $\sin(\pi x) = \pm 1$, which occurs at $x = \frac{1}{2}$ (the middle of the string).
- d. For fixed x , the quantity u_t is largest when $\cos\left(\frac{\pi t}{2}\right) = \pm 1$, which occurs at $t = 2k$ for any integer k . At these times the string is in its rest position (no displacement).
- e. We have $u_x = 2\pi \cos(\pi x) \cos\left(\frac{\pi t}{2}\right)$.
- f. For fixed t , the slope is greatest when $\cos(\pi x) = \pm 1$, which occurs at the endpoints of the string ($x = 0, 1$).

15.3.87 Observe that $\frac{\partial^2 u}{\partial t^2} = -4c^2 \cos(2(x + ct)) = c^2 \frac{\partial^2 u}{\partial x^2}$.

15.3.88 Observe that $\frac{\partial^2 u}{\partial t^2} = -20c^2 \cos(2(x + ct)) - 3c^2 \sin(x - ct) = c^2 \frac{\partial^2 u}{\partial x^2}$.

15.3.89 Observe that $\frac{\partial^2 u}{\partial t^2} = Ac^2 f''(x + ct) + Bc^2 g''(x - ct) = c^2 \frac{\partial^2 u}{\partial x^2}$.

15.3.90 Observe that $u_{xx} + u_{yy} = e^{-x} \sin y + e^{-x} (-\sin y) = 0$.

15.3.91 Observe that $u_{xx} + u_{yy} = 6x - 6x = 0$.

15.3.92 Observe that $u_{xx} + u_{yy} = a^2 e^{ax} \cos ay + e^{ax} (-a^2 \cos ay) = 0$

15.3.93 Observe that

$$u_{xx} = \frac{2(x-1)y}{[(x-1)^2 + y^2]^2} - \frac{2(x+1)y}{[(x+1)^2 + y^2]^2},$$

and

$$u_{yy} = -\frac{2(x-1)y}{[(x-1)^2 + y^2]^2} + \frac{2(x+1)y}{[(x+1)^2 + y^2]^2};$$

so

$$u_{xx} + u_{yy} = 0.$$

15.3.94 We have $f(0, 0) = 0$, $f_x(0, 0) = f_y(0, 0) = 1$, and $f(\Delta x, \Delta y) = 1 \cdot \Delta x + 1 \cdot \Delta y$, so we can take $\epsilon_1 = \epsilon_2 = 0$.

15.3.95 We have $f(0, 0) = 0$, $f_x(0, 0) = f_y(0, 0) = 0$, and $f(\Delta x, \Delta y) = \Delta x \cdot \Delta y$, so we can take $\epsilon_1 = \Delta y$, $\epsilon_2 = 0$ or $\epsilon_1 = 0$, $\epsilon_2 = \Delta x$.

15.3.96

- a. Observe that $\lim_{(x,y) \rightarrow (0,0)} (1 - |xy|) = 1 = f(0, 0)$, so f is continuous at $(0, 0)$.
- b. Let $(a, b) = (0, 0)$; then $f(a + \Delta x, b + \Delta y) - f(a, b) = -|\Delta x||\Delta y| = \epsilon_1 \Delta x$ where $\epsilon_1 = \pm \Delta y$ (depending on the sign of x). Because $\epsilon_1 \rightarrow 0$ as $\Delta y \rightarrow 0$, we see that f is differentiable at $(0, 0)$.

- c. Because f is identically equal to 1 on the coordinate axes, $f_x(0,0) = f_y(0,0) = 0$.
- d. Because $f(x,y) = 1 - |x||y|$,

$$f_x(x,y) = \begin{cases} |y| & \text{if } x < 0, \\ -|y| & \text{if } x > 0. \end{cases}$$

So $|f_x(x,y)| = |y|$ if $x \neq 0$. If $y \neq 0$, $f_x(0,y)$ is undefined because $\frac{d}{dx}|x|$ is undefined at $x = 0$. By part c), $f_x(0,0)$ is defined and equals 0. So the domain of f_x is all of $\mathbb{R}^2$ except points (x,y) where $x = 0$ and $y \neq 0$.

Let $\epsilon > 0$ be given. Assume (x,y) is in the domain of f_x and $\sqrt{(x-0)^2 + (y-0)^2} < \delta$, where $\delta = \epsilon$. Then $x \neq 0$, $y \neq 0$ and $|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} < \epsilon$. This implies that

$$|f_x(x,y) - f_x(0,0)| = |f_x(x,y)| = |y| < \epsilon.$$

So $\lim_{(x,y) \rightarrow (0,0)} f_x(x,y) = f_x(0,0) = 0$. Therefore f_x is continuous at $(0,0)$. A similar argument shows that f_y is continuous at $(0,0)$.

- e. Theorem 15.5 does not apply because we have shown that f_x and f_y are undefined at points in any open set containing $(0,0)$. Theorem 15.6 holds by parts c) and d). This exercise shows that the converse of Theorem 15.5 doesn't hold in general.

15.3.97

- a. Observe that $\lim_{(x,y) \rightarrow (0,0)} \sqrt{|xy|} = 0 = f(0,0)$, so f is continuous at $(0,0)$.
- b. Let $(a,b) = (0,0)$, and suppose that $f(a + \Delta x, b + \Delta y) - f(a,b) = \sqrt{|\Delta x \Delta y|} = \epsilon_1 \Delta x + \epsilon_2 \Delta y$. Let $\Delta x = \Delta y$; then we obtain $\sqrt{|\Delta x|^2} = |\Delta x| = (\epsilon_1 + \epsilon_2) \Delta x$ which implies that $\epsilon_1 + \epsilon_2 = \pm 1$, and so we cannot have $\epsilon_1, \epsilon_2 \rightarrow 0$, as $\Delta x, \Delta y \rightarrow 0$. Therefore f is not differentiable at $(0,0)$.
- c. Because f is identically equal to 0 on the coordinate axes, $f_x(0,0) = f_y(0,0) = 0$.
- d. Because $f_x(x,y) = 1 - \sqrt{|x|}\sqrt{|y|}$,

$$f_x(x,y) = \begin{cases} -\frac{\sqrt{|y|}}{2\sqrt{|x|}} & \text{if } x < 0 \\ \frac{\sqrt{|y|}}{2\sqrt{|x|}} & \text{if } x > 0. \end{cases}$$

Observe that along $y = x$, $x > 0$, $\lim_{(x,y) \rightarrow (0,0)} f_x(x,y) = \frac{1}{2}$, and along $y = x$, $x < 0$, $\lim_{(x,y) \rightarrow (0,0)} f_x(x,y) = -\frac{1}{2}$. So by the Two Path test, $\lim_{(x,y) \rightarrow (0,0)} f_x(x,y)$ doesn't exist. Therefore f_x is not continuous at $(0,0)$.

- e. Theorem 15.5 does not apply because the partials f_x and f_y are not defined at $(0,0)$. Theorem 15.6 does not apply because f is not differentiable at $(0,0)$. This exercise shows that the converse of Theorem 15.6 isn't true in general.

15.3.98

- a. Observe that $u_x = 2x = v_y$ and $u_y = -2y = -v_x$.
- b. Observe that $u_x = 3x^2 - 3y^2 = v_y$ and $u_y = -6xy = -v_x$.
- c. We have $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$, so $u_{xx} + u_{yy} = 0$. The proof that $v_{xx} + v_{yy} = 0$ is similar.

15.3.99 Let $H(s)$ be an antiderivative of $h(s)$; then $f(x,y) = H(xy) - H(1)$, so $f_x(x,y) = y h(xy)$, $f_y(x,y) = x h(xy)$.

15.4 The Chain Rule

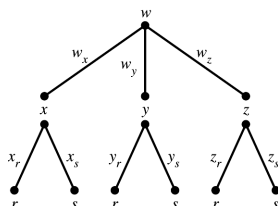
15.4.1 There is one dependent variable (z), two intermediate variables (x and y) and one independent variable (t).

15.4.2 Multiply each of the partial derivatives of z by the t -derivative of the corresponding function, and add all these expressions.

15.4.3 Multiply each of the partial derivatives of w by the t -derivative of the corresponding function, and add all these expressions.

15.4.4 Multiply each of the partial derivatives of z by the t -partial derivative of the corresponding function, and add all these expressions.

15.4.5



15.4.6 Use Theorem 15.9: $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

15.4.7 We have $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2x(2t) + 3y^2(1) = 4t^3 + 3t^2$.

To check, we write $z = x^2 + y^3 = t^4 + t^3$, so $\frac{dz}{dt} = 4t^3 + 3t^2$.

15.4.8 We have $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = y^2(2t) + 2xy(1) = 2t^3 + 2t^3 = 4t^3$.

To check, we write $z = t^2 t^2 = t^4$, so $\frac{dz}{dt} = 4t^3$.

15.4.9 We have $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (\sin y) 2t + (x \cos y) 12t^2 = 2t \sin(4t^3) + 12t^4 \cos(4t^3)$.

15.4.10 We have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (2xy - y^3) 2t + (x^2 - 3xy^2) (-2t^{-3}) = 2t(2 - t^{-6}) - 2t^{-3}(t^4 - 3t^{-2}) = 2t + 4t^{-5}.$$

15.4.11 We have $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = (-2 \sin 2x \sin 3y) \left(\frac{1}{2}\right) + (3 \cos 2x \cos 3y) 4t^3 = -\sin t \sin 3t^4 + 12t^3 \cos t \cos 3t^4$.

15.4.12 We have

$$\frac{dz}{dt} = \frac{\partial z}{\partial r} \frac{dr}{dt} + \frac{\partial z}{\partial s} \frac{ds}{dt} = \frac{r}{\sqrt{r^2 + s^2}} (-2 \sin 2t) + \frac{s}{\sqrt{r^2 + s^2}} (2 \cos 2t) = \frac{-2 \cos 2t \sin 2t + 2 \sin 2t \cos 2t}{1} = 0.$$

15.4.13

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= 10(x + 2y)^9 2 \sin t \cos t + 10(x + 2y)^9 \cdot 2 \cdot 5(3t + 4)^4 (3) \\ &= 20(\sin^2 t + 2(3t + 4)^5)^9 (\sin t \cos t + 15(3t + 4)^4). \end{aligned}$$

15.4.14

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\
&= \frac{20x^{19}}{y^{10}} \cdot \frac{1}{1+t^2} + \frac{-10x^{20}}{y^{11}} \cdot \frac{2t}{t^2+1} \\
&= \frac{20x^{19}y - 20x^{20}t}{y^{11}(t^2+1)} = \frac{20x^{19}(y-tx)}{y^{11}(t^2+1)} \\
&= \frac{20(\tan^{-1}t)^{19}(\ln(t^2+1) - t \tan^{-1}t)}{(t^2+1)\ln^{11}(t^2+1)}
\end{aligned}$$

15.4.15 We have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\
&= (y \sin z) 2t + (x \sin z) 12t^2 + (xy \cos z) \cdot 1 \\
&= (2ty + 12t^2x) \sin z + xy \cos z \\
&= 20t^4 \sin(t+1) + 4t^5 \cos(t+1).
\end{aligned}$$

15.4.16 We have

$$\begin{aligned}
\frac{dQ}{dt} &= \frac{\partial Q}{\partial x} \frac{dx}{dt} + \frac{\partial Q}{\partial y} \frac{dy}{dt} + \frac{\partial Q}{\partial z} \frac{dz}{dt} \\
&= \frac{x}{\sqrt{x^2+y^2+z^2}} \cos t + \frac{y}{\sqrt{x^2+y^2+z^2}} (-\sin t) + \frac{z}{\sqrt{x^2+y^2+z^2}} (-\sin t) \\
&= -\frac{\cos t \sin t}{\sqrt{1+\cos^2 t}}.
\end{aligned}$$

15.4.17

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} + \frac{\partial V}{\partial z} \frac{dz}{dt} \\
&= yze^t + 2xz + xy \cos t = e^t(2t+3) \sin t + 2e^t \sin t + e^t(2t+3) \cos t \\
&= e^t((2t+5) \sin t + (2t+3) \cos t).
\end{aligned}$$

15.4.18

$$\begin{aligned}
\frac{dU}{dt} &= \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt} + \frac{\partial U}{\partial z} \frac{dz}{dt} \\
&= \frac{y^2}{z^8} e^t + \frac{2xy}{z^8} 3 \cos 3t - \frac{8xy^2}{z^9} \cdot 4 \\
&= \frac{xy(yz + 6z \cos 3t - 32y)}{z^9} \\
&= \frac{e^t \sin 3t((4t+1) \sin 3t + 6(4t+1) \cos 3t - 32 \sin 3t)}{z^9} \\
&= \frac{e^t \sin 3t(4t \sin 3t - 31 \sin 3t + 24t \cos 3t + 6 \cos 3t)}{(4t+1)^9}.
\end{aligned}$$

15.4.19 $z_s = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 2x \sin y + x^2 \cos y \cdot 0 = 2(s-t) \sin t^2$ and $z_t = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (2x \sin y)(-1) + x^2 \cos y(2t) = 2(t-s) \sin t^2 + 2t(s-t)^2 \cos t^2$.

15.4.20 $z_s = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 2 \cos(2x+y)(2s) + \cos(2x+y)(2s) = 6s \cos(3s^2-t^2)$ and $z_t = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = 2 \cos(2x+y)(-2t) + \cos(2x+y)(2t) = -2t \cos(2(s^2-t^2) + s^2 + t^2) = -2t \cos(3s^2-t^2)$.

$$\begin{aligned} 15.4.21 \quad z_s &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (y - 2xy) \cdot 1 + (x - x^2) \cdot 1 = s - t - 2(s^2 - t^2) + (s + t) - (s + t)^2 = 2s - 3s^2 - \\ &2st + t^2 \text{ and } z_t = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (y - 2xy) \cdot 1 + (x - x^2) \cdot (-1) = s - t - 2(s^2 - t^2) - (s + t) + (s + t)^2 = \\ &-s^2 - 2t + 2st + 3t^2. \end{aligned}$$

15.4.22

$$z_s = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \cos x \cos 2y - 2 \sin x \sin 2y = \cos(s + t) \cos[2(s - t)] - 2 \sin(s + t) \sin[2(s - t)]$$

and

$$z_t = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \cos x \cos 2y - 2 \sin x \sin 2y \cdot (-1) = \cos(s + t) \cos[2(s - t)] + 2 \sin(s + t) \sin[2(s - t)]$$

$$15.4.23 \quad z_s = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = e^{x+y} \cdot t + e^{x+y} \cdot 1 = (t + 1) e^{st+s+t} \text{ and } z_t = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = e^{x+y} \cdot s + e^{x+y} \cdot 1 = (s + 1) e^{st+s+t}.$$

$$\begin{aligned} 15.4.24 \quad z_s &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (y \cos xy)(2st) + (x \cos xy)(10(s + t)^9) = (2yst + 10s^2t(s + t)^9)(\cos xy) = \\ &2st((s + t)^{10} + 5s(s + t)^9)(\cos xy) = 2st(s + t)^9(6s + t) \cos(s^2t(s + t)^{10}) \text{ and} \\ z_t &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (y \cos xy)s^2 + (x \cos xy)10(s + t)^9 = ((s + t)^{10}s^2 + 10s^2t(s + t)^9) \cos xy = \\ &s^2(s + t)^9(s + 11t) \cos(s^2t(s + t)^{10}). \end{aligned}$$

$$\begin{aligned} 15.4.25 \quad w_s &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} = \frac{1}{y + z} \cdot 1 + \frac{z - x}{(y + z)^2} \cdot t - \frac{x + y}{(y + z)^2} \cdot 1 = \frac{(1 + t)(z - x)}{(y + z)^2} = \\ &-\frac{2t(1 + t)}{(st + s - t)^2} \text{ and} \\ w_t &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} = \frac{1}{y + z} \cdot 1 + \frac{z - x}{(y + z)^2} \cdot s - \frac{x + y}{(y + z)^2} \cdot (-1) = \frac{(1 - s)x + 2y + (1 + s)z}{(y + z)^2} = \\ &\frac{2s}{(st + s - t)^2}. \end{aligned}$$

$$15.4.26 \quad w_r = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot 0 + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \cdot s + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \cdot t = \frac{r(s^2 + t^2)}{\sqrt{r^2s^2 + r^2t^2 + s^2t^2}}.$$

$$\text{Similarly, } w_s = \frac{s(r^2 + t^2)}{\sqrt{r^2s^2 + r^2t^2 + s^2t^2}} \quad \text{and} \quad w_t = \frac{t(r^2 + s^2)}{\sqrt{r^2s^2 + r^2t^2 + s^2t^2}}.$$

15.4.27

- By the chain rule, $V'(t) = 2\pi r(t)h(t)r'(t) + \pi[r(t)]^2h'(t)$.
- Substituting $r(t) = e^t$ and $h(t) = e^{-2t}$ gives $V'(t) = 2\pi e^t e^{-2t} e^t + \pi e^{2t} (-2e^{-2t}) = 0$.
- Because $V'(t) = 0$, the volume remains constant.

15.4.28

- By the chain rule, $V'(t) = \frac{2}{3}x(t)h(t)x'(t) + \frac{1}{3}[x(t)]^2h'(t)$.

- Substituting $x(t) = \frac{t}{t+1}$ and $h(t) = \frac{1}{t+1}$ gives

$$V'(t) = \frac{2}{3} \frac{t}{t+1} \frac{1}{t+1} \frac{1}{(t+1)^2} + \frac{1}{3} \left(\frac{t}{t+1} \right)^2 \left(-\frac{1}{(t+1)^2} \right) = \frac{1}{3} \frac{t(2-t)}{(t+1)^4}.$$

c. The volume increases for $0 \leq t \leq 2$ and decreases for $t \geq 2$.

15.4.29

a. $z(t) = \frac{1}{t^2 + 2t} + \frac{1}{t^3 - 2}$, so $z'(t) = -\frac{1}{(t^2 + 2t)^2} \cdot (2t + 2) - \frac{1}{(t^3 - 2)^2} \cdot 3t^2$.

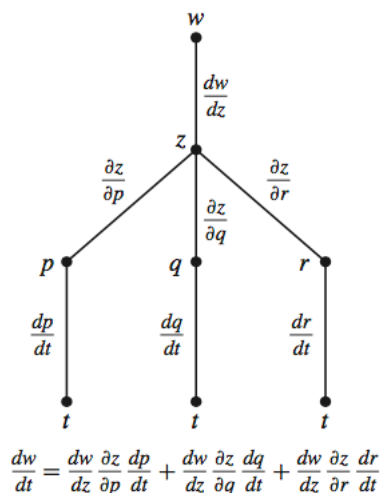
b. $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = -\frac{1}{x^2} \cdot (2t + 2) - \frac{1}{y^2} \cdot 3t^2 = -\frac{1}{(t^2 + 2t)^2} \cdot (2t + 2) - \frac{1}{(t^3 - 2)^2} \cdot 3t^2$.

15.4.30

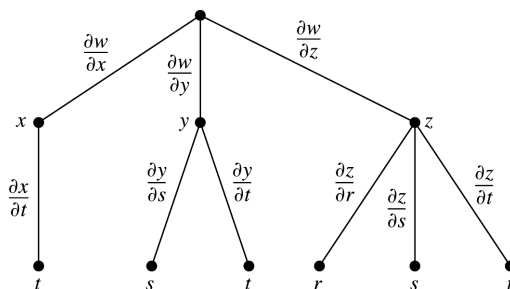
a. $z(t) = \ln(te^t + e^t)$, so $z'(t) = \frac{1}{te^t + e^t} (e^t + te^t + e^t) = \frac{te^t + 2e^t}{te^t + e^t}$.

b. $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{1}{x + y} \cdot (e^t + te^t) + \frac{1}{x + y} \cdot e^t = \frac{te^t + 2e^t}{te^t + e^t}$.

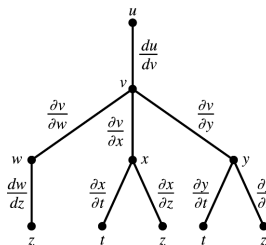
15.4.31 $\frac{dw}{dt} = \frac{dw}{dz} \frac{\partial z}{\partial p} \frac{dp}{dt} + \frac{dw}{dz} \frac{\partial z}{\partial q} \frac{dq}{dt} + \frac{dw}{dz} \frac{\partial z}{\partial r} \frac{dr}{dt}$.



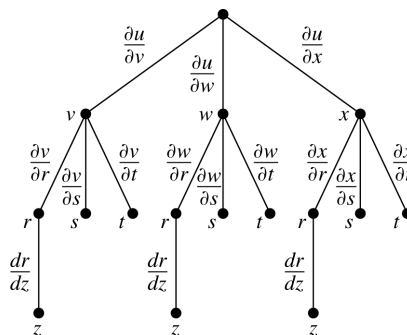
15.4.32 $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}$



15.4.33 $\frac{\partial u}{\partial z} = \frac{du}{dv} \left(\frac{\partial v}{\partial w} \frac{dw}{dz} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial z} \right)$



$$15.4.34 \quad \frac{du}{dz} = \left(\frac{\partial u}{\partial v} \frac{dv}{dr} + \frac{\partial u}{\partial w} \frac{dw}{dr} + \frac{\partial u}{\partial x} \frac{dx}{dr} \right) \frac{dr}{dz}$$



$$15.4.35 \quad \text{Let } F(x, y) = x^2 - 2y^2 - 1; \text{ then by Theorem 15.9, we have } \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{-4y} = \frac{x}{2y}.$$

$$15.4.36 \quad \text{Let } F(x, y) = x^3 + 3xy^2 - y^5; \text{ then by Theorem 15.9, we have } \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 + 3y^2}{6xy - 5y^4} = \frac{3x^2 + 3y^2}{5y^4 - 6xy}.$$

$$15.4.37 \quad \text{Let } F(x, y) = 2 \sin(xy) - 1; \text{ then by Theorem 15.9, we have } \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y \cos(xy)}{2x \cos(xy)} = -\frac{y}{x}.$$

$$15.4.38 \quad \text{Let } F(x, y) = ye^{xy} - 2; \text{ then by Theorem 15.9, we have } \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y^2 e^{xy}}{e^{xy} + xy e^{xy}} = -\frac{y^2}{1 + xy}.$$

$$15.4.39 \quad \text{Note that we can simplify this equation to } x^2 + 2xy + y^4 = 9, \text{ so let } F(x, y) = x^2 + 2xy + y^4 - 9; \text{ then by Theorem 15.9, we have } \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x + 2y}{2x + 4y^3} = -\frac{x + y}{x + 2y^3}.$$

$$15.4.40 \quad \text{Let } F(x, y) = y \ln(x^2 + y^2 + 4) - 3; \text{ then by Theorem 15.9, we have } \frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-\left(\frac{2xy}{x^2 + y^2 + 4}\right)}{\left(\ln(x^2 + y^2 + 4) + \frac{2y^2}{x^2 + y^2 + 4}\right)} = -\frac{2xy}{2y^2 + (x^2 + y^2 + 4) \ln(x^2 + y^2 + 4)}.$$

15.4.41 The chain rule gives

$$\frac{\partial s}{\partial x} = \frac{\partial s}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial s}{\partial v} \frac{\partial v}{\partial x} = \frac{u}{\sqrt{u^2 + v^2}} \cdot 0 + \frac{v}{\sqrt{u^2 + v^2}} \cdot (-2) = \frac{4x}{\sqrt{4(x^2 + y^2)}} = \frac{2x}{\sqrt{x^2 + y^2}}$$

and

$$\frac{\partial s}{\partial y} = \frac{\partial s}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial s}{\partial v} \frac{\partial v}{\partial y} = \frac{u}{\sqrt{u^2 + v^2}} \cdot 2 + \frac{v}{\sqrt{u^2 + v^2}} \cdot 0 = \frac{4y}{\sqrt{4(x^2 + y^2)}} = \frac{2y}{\sqrt{x^2 + y^2}}.$$

15.4.42 The chain rule gives

$$\begin{aligned}\frac{\partial s}{\partial x} &= \frac{\partial s}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial s}{\partial v} \frac{\partial v}{\partial x} = \frac{u}{\sqrt{u^2 + v^2}} \cdot (1 - 2x)(1 - 2y) + \frac{v}{\sqrt{u^2 + v^2}} \cdot y(y - 1)(-2) \\ &= \frac{x(1 - x)(1 - 2x)(1 - 2y)^2 - 2y^2(1 - 2x)(y - 1)^2}{\sqrt{x^2(1 - x)^2(1 - 2y)^2 + y^2(y - 1)^2(1 - 2x)^2}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial s}{\partial y} &= \frac{\partial s}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial s}{\partial v} \frac{\partial v}{\partial y} \\ &= \frac{u}{\sqrt{u^2 + v^2}} \cdot x(1 - x)(-2) + \frac{v}{\sqrt{u^2 + v^2}} \cdot (2y - 1)(1 - 2x) \\ &= \frac{y(1 - y)(1 - 2y)(1 - 2x)^2 - 2x^2(1 - 2y)(1 - x)^2}{\sqrt{x^2(1 - x)^2(1 - 2y)^2 + y^2(1 - y)^2(1 - 2x)^2}}\end{aligned}$$

15.4.43 Using the derivation from Example 7, we have

$$f_{ss} = f_{xx}x_s^2 + 2f_{xy}x_sy_s + f_{yy}y_s^2 + f_{xx}x_{ss} + f_yy_{ss},$$

$$f_{st} = f_{xx}x_sy_t + f_{xy}x_sy_t + f_{xy}x_ty_s + f_{yy}y_sy_t + f_{xx}x_{st} + f_yy_{st},$$

$$f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_ty_t + f_{yy}y_t^2 + f_{xx}x_{tt} + f_yy_{tt}.$$

For $f = x^2y$ and $x = s + t$ and $y = s - t$, we have $f_x = 2xy$, $f_{xx} = 2y$, $f_y = x^2$, $f_{yx} = 2x$, and $f_{yy} = 0$. Also, $x_s = 1$, $x_t = 1$, $y_s = 1$, and $y_t = -1$, so all the second partials of x and y are zero. Therefore we have

$$f_{ss} = 2y + 4x + 0 + 0 + 0 = 2s - 2t + 4s + 4t = 6s + 2t = 2(3s + t).$$

$$f_{st} = 2y - 2x + 2x + 0 + 0 + 0 = 2y = 2(s - t).$$

$$f_{tt} = 2y - 4x + 0 + 0 + 0 = -2(2x - y) = -2(s + 3t).$$

15.4.44 Using the derivation from Example 7, we have

$$f_{ss} = f_{xx}x_s^2 + 2f_{xy}x_sy_s + f_{yy}y_s^2 + f_{xx}x_{ss} + f_yy_{ss},$$

$$f_{st} = f_{xx}x_sy_t + f_{xy}x_sy_t + f_{xy}x_ty_s + f_{yy}y_sy_t + f_{xx}x_{st} + f_yy_{st},$$

$$f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_ty_t + f_{yy}y_t^2 + f_{xx}x_{tt} + f_yy_{tt}.$$

For $f = x^2y - xy^2$ and $x = st$ and $y = \frac{s}{t}$, we have $f_x = 2xy - y^2$, $f_{xx} = 2y$, $f_y = x^2 - 2xy$, $f_{yx} = 2x - 2y$, and $f_{yy} = -2x$. Also, $x_s = t$, $x_t = s$, $y_s = \frac{1}{t}$, and $y_t = -\frac{s}{t^2}$. Also $x_{ss} = 0$, $y_{ss} = 0$, and $x_{st} = 1$, while $y_{ss} = 0$, $y_{tt} = \frac{2s}{t^3}$, and $y_{st} = -\frac{1}{t^2}$. Therefore we have

$$f_{ss} = 2yt^2 + 2(2x - 2y) - 2x \cdot \frac{1}{t^2} = 2st + 4st - \frac{4s}{t^2} - \frac{2s}{t} = 6st - \frac{6s}{t}.$$

$$f_{st} = 2y(st) + (2x - 2y)\left(-\frac{s}{t}\right) + (2x - 2y)\left(\frac{s}{t}\right) - 2x\left(-\frac{s}{t^3}\right) + (x^2 - 2xy)\left(-\frac{1}{t^2}\right) = 3s^2 + \frac{3s^2}{t^2}.$$

$$f_{tt} = 2ys^2 + 4(x - y)\left(-\frac{s^2}{t^2}\right) - 2x\frac{s^2}{t^4} + 0 + (x^2 - 2xy)\frac{2s}{t^3} = -\frac{2s^3}{t^3}.$$

15.4.45 Using the derivation from Example 7, we have

$$f_{ss} = f_{xx}x_s^2 + 2f_{xy}x_sy_s + f_{yy}y_s^2 + f_{xx}x_{ss} + f_yy_{ss},$$

$$f_{st} = f_{xx}x_sy_t + f_{xy}x_sy_t + f_{xy}x_ty_s + f_{yy}y_sy_t + f_{xx}x_{st} + f_yy_{st},$$

$$f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_ty_t + f_{yy}y_t^2 + f_{xx}x_{tt} + f_yy_{tt}.$$

For $f = \frac{y}{x}$ we have $f_x = -\frac{y}{x^2}$, $f_{xy} = -\frac{1}{x^2}$, $f_{xx} = \frac{2y}{x^3}$, $f_y = \frac{1}{x}$, $f_{yy} = 0$, $x_s = 2s$, $x_{ss} = 2$, $x_t = 2t$, $x_{tt} = 2$, $x_{ts} = 0$, $y_s = 2s$, $y_{ss} = 2$, $y_{st} = 0$, $y_t = -2t$, and $y_{tt} = -2$. Then we have

$$f_{ss} = \frac{2y}{x^3} \cdot 4s^2 - \frac{2}{x^2} \cdot 4s^2 + 0 + 2 \left(-\frac{y}{x^2} \right) + \frac{2}{x} = \frac{4t^2(t^2 - 3s^2)}{(s^2 + t^2)^3}.$$

$$f_{st} = \frac{2y}{x^3}(2s)(2t) + \frac{-1}{x^2}(-4ts) + \frac{-1}{x^2}(4ts) + 0 - \frac{y}{x^2}(0) + 0 = \frac{2(s^2 - t^2)(4st)}{(s^2 + t^2)^3} = \frac{8st(s^2 - t^2)}{(s^2 + t^2)^3}.$$

$$f_{ttt} = \frac{2y}{x^3}(4t^2) + 2 \left(\frac{-1}{x^2} \right) (-4t^2) + 0 + 2 \left(\frac{-1}{x^2} \right) + -2 \left(\frac{1}{x} \right) = \frac{-4(s^4 - 3s^2t^2)}{(s^2 + t^2)^3}.$$

15.4.46 Using the derivation from Example 7, we have

$$f_{ss} = f_{xx}x_s^2 + 2f_{xy}x_sy_s + f_{yy}y_s^2 + f_{xx}x_{ss} + f_yy_{ss},$$

$$f_{st} = f_{xx}x_sy_t + f_{xy}x_sy_t + f_{xy}x_ty_s + f_{yy}y_sy_t + f_{xx}x_{st} + f_yy_{st},$$

$$f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_ty_t + f_{yy}y_t^2 + f_{xx}x_{tt} + f_yy_{tt}.$$

For $f = e^{x-y}$ and $x = s^2$ and $y = 3t^2$ we have $f_x = e^{x-y}$, $f_y = -e^{x-y}$, $f_{xx} = e^{x-y}$, $f_{xy} = -e^{x-y}$, and $f_{yy} = e^{x-y}$. Also, $x_s = 2s$, $x_t = 0$, $x_{ts} = 0$, $y_t = 6t$, $y_s = 0$, $x_{ss} = 2$, $y_{tt} = 6$ and the others are 0. Then

$$f_{ss} = e^{x-y}(4s^2) + 0 + 0 + 2e^{x-y} + 0 = 4s^2e^{s^2-3t^2} + 2e^{s^2-3t^2}.$$

$$f_{st} = e^{x-y}(12st) + 0 + 0 + 0 + 0 + 0 + 0 = 12ste^{s^2-3t^2}.$$

$$f_{tt} = 0 + 0 + 36t^2e^{x-y} - 6e^{x-y} = 36t^2e^{s^2-3t^2} - 6e^{s^2-3t^2}.$$

15.4.47 Note that $f_x = y + z$, $f_y = x - z$, $f_z = x - y$, $f_{xx} = 0$, $f_{xy} = 1$, $f_{xz} = -1$, $f_{yy} = 0$, $f_{yz} = -1$, and $f_{zz} = 0$. Also, $x_s = 2s - 2$, $y_s = \frac{-4}{s^3}$, $z_s = 6s$, etc.

Omitting the terms that are equal have factors of 0, we have

$$f_{ss} = 2f_{xy}x_sy_s + 2f_{xz}x_sz_s + 2f_{yz}y_sz_s + f_{xx}x_{ss} + f_yy_{ss} + f_zz_{ss}.$$

We have

$$f_{ss} = 2(2s - 2) \left(\frac{-4}{s^3} \right) + 2(2s - 2)(6s) + 2 \left(\frac{4}{s^3} \right) (6s) + 2(y + z) + \frac{12}{s^4}(x - z) + (x - y)6.$$

This can be written as

$$4 \left(\frac{6}{s^4} - \frac{2}{s^3} - 1 - 9s + 9s^2 \right).$$

15.4.48 Note that all of f_{xx} , f_{yy} , and all the mixed second partials for x and y are 0. So we have $f_{ss} = 2f_{xy}x_sy_s = 2$, $f_{st} = f_{xy}x_sy_t + f_{xy}x_ty_s = 2 + 2 = 4$, $f_{su} = f_{xy}x_sy_u + f_{xy}x_u y_s = 1 - 1 = 0$, $f_{tt} = 2f_{xy}x_ty_t = 8$, $f_{tu} = f_{xy}x_ty_u + f_{xy}x_u y_t = 0$, and $f_{uu} = 2f_{xy}x_u y_u = -2$.

15.4.49

a. False. The correct equation is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{dx}{ds} + \frac{\partial z}{\partial y} \frac{dy}{ds}$.

- b. False. w is a function of both s and t , so the rate of change of w with respect to t is the partial derivative $\frac{\partial w}{\partial t}$.

15.4.50 The chain rule gives $\frac{\partial z}{\partial p} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial p} = \frac{1}{y} \cdot 1 - \frac{x}{y^2} \cdot 1 = \frac{y-x}{y^2} = -\frac{2q}{(p-q)^2}$.

15.4.51 The chain rule gives $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = yz \cdot 8t^3 + xz(-3t^{-2}) + xy(-12t^{-4}) = \frac{12}{t^4} \cdot 8t^3 + 8t \left(-\frac{3}{t^2}\right) + 6t^3 \left(-\frac{12}{t^4}\right) = 0$. This can also be seen by expressing w in terms of t : $w = 2t^4 3t^{-1} 4t^{-3} = 24$, so $\frac{dw}{dt} = 0$.

15.4.52 Observe that $w = \cos(x+y) - (\cos x \cos y - \sin x \sin y) = 0$, $\frac{\partial w}{\partial x} = 0$.

15.4.53 The chain rule gives $-\frac{1}{x^2} - \frac{1}{z^2} \frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{z^2}{x^2}$.

15.4.54 Observe that $z = xy - 1$, so $\frac{\partial z}{\partial x} = y$.

15.4.55

a. The chain rule gives $w'(t) = aw_x + bw_y + cw_z$.

b. Using part (a), $w'(t) = ayz + bxz + cxy = 3abct^2$.

c. Using part (a), $w'(t) = \frac{ax}{\sqrt{x^2 + y^2 + z^2}} + \frac{by}{\sqrt{x^2 + y^2 + z^2}} + \frac{cz}{\sqrt{x^2 + y^2 + z^2}} = \frac{ax + by + cz}{\sqrt{x^2 + y^2 + z^2}} = \frac{\sqrt{a^2 + b^2 + c^2} \cdot t}{|t|}$

d. Differentiate the result from part (a) one more time:

$$w''(t) = a(aw_{xx} + bw_{xy} + cw_{xz}) + b(aw_{yx} + bw_{yy} + cw_{yz}) + c(aw_{zx} + bw_{zy} + cw_{zz}) \text{ which simplifies to } w''(t) = a^2w_{xx} + b^2w_{yy} + c^2w_{zz} + 2abw_{xy} + 2acw_{xz} + 2bcw_{yz}.$$

15.4.56 Differentiate $F(x, y, z(x, y)) = 0$ using the chain rule: $F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$.

The proof that $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ is similar.

15.4.57 Let $F(x, y, z) = xy + xz + yz - 3$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y+z}{x+y}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{x+y}$.

15.4.58 Let $F(x, y, z) = x^2 + 2y^2 - 3z^2 - 1$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{-6z} = \frac{x}{3z}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{-6z} = \frac{2y}{3z}$.

15.4.59 Let $F(x, y, z) = xyz + x + y - z$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz+1}{xy-1}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz+1}{xy-1}$.

15.4.60

- a. Let $F(x, y, z) = e^{xyz} - 2$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yze^{xyz}}{xye^{xyz}} = -\frac{z}{x}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{xyz}}{xye^{xyz}} = -\frac{z}{y}$.
- b. Take logarithms of both sides to obtain $xyz - \ln 2 = 0$; then the result from Exercise 56 applied to $F(x, y, z) = xyz - \ln 2$ gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz}{xy} = -\frac{z}{x}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz}{xy} = -\frac{z}{y}$.
- c. Solve for z to obtain $z = \frac{\ln 2}{xy}$; therefore $\frac{\partial z}{\partial x} = -\frac{\ln 2}{x^2 y} = -\frac{z}{x}$, $\frac{\partial z}{\partial y} = -\frac{\ln 2}{xy^2} = -\frac{z}{y}$.

15.4.61

- a. The chain rule gives $z'(t) = 2x(-\sin t) + 8y \cos t = 6 \sin t \cos t = 3 \sin 2t$.
- b. Observe that for $0 \leq t \leq 2\pi$, $z'(t) = 3 \sin 2t > 0$ when $0 < t < \frac{\pi}{2}$ or $\pi < t < \frac{3\pi}{2}$.

15.4.62

- a. The chain rule gives $z'(t) = 8x(-\sin t) - 2y \cos t = -10 \sin t \cos t = -5 \sin 2t$.
- b. Observe that for $0 \leq t \leq 2\pi$, $z'(t) = -5 \sin 2t > 0$ when $\frac{\pi}{2} < t < \pi$ or $\frac{3\pi}{2} < t < 2\pi$.

15.4.63

- a. The chain rule gives $z'(t) = -\frac{x}{\sqrt{1-x^2-y^2}}(-e^{-t}) + -\frac{y}{\sqrt{1-x^2-y^2}}(-e^{-t}) = \frac{2e^{-2t}}{\sqrt{1-e^{-2t}}}$.
- b. Observe that $z'(t) > 0$ for all t where defined, so the function $z(t)$ is increasing for all $t \geq \frac{1}{2} \ln 2$.

15.4.64

- a. The chain rule gives $z'(t) = 4x(-\sin t) + 2y \cos t = -4(1 + \cos t) \sin t + 2 \sin t \cos t = -2 \sin t(2 + \cos t)$.
- b. Observe that $2 + \cos t > 0$ for all t , so for $0 \leq t \leq 2\pi$, $z'(t) > 0$ if and only if $\sin t < 0$, which occurs when $\pi < t < 2\pi$.

15.4.65 The chain rule gives $E'(t) = m(uu' + vv') + mgy' = m(x'x'' + y'y'' + gy') = m(u_0 \cdot 0 + y'(y'' + g)) = 0$. Therefore, the energy of the projectile remains constant during the motion.

15.4.66

- a. The marginal utilities are $\frac{\partial U}{\partial x} = ax^{a-1}y^{1-a}$, $\frac{\partial U}{\partial y} = (1-a)x^ay^{-a}$.
- b. Using Theorem 15.9, we find that the slope of the indifference curve $U(x, y) - c = 0$ is $\text{MRS} = -\frac{U_x}{U_y} = -\frac{a}{1-a} \frac{y}{x}$.
- c. The result from part (b) above gives $\text{MRS} = -\frac{0.4}{0.6} \frac{12}{8} = -1$.

15.4.67

- a. If r and R increase at the same rate then $R - r$ is a constant C , so $V = \frac{C^2 \pi^2}{4} (R + r)$ is increasing.
- b. Similarly, if r and R decrease at the same rate then V is decreasing.

15.4.68

- a. The chain rule gives $S'(t) = \frac{1}{60} \left(\frac{w}{2\sqrt{hw}} h'(t) + \frac{h}{2\sqrt{hw}} w'(t) \right) = \frac{w h'(t) + h w'(t)}{120\sqrt{hw}}$.
- b. From part (a), the condition that $S(t)$ is constant is $w h'(t) + h w'(t) = 0$.
- c. If S is constant then we must have hw constant, so h and w are inversely proportional.

15.4.69

- a. Consider P as a function of T and V and differentiate with respect to V : $\frac{\partial P}{\partial V} V + P \cdot 1 = 0$, so $\frac{\partial P}{\partial V} = -\frac{P}{V}$. Next, consider T as a function of P and V and differentiate with respect to P : $1 \cdot V = k \frac{\partial T}{\partial P}$, so $\frac{\partial T}{\partial P} = \frac{V}{k}$. Lastly, consider V as a function of T and P and differentiate with respect to T : $P \frac{\partial V}{\partial T} = k$, so $\frac{\partial V}{\partial T} = \frac{k}{P}$.
- b. Observe that $\frac{\partial P}{\partial V} \frac{\partial T}{\partial P} \frac{\partial V}{\partial T} = -\frac{P}{V} \frac{V}{k} \frac{k}{P} = -1$.

15.4.70

- a. The chain rule gives $\frac{d\rho}{dt} = y(-2\sin t) + x \cdot 2\cos t = 4(\cos^2 t - \sin^2 t) = 4\cos 2t$.
- b. The density as a function of t is given by $\rho(t) = 4 + 4\cos t \sin t = 4 + 2\sin 2t$, which attains its maximum at $t = \frac{\pi}{4}, \frac{5\pi}{4}$. (This also follows from part (a) and the second derivative test.) The corresponding points on the plate are $\pm(\sqrt{2}, \sqrt{2})$.

15.4.71

- a. The chain rule gives $w'(t) = \frac{yz}{z^2+1}(-\sin t) + \frac{xz}{z^2+1}(\cos t) + \frac{xy(1-z^2)}{(z^2+1)^2} \cdot 1 = -\frac{(\sin t)t}{t^2+1} \sin t + \frac{(\cos t)t}{t^2+1} \cos t + \frac{(\cos t)(\sin t)(1-t^2)}{(t^2+1)^2}$.
- b. The function $w(t) = \frac{t \cos t \sin t}{1+t^2}$ takes its maximum value on $[0, \infty)$ approximately at $t = 0.838$, which gives the point $(0.669, 0.743, 0.838)$ on the spiral.

15.4.72

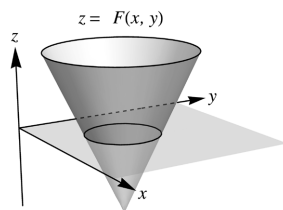
- a. We have $x_r = \cos \theta$, $y_r = \sin \theta$, $x_\theta = -r \sin \theta$, $y_\theta = r \cos \theta$.
- b. We have $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1} \left(\frac{y}{x} \right)$, so $r_x = \frac{x}{\sqrt{x^2 + y^2}}$, $r_y = \frac{y}{\sqrt{x^2 + y^2}}$, and $\theta_x = -\frac{y}{x^2 + y^2}$, $\theta_y = \frac{x}{x^2 + y^2}$.
- c. The chain rule gives $z_r = f_x \cos \theta + f_y \sin \theta$, $z_\theta = -r f_x \sin \theta + r f_y \cos \theta$.
- d. The chain rule gives $z_x = g_r \frac{x}{\sqrt{x^2 + y^2}} - g_\theta \frac{y}{x^2 + y^2}$, $z_y = g_r \frac{y}{\sqrt{x^2 + y^2}} + g_\theta \frac{x}{x^2 + y^2}$.
- e. Observe that $z_x^2 + z_y^2 = g_r^2 \left(\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} \right) + g_\theta^2 \left(\frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2} \right) = g_r^2 + \frac{1}{r^2} g_\theta^2 = z_r^2 + \frac{1}{r^2} z_\theta^2$.

15.4.73

- a. From problem 72 part (d) we have $z_x = \frac{x}{r}z_r - \frac{y}{r^2}z_\theta$, $z_y = \frac{y}{r}z_r + \frac{x}{r^2}z_\theta$
- b. Differentiating the equation for z_x in part (a) with respect to x gives $z_{xx} = \frac{1}{r}z_r + x\left(-\frac{1}{r^2}\right)r_z z_r + \frac{x}{r}(z_r)_x + \frac{2y}{r^3}r_x z_\theta - \frac{y}{r^2}(z_\theta)_x = \frac{x}{r}(z_r)_x - \frac{y}{r^2}(z_\theta)_x + \left(\frac{r^2}{r^3} - \frac{x^2}{r^3}\right)z_r + \frac{2xy}{r^4}z_\theta = \frac{x}{r}\left(\frac{x}{r}z_{rr} - \frac{y}{r^2}z_{r\theta}\right) - \frac{y}{r^2}\left(\frac{x}{r}z_{\theta r} - \frac{y}{r^2}z_{\theta\theta}\right) + \frac{y^2}{r^3}z_r + \frac{2xy}{r^4}z_\theta = \frac{x^2}{r^2}z_{rr} + \frac{y^2}{r^4}z_{\theta\theta} - \frac{2xy}{r^3}z_{r\theta} + \frac{y^2}{r^3}z_r + \frac{2xy}{r^4}z_\theta$.
- c. Differentiating the equation for z_y in part (a) with respect to y gives $z_{yy} = \frac{1}{r}z_r + y\left(-\frac{1}{r^2}\right)r_y z_r + \frac{y}{r}(z_r)_y - \frac{2x}{r^3}r_y z_\theta + \frac{x}{r^2}(z_\theta)_y = \frac{y}{r}(z_r)_y + \frac{x}{r^2}(z_\theta)_y + \left(\frac{r^2}{r^3} - \frac{y^2}{r^3}\right)z_r - \frac{2xy}{r^4}z_\theta = \frac{y}{r}\left(\frac{y}{r}z_{rr} + \frac{x}{r^2}z_{r\theta}\right) + \frac{x}{r^2}\left(\frac{y}{r}z_{\theta r} + \frac{x}{r^2}z_{\theta\theta}\right) + \frac{x^2}{r^3}z_r - \frac{2xy}{r^4}z_\theta = \frac{y^2}{r^2}z_{rr} + \frac{x^2}{r^4}z_{\theta\theta} + \frac{2xy}{r^3}z_{r\theta} + \frac{x^2}{r^3}z_r - \frac{2xy}{r^4}z_\theta$.
- d. Adding the results from (b) and (c) gives $z_{xx} + z_{yy} = z_{rr} + \frac{1}{r}z_r + \frac{1}{r^2}z_{\theta\theta}$.

15.4.74

- a. The tangent plane to the surface $z = F(x, y)$ has normal vector $\mathbf{n} = \langle F_x, F_y, -1 \rangle$, so we see that the projection of $\mathbf{n}$ into the xy -plane is orthogonal to the curve $F(x, y) = 0$.



- b. The curve $F(x, y) = 0$ has a vertical tangent at a point where $F_y(x, y) = 0$.

15.4.75

- a. Assuming y is fixed, the chain rule gives $F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \left(\frac{\partial z}{\partial x}\right)_y = 0$, so $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$.
- b. Similarly we find that $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$ and $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}$.
- c. From (a) and (b) we see that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1$.
- d. Let $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ denote the partial derivative of w with respect to x holding y and z constant, with similar notation for the other possible pairs of variables. A similar derivation as in part (a) and (b) above for $F(w, x, y, z) = 0$ shows that $\left(\frac{\partial w}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial y}\right)_{w,z} \left(\frac{\partial y}{\partial z}\right)_{w,x} \left(\frac{\partial z}{\partial w}\right)_{x,y} = \left(-\frac{F_x}{F_w}\right) \left(-\frac{F_y}{F_x}\right) \left(-\frac{F_z}{F_y}\right) \left(-\frac{F_w}{F_z}\right) = 1$.

15.4.76

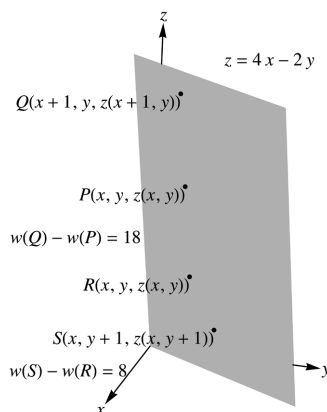
- a. We know that $y'(x) = -\frac{f_x}{f_y}$; differentiating again with respect to x gives

$$y''(x) = -\frac{f_y(f_x)' - f_x(f_y)'}{f_y^2} = -\frac{f_y(f_{xx} + f_{xy}y') - f_x(f_{yx} + f_{yy}y')}{f_y^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}.$$

- b. For $f(x, y) = xy - 1$ we have $f_x = y$, $f_y = x$, $f_{xy} = 1$, $f_{xx} = f_{yy} = 0$, so we obtain $y''(x) = -\frac{2xy}{x^3} = -\frac{2}{x^2}$ because $xy = 1$. In this case we can solve to find $y(x) = \frac{1}{x}$, so $y''(x) = -\frac{2}{x^3}$ is correct.

15.4.77

- a. We have $\left(\frac{\partial w}{\partial x}\right)_y = f_x + f_z \frac{dz}{dx} = 2 + 4 \cdot 4 = 18$.
- b. Rewrite $z = 4x - 2y$ as $y = 2x - \frac{z}{2}$; therefore $\left(\frac{\partial w}{\partial x}\right)_z = f_x + f_y \frac{dy}{dx} = 2 + 3 \cdot 2 = 8$.
- c.



- d. Hold x constant; then

$$\left(\frac{\partial w}{\partial y}\right)_x = f_y + f_z \frac{dz}{dy} = 3 + 4(-2) = -5.$$

Hold z constant; then

$$\left(\frac{\partial w}{\partial y}\right)_z = f_x \frac{dx}{dy} + f_y = 2 \cdot \frac{1}{2} + 3 = 4.$$

Hold x constant; then

$$\left(\frac{\partial w}{\partial z}\right)_x = f_y \frac{dy}{dz} + f_z = 3 \left(-\frac{1}{2}\right) + 4 = \frac{5}{2}.$$

Hold y constant; then

$$\left(\frac{\partial w}{\partial z}\right)_y = f_x \frac{dx}{dz} + f_z = 2 \cdot \frac{1}{4} + 4 = \frac{9}{2}.$$

15.5 Directional Derivatives and the Gradient

15.5.1 Take the dot product of the unit direction vector $\mathbf{u}$ and the gradient of the function.

15.5.2 The gradients $\nabla f = \langle f_x, f_y \rangle$ and $\nabla f = \langle f_x, f_y, f_z \rangle$.

15.5.3 The direction of the gradient vector is the direction in which the function is increasing the most (steepest ascent).

15.5.4 The magnitude of the gradient vector is the largest possible directional derivative of the function at a point.

15.5.5 The gradient is perpendicular to the level curves.

15.5.6 The direction must be perpendicular to the level curves and in a direction in which the function is increasing; therefore the gradients have directions $\pm \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$ at the points $\pm(1, 1)$ respectively.

15.5.7 $\langle -\sqrt{3}, 1 \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = -\frac{3}{2} - \frac{1}{2} = -2.$

15.5.8 It is $\langle 3, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle = \frac{2}{\sqrt{2}} = \sqrt{2}.$

15.5.9 Because $\mathbf{v}$ is in the opposite direction of $\mathbf{u}$, $D_{\mathbf{v}}f(3, 4) = -7$. $D_{\mathbf{w}}f(3, 4) = 0$, because $\mathbf{w} \cdot \nabla f(3, 4) = 0$.

15.5.10 We are looking for the slope of a line which is perpendicular to a line of slope $\frac{4}{3}$, so the slope we seek is $-\frac{3}{4}$.

15.5.11

a. Note that $f_x = -x$ and $f_y = -2y$. So $\nabla f(2, 0) = \langle -2, 0 \rangle$, $\nabla f(0, 2) = \langle 0, -4 \rangle$, and $\nabla f(1, 1) = \langle -1, -2 \rangle$.

	$(a, b) = (2, 0)$	$(a, b) = (0, 2)$	$(a, b) = (1, 1)$
$\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$	$-\sqrt{2}$	$-2\sqrt{2}$	$-3\sqrt{2}/2$
$\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$	$\sqrt{2}$	$-2\sqrt{2}$	$-\sqrt{2}/2$
$\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}/2$

b. The function is decreasing at $(2, 0)$ in the direction of $\mathbf{u}$ and increasing at $(2, 0)$ in the direction of $\mathbf{v}$ and $\mathbf{w}$.

15.5.12

a. Note that $f_x = 4x$ and $f_y = 2y$. So $\nabla f(1, 0) = \langle 4, 0 \rangle$, $\nabla f(1, 1) = \langle 4, 2 \rangle$, and $\nabla f(1, 2) = \langle 4, 4 \rangle$.

	$(a, b) = (1, 0)$	$(a, b) = (1, 1)$	$(a, b) = (1, 2)$
$\mathbf{u} = \langle 1, 0 \rangle$	4	4	4
$\mathbf{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$	$2\sqrt{2}$	$3\sqrt{2}$	$4\sqrt{2}$
$\mathbf{w} = \langle 0, 1 \rangle$	0	2	4

b. The function is increasing at $(1, 0)$ in the direction of $\mathbf{u}$ and $\mathbf{v}$ and is constant in the direction of $\mathbf{w}$.

15.5.13 $\nabla f(x, y) = \langle 6x, -10y \rangle$ so $\nabla f(2, -1) = \langle 12, 10 \rangle$.

15.5.14 $\nabla f(x, y) = \langle 8x - 2y, -2x + 2y \rangle$, so $\nabla f(-1, -5) = \langle 2, -8 \rangle$.

15.5.15 $\nabla g(x, y) = \langle 2x - 8xy - 8y^2, -4x^2 - 16xy \rangle$, so $\nabla g(-1, 2) = \langle -18, 28 \rangle$.

15.5.16 $\nabla p(x, y) = \left\langle -\frac{4x}{\sqrt{12 - 4x^2 - y^2}}, -\frac{y}{\sqrt{12 - 4x^2 - y^2}} \right\rangle$, so $\nabla p(-1, -1) = \left\langle \frac{4}{\sqrt{7}}, \frac{1}{\sqrt{7}} \right\rangle$.

15.5.17 $\nabla f(x, y) = \langle 2xye^{2xy} + e^{2xy}, 2x^2e^{2xy} \rangle$, so $\nabla f(1, 0) = \langle 1, 2 \rangle$.

15.5.18 $\nabla f(x, y) = \langle 3 \cos(3x + 2y), 2 \cos(3x + 2y) \rangle$, so $\nabla f(\pi, 3\pi/2) = \langle 3, 2 \rangle$.

15.5.19 $\nabla F(x, y) = \langle -2xe^{-x^2-2y^2}, -4ye^{-x^2-2y^2} \rangle$, so $\nabla F(-1, 2) = 2e^{-9} \langle 1, -4 \rangle$.

15.5.20 $\nabla h(x, y) = \left\langle \frac{2x}{1+x^2+2y^2}, \frac{4y}{1+x^2+2y^2} \right\rangle$, so $\nabla h(2, -3) = \frac{2}{23} \langle 2, -6 \rangle$.

15.5.21 $\nabla f(x, y) = \langle 2x, -2y \rangle$, so $\nabla f(-1, -3) = \langle -2, 6 \rangle$. We have $D_u f(-1, -3) = \langle -2, 6 \rangle \cdot \langle 3/5, -4/5 \rangle = -6/5 - 24/5 = -30/5 = -6$.

15.5.22 $\nabla f(x, y) = \langle 6x, 3y^2 \rangle$, so $\nabla f(3, 2) = \langle 18, 12 \rangle$. We have $D_u f(3, 2) = \langle 18, 12 \rangle \cdot \langle 5/13, 12/13 \rangle = 90/13 + 144/13 = 234/13 = 18$.

15.5.23 $\nabla f(x, y) = \langle -6x, y^3 \rangle$, so $\nabla f(2, -3) = \langle -12, -27 \rangle$. We have

$$D_u f(2, -3) = \langle -12, -27 \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = \frac{27}{2} - 6\sqrt{3}.$$

15.5.24 $\nabla g(x, y) = \langle 2\pi \cos \pi(2x - y), -\pi \cos \pi(2x - y) \rangle$, so $\nabla g(-1, -1) = \langle -2\pi, \pi \rangle$. We have $D_u g(-1, -1) = \langle -2\pi, \pi \rangle \cdot \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle = -\frac{22}{13}\pi$.

15.5.25 $\nabla f(x, y) = \left\langle -\frac{x}{\sqrt{4-x^2-2y}}, -\frac{1}{\sqrt{4-x^2-2y}} \right\rangle$, so $\nabla f(2, -2) = \left\langle -1, -\frac{1}{2} \right\rangle$. Thus,

$$D_u f(2, -2) = \left\langle -1, -\frac{1}{2} \right\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = -\frac{2}{\sqrt{5}}.$$

15.5.26 $\nabla f(x, y) = \langle 13ye^{xy}, 13xe^{xy} \rangle$, so $\nabla f(1, 0) = \langle 0, 13 \rangle$. A unit vector in the direction given is $\langle 5/13, 12/13 \rangle$, so $D_u f = \langle 0, 13 \rangle \cdot \langle 5/13, 12/13 \rangle = 12$.

15.5.27 $\nabla f(x, y) = \langle 6x, 2 \rangle$, so $\nabla f(1, 2) = \langle 6, 2 \rangle$. A unit vector in the direction given is $\langle -3/5, 4/5 \rangle$, so $D_u f = \langle 6, 2 \rangle \cdot \langle -3/5, 4/5 \rangle = -2$.

15.5.28 $\nabla h(x, y) = \langle -e^{-x-y}, -e^{-x-y} \rangle$, so $\nabla h(\ln 2, \ln 3) = -\left\langle \frac{1}{6}, \frac{1}{6} \right\rangle$. A unit vector in the direction given is $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. We have $D_u h(\ln 2, \ln 3) = -\left\langle \frac{1}{6}, \frac{1}{6} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = -\frac{2}{6\sqrt{2}} = -\frac{1}{3\sqrt{2}}$.

15.5.29 $\nabla g(x, y) = \left\langle \frac{2x}{4+x^2+y^2}, \frac{2y}{4+x^2+y^2} \right\rangle$, so $\nabla g(-1, 2) = \left\langle -\frac{2}{9}, \frac{4}{9} \right\rangle$. A unit vector in the direction given is $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$. We have $D_u g(-1, 2) = \left\langle -\frac{2}{9}, \frac{4}{9} \right\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = 0$.

15.5.30 $\nabla f(x, y) = \left\langle -\frac{y}{(x-y)^2}, \frac{x}{(x-y)^2} \right\rangle$, so $\nabla f(4, 1) = \left\langle -\frac{1}{9}, \frac{4}{9} \right\rangle$. A unit vector in the direction given is $\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$. We have $D_u f(4, 1) = \left\langle -\frac{1}{9}, \frac{4}{9} \right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{1}{\sqrt{5}}$.

15.5.31

- a. At the point $(1, -2)$ the value of the gradient is $\nabla f(1, -2) = \langle 2x, -8y \rangle \Big|_{(1, -2)} = \langle 2, 16 \rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{65}} \langle 1, 8 \rangle$ and the direction of steepest descent is $-\mathbf{u}$.

- b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{65}} \langle -8, 1 \rangle$.

15.5.32

- a. At the point $(2, 1)$ the value of the gradient is $\nabla f(2, 1) = \langle 2x + 4y, 4x - 2y \rangle \Big|_{(2,1)} = \langle 8, 6 \rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \langle 4/5, 3/5 \rangle$ and the direction of steepest descent is $-\mathbf{u}$.
- b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \langle -3/5, 4/5 \rangle$.

15.5.33

- a. At the point $(-1, 1)$ the value of the gradient is $\nabla f(-1, 1) = \langle 4x^3 - 2xy, -x^2 + 2y \rangle \Big|_{(-1,1)} = \langle -2, 1 \rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{5}} \langle -2, 1 \rangle$ and the direction of steepest descent is $-\mathbf{u}$.
- b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$.

15.5.34

- a. At the point $(1, 2)$ the value of the gradient is

$$\nabla P(1, 2) = \left\langle \frac{x+y}{\sqrt{20+x^2+2xy-y^2}}, \frac{x-y}{\sqrt{20+x^2+2xy-y^2}} \right\rangle \Big|_{(1,2)} = \left\langle \frac{3}{\sqrt{21}}, -\frac{1}{\sqrt{21}} \right\rangle.$$

Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{10}} \langle 3, -1 \rangle$ and the direction of steepest descent is $-\mathbf{u}$.

- b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$.

15.5.35

- a. At the point $(-1, 1)$ the value of the gradient is

$$\nabla f(-1, 1) = \left\langle -xe^{-x^2/2-y^2/2}, -ye^{-x^2/2-y^2/2} \right\rangle \Big|_{(-1,1)} = e^{-1} \langle 1, -1 \rangle.$$

Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$ and the direction of steepest descent is $-\mathbf{u}$.

- b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$.

15.5.36

- a. At the point $(0, \pi)$ the value of the gradient is

$$\nabla f(0, \pi) = \langle 4 \cos(2x - 3y), -6 \cos(2x - 3y) \rangle \Big|_{(0,\pi)} = \langle -4, 6 \rangle.$$

Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{13}} \langle -2, 3 \rangle$ and the direction of steepest descent is $-\mathbf{u}$.

- b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{13}} \langle 3, 2 \rangle$.

15.5.37

- a. The gradient of f at P is $\nabla f(3, 2) = \langle -4x, -6y \rangle \Big|_{(3,2)} = \langle -12, -12 \rangle$.

- b. The direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$ which makes angle $\theta = \frac{5\pi}{4}$ with the x -axis; therefore, the angle of maximum decrease is $\theta = \frac{\pi}{4}$ and the angles of zero change are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.
- c. We have $g(\theta) = \langle -12, -12 \rangle \cdot \langle \cos \theta, \sin \theta \rangle = -12 \cos \theta - 12 \sin \theta$.
- d. The critical points for $g(\theta)$ satisfy $g'(\theta) = 12(\sin \theta - \cos \theta) = 0$, which gives $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. By inspection we see that the maximum occurs at $\frac{5\pi}{4}$, and we have $g\left(\frac{5\pi}{4}\right) = 12\sqrt{2}$.
- e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(3, 2)| = 12\sqrt{2} = g\left(\frac{5\pi}{4}\right)$.

15.5.38

- a. The gradient of f at p is $\nabla f(-3, -1) = \langle 2x, 6y \rangle \Big|_{(-3, -1)} = \langle -6, -6 \rangle$.
- b. The direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$ which makes angle $\theta = \frac{5\pi}{4}$ with the x -axis; therefore, the angle of maximum decrease is $\theta = \frac{\pi}{4}$ and the angles of zero change are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.
- c. We have $g(\theta) = \langle -6, -6 \rangle \cdot \langle \cos \theta, \sin \theta \rangle = -6 \cos \theta - 6 \sin \theta$.
- d. The critical points for $g(\theta)$ satisfy $g'(\theta) = 6(\sin \theta - \cos \theta) = 0$, which gives $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. By inspection we see that the maximum occurs at $\frac{5\pi}{4}$, and we have $g\left(\frac{5\pi}{4}\right) = 6\sqrt{2}$.
- e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(-3, -1)| = 6\sqrt{2} = g\left(\frac{5\pi}{4}\right)$.

15.5.39

- a. The gradient of f at P is $\nabla f(\sqrt{3}, 1) = \left\langle \frac{x}{\sqrt{2+x^2+y^2}}, \frac{y}{\sqrt{2+x^2+y^2}} \right\rangle \Big|_{(\sqrt{3}, 1)} = \left\langle \frac{\sqrt{3}}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle = \frac{\sqrt{6}}{6} \langle \sqrt{3}, 1 \rangle$.
- b. The direction of steepest ascent is $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ which makes angle $\theta = \frac{\pi}{6}$ with the x -axis; therefore, the angle of maximum decrease is $\theta = \frac{7\pi}{6}$ and the angles of zero change are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.
- c. We have $g(\theta) = \frac{\sqrt{6}}{6} \langle \sqrt{3}, 1 \rangle \cdot \langle \cos \theta, \sin \theta \rangle = \frac{\sqrt{18}}{6} \cos \theta + \frac{\sqrt{6}}{6} \sin \theta$.
- d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \frac{\sqrt{6}}{6} (-\sqrt{3} \sin \theta + \cos \theta) = 0$, which gives $\tan \theta = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$. By inspection we see that the maximum occurs at $\frac{\pi}{6}$, and we have $g\left(\frac{\pi}{6}\right) = \frac{\sqrt{6}}{3}$.
- e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(3, 1)| = \frac{\sqrt{6}}{3} = g\left(\frac{\pi}{6}\right)$.

15.5.40

- a. The gradient of f at P is $\nabla f\left(-1, -\frac{1}{\sqrt{3}}\right) = \left\langle -\frac{x}{\sqrt{12-x^2-y^2}}, -\frac{y}{\sqrt{12-x^2-y^2}} \right\rangle \Big|_{(-1, -1/\sqrt{3})} = \frac{1}{4\sqrt{2}} \langle \sqrt{3}, 1 \rangle$.
- b. The direction of steepest ascent is $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ which makes angle $\theta = \frac{\pi}{6}$ with the x -axis; therefore, the angle of maximum decrease is $\theta = \frac{7\pi}{6}$ and the angles of zero change are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.
- c. We have $g(\theta) = \frac{1}{4\sqrt{2}} \langle \sqrt{3}, 1 \rangle \cdot \langle \cos \theta, \sin \theta \rangle = \frac{1}{4\sqrt{2}} (\sqrt{3} \cos \theta + \sin \theta)$.
- d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \frac{1}{4\sqrt{2}} (-\sqrt{3} \sin \theta + \cos \theta) = 0$, which gives $\tan \theta = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$. By inspection we see that the maximum occurs at $\frac{\pi}{6}$, and we have $g\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}}{4}$.
- e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $\left| \nabla f\left(-1, -\frac{1}{\sqrt{3}}\right) \right| = \frac{\sqrt{2}}{4} = g\left(\frac{\pi}{6}\right)$.

15.5.41

- a. The gradient of f at P is $\nabla f(-1, 0) = \langle -2xe^{-x^2-2y^2}, -4ye^{-x^2-2y^2} \rangle \Big|_{(-1, 0)} = \langle 2e^{-1}, 0 \rangle$.
- b. The direction of steepest ascent is $\mathbf{u} = \langle 1, 0 \rangle$ which makes angle $\theta = 0$ with the x -axis; therefore the angle of maximum decrease is $\theta = \pi$ and the angles of zero change are $\pm \frac{\pi}{2}$.
- c. We have $g(\theta) = \frac{2}{e} \langle 1, 0 \rangle \cdot \langle \cos \theta, \sin \theta \rangle = \frac{2}{e} \cos \theta$.
- d. The maximum value of $g(\theta)$ occurs at $\theta = 0$, and we have $g(0) = \frac{2}{e}$.
- e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(-1, 0)| = \frac{2}{e} = g(0)$.

15.5.42

- a. The gradient of f at P is

$$\nabla f\left(\frac{3}{4}, -\sqrt{3}\right) = \left\langle \frac{4x}{1+2x^2+3y^2}, \frac{6y}{1+2x^2+3y^2} \right\rangle \Big|_{(3/4, -\sqrt{3})} = \frac{24}{89} \langle 1, -2\sqrt{3} \rangle.$$

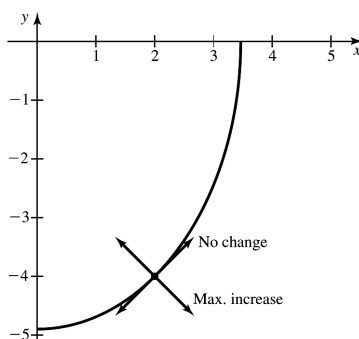
- b. The direction of steepest ascent is $\mathbf{u} = \left\langle \frac{1}{\sqrt{13}}, -\frac{2\sqrt{3}}{\sqrt{13}} \right\rangle$ which makes angle $\theta = -\tan^{-1}(2\sqrt{3})$ with the x -axis; therefore, the angle of maximum decrease is $\theta = \pi - \tan^{-1}(2\sqrt{3})$ and the angles of zero change are $\pm \frac{\pi}{2} - \tan^{-1}(2\sqrt{3})$.
- c. We have $g(\theta) = \frac{24}{89} \langle 1, -2\sqrt{3} \rangle \cdot \langle \cos \theta, \sin \theta \rangle = \frac{24}{89} (\cos \theta - 2\sqrt{3} \sin \theta)$.

- d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \frac{24}{89}(-\sin\theta - 2\sqrt{3}\cos\theta) = 0$, which gives $\tan\theta = -2\sqrt{3}$, so $\theta = -\tan^{-1}(2\sqrt{3}), \pi - \tan^{-1}(2\sqrt{3})$. By inspection we see that the maximum occurs at $-\tan^{-1}(2\sqrt{3})$, and we have $g(-\tan^{-1}(2\sqrt{3})) = \frac{24\sqrt{13}}{89}$.

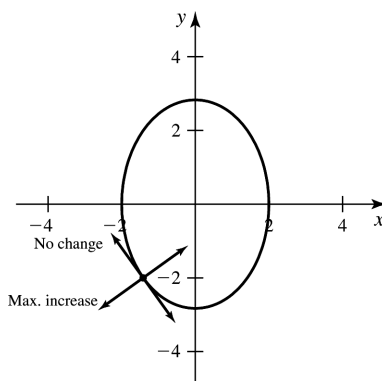
- e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that

$$\left| \nabla f\left(\frac{3}{4}, -\sqrt{3}\right) \right| = \frac{24\sqrt{13}}{89} = g\left(-\tan^{-1}(2\sqrt{3})\right).$$

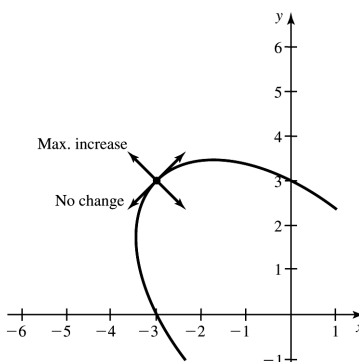
15.5.43 The gradient of f at P is $\nabla f(2, -4) = \langle 8x, 4y \rangle \Big|_{(2, -4)} = \langle 16, -16 \rangle$, which gives the direction of maximum increase.



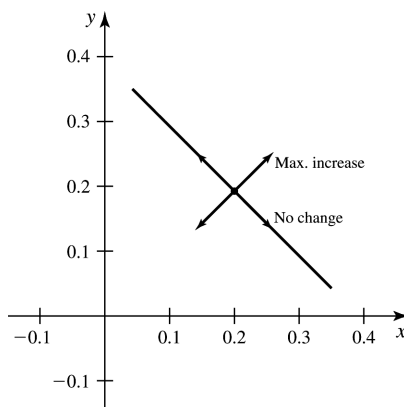
15.5.44 The gradient of f at P is $\nabla f(-1, -2) = \langle 12x, 6y \rangle \Big|_{(-1, -2)} = \langle -12, -12 \rangle$, which gives the direction of maximum increase.



15.5.45 The gradient of f at P is $\nabla f(-3, 3) = \langle 2x + y, x + 2y \rangle \Big|_{(-3, 3)} = \langle -3, 3 \rangle$, which gives the direction of maximum increase.



15.5.46 The gradient of f at P is $\nabla f\left(\frac{\pi}{16}, \frac{\pi}{16}\right) = \langle 2\sec^2(2x+2y), 2\sec^2(2x+2y) \rangle \Big|_{(\pi/16, \pi/16)} = \langle 4, 4 \rangle$, which gives the direction of maximum increase.



15.5.47 The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope 0 at $(0, 16)$. The gradient of f at this point is $\nabla f(0, 16) = \left\langle -\frac{x}{2}, -\frac{y}{8} \right\rangle \Big|_{(0, 16)} = \langle 0, -2 \rangle$, which is perpendicular to the tangent line.

15.5.48 The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, which is undefined at $(8, 0)$, so the tangent line is vertical at $(8, 0)$. The gradient of f at this point is $\nabla f(8, 0) = \left\langle -\frac{x}{2}, -\frac{y}{8} \right\rangle \Big|_{(8, 0)} = \langle -4, 0 \rangle$, which is perpendicular to the tangent line.

15.5.49 The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, which is undefined at $(4, 0)$, so the tangent line is vertical at $(4, 0)$. The gradient of f at this point is $\nabla f(4, 0) = \left\langle -\frac{x}{2}, -\frac{y}{8} \right\rangle \Big|_{(4, 0)} = \langle -2, 0 \rangle$, which is perpendicular to the tangent line.

15.5.50 The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope $-2\sqrt{3}$ at $(2\sqrt{3}, 4)$, and its direction is parallel to the vector $\langle 1, -2\sqrt{3} \rangle$. The gradient of f at this point is $\nabla f(2\sqrt{3}, 4) = \left\langle -\frac{x}{2}, -\frac{y}{8} \right\rangle \Big|_{(2\sqrt{3}, 4)} = \left\langle -\sqrt{3}, -\frac{1}{2} \right\rangle$, which is perpendicular to the tangent direction.

15.5.51 Let $z = f(x, y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$. The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope $-\frac{2}{\sqrt{3}}$ at $\left(\frac{1}{2}, \sqrt{3}\right)$, and its direction is parallel to the vector $\left\langle 1, -\frac{2}{\sqrt{3}} \right\rangle$. The gradient of f at this point is $\nabla f\left(\frac{1}{2}, \sqrt{3}\right) = \left\langle -\frac{x}{4z}, -\frac{y}{16z} \right\rangle \Big|_{(1/2, \sqrt{3})} = -\frac{1}{8} \left\langle \frac{2}{\sqrt{3}}, 1 \right\rangle$, which is perpendicular to the tangent direction.

15.5.52 Let $z = f(x, y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$. The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope 0 at $(0, \sqrt{8})$, and its direction is parallel to the

vector $\langle 1, 0 \rangle$. The gradient of f at this point is $\nabla f(0, \sqrt{8}) = \left\langle -\frac{x}{4z}, -\frac{y}{16z} \right\rangle \Big|_{(0, \sqrt{8})} = \left\langle 0, -\frac{1}{4} \right\rangle$, which is perpendicular to the tangent direction.

15.5.53 Let $z = f(x, y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also, $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$. The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line is vertical at $(\sqrt{2}, 0)$, and its direction is parallel to the vector $\langle 0, 1 \rangle$. The gradient of f at this point is $\nabla f(\sqrt{2}, 0) = \left\langle -\frac{x}{4z}, -\frac{y}{16z} \right\rangle \Big|_{(\sqrt{2}, 0)} = \left\langle -\frac{1}{2}, 0 \right\rangle$, which is perpendicular to the tangent direction.

15.5.54 Let $z = f(x, y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also, $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$. The slope of the level curves for $f(x, y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope -2 at $(1, 2)$, and its direction is parallel to the vector $\langle 1, -2 \rangle$. The gradient of f at this point is $\nabla f(1, 2) = \left\langle -\frac{x}{4z}, -\frac{y}{16z} \right\rangle \Big|_{(1, 2)} = -\frac{\sqrt{2}}{8} \langle 2, 1 \rangle$, which is perpendicular to the tangent direction.

15.5.55

- We have $\nabla f = \langle f_x, f_y \rangle = \langle 1, 0 \rangle$.
- Let $(x(t), y(t))$ be the projection into the xy -plane of the path of steepest descent starting at $(x(0), y(0)) = (4, 4)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle -1, 0 \rangle$ which together with the initial conditions gives $x = 4 - t$, $y = 4$ for $t \geq 0$.
- We have $z = 4 + x$, so $\mathbf{r}(t) = \langle 4 - t, 4, 8 - t \rangle$.

15.5.56

- We have $\nabla f = \langle f_x, f_y \rangle = \langle 1, 1 \rangle$.
- Let $(x(t), y(t))$ be the projection into the xy -plane of the path of steepest descent starting at $(x(0), y(0)) = (2, 2)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle -1, -1 \rangle$ which together with the initial conditions gives $x = 2 - t$, $y = 2 - t$ for $t \geq 0$.
- We have $z = x + y$, so $\mathbf{r}(t) = \langle 2 - t, 2t, 2 - t \rangle$.

15.5.57

- We have $\nabla f = \langle f_x, f_y \rangle = \langle -2x, -4y \rangle$.
- Let $(x(t), y(t))$ be the projection into the xy -plane of the path of steepest descent starting at $(x(0), y(0)) = \left(\frac{\pi}{2}, 1\right)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle 2x, 4y \rangle$ with the initial conditions $x(0) = 1$ and $y(0) = 1$. The differential equation $\frac{dx}{dt} = -2x$ is separable: we have $\int \frac{1}{x} dx = \int 2 dt$ which implies that $\ln|x| = 2t + C$. The initial condition $x(0) = 1$ gives $C = 0$ so $x(t) = e^{2t}$. Similarly, $y(t) = e^{4t}$. Thus, $y = x^2$ for $x \geq 1$.
- So if we let $x = t$, we have $y = t^2$ and $z = 4 - x^2 - 2y^2$, so $\mathbf{r}(t) = \langle t, t^2, 4 - t^2 - 2t^4 \rangle$.

15.5.58

- We have $\nabla f = \langle f_x, f_y \rangle = \langle -x^{-2}, 1 \rangle$.

- b. Let $(x(t), y(t))$ be the projection into the xy -plane of the path of steepest descent starting at $(x(0), y(0)) = (1, 2)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle x^{-2}, -1 \rangle$ with the initial conditions $x(0) = 1$ and $y(0) = 2$. The differential equation $\frac{dx}{dt} = \frac{1}{x^2}$ is separable: we have $\int x^2 dx = \int dt$, which implies that $\frac{x^3}{3} = t + C$; the initial condition gives $C = \frac{1}{3}$ so $x(t) = (3t + 1)^{1/3}$ and $y(t) = 2 - t$ for $t \geq 0$.
- c. $z = y + \frac{1}{x}$ so $\mathbf{r}(t) = \left\langle (3t + 1)^{1/3}, 2 - t, 2 - t + \frac{1}{(3t + 1)^{1/3}} \right\rangle$.

15.5.59

- a. We have $\nabla f = 2x\mathbf{i} + 4y\mathbf{j} + 8z\mathbf{k}$; $\nabla f(1, 0, 4) = 2\mathbf{i} + 32\mathbf{k}$.
- b. The vector $2\mathbf{i} + 32\mathbf{k}$ has length $2\sqrt{257}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{257}}(\mathbf{i} + 16\mathbf{k})$.
- c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1, 0, 4)| = 2\sqrt{257}$.
- d. The vector $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_{\mathbf{u}}f(1, 0, 4) = \langle 2, 0, 32 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle = \frac{34}{\sqrt{2}} = 17\sqrt{2}$.

15.5.60

- a. We have $\nabla f = -2x\mathbf{i} + 6y\mathbf{j} + z\mathbf{k}$; $\nabla f(0, 2, -1) = 12\mathbf{j} - \mathbf{k}$.
- b. The vector $12\mathbf{j} - \mathbf{k}$ has length $\sqrt{145}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{145}}(12\mathbf{j} - \mathbf{k})$.
- c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(0, 2, -1)| = \sqrt{145}$.
- d. The vector $\mathbf{u} = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_{\mathbf{u}}f(0, 2, -1) = \langle 0, 12, -1 \rangle \cdot \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \frac{13}{\sqrt{2}}$.

15.5.61

- a. We have $\nabla f = 4yz\mathbf{i} + 4xz\mathbf{j} + 4xy\mathbf{k}$; $\nabla f(1, -1, -1) = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$.
- b. The vector $4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ has length $4\sqrt{3}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$.
- c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1, -1, -1)| = 4\sqrt{3}$.
- d. The vector $\mathbf{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_{\mathbf{u}}f(1, -1, -1) = \langle 4, -4, -4 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle = \frac{4}{\sqrt{3}}$.

15.5.62

- a. We have $\nabla f = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$; $\nabla f(2, -2, 1) = -\mathbf{i} + 3\mathbf{j}$.
- b. The vector $-\mathbf{i} + 3\mathbf{j}$ has length 10, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{10}}(-\mathbf{i} + 3\mathbf{j})$.
- c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(2, -2, 1)| = \sqrt{10}$.

- d. The vector $\mathbf{u} = \left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_{\mathbf{u}}f(2, -2, 1) = \langle -1, 3, 0 \rangle \cdot \left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = -\frac{3}{\sqrt{2}}$.

15.5.63

- a. We have $\nabla f = \cos(x + 2y - z)(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$; $\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{3}\right)(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$.
- b. The vector $-\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$ has length $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{6}}(-\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.
- c. The rate of change of f in the direction of maximum increase at P is $\left| \nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) \right| = \frac{\sqrt{6}}{2}$.
- d. The vector $\mathbf{u} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_{\mathbf{u}}f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \cdot \left\langle -\frac{1}{2}, -1, \frac{1}{2} \right\rangle = -\frac{1}{2}$.

15.5.64

- a. We have $\nabla f = e^{xyz-1}(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$; $\nabla f(0, 1, -1) = -e^{-1}\mathbf{i}$.
- b. The unit vector in the direction of $-e^{-1}\mathbf{i}$ is $\mathbf{u} = -\mathbf{i}$.
- c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(0, 1, -1)| = e^{-1}$.
- d. The vector $\mathbf{u} = \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_{\mathbf{u}}f(0, 1, -1) = \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle \cdot \langle -e^{-1}, 0, 0 \rangle = \frac{2}{3e}$.

15.5.65

- a. We have $\nabla f = \frac{2}{1+x^2+y^2+z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$; $\nabla f(1, 1, -1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$.
- b. The vector $\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$ has length $\frac{\sqrt{3}}{2}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$.
- c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1, 1, -1)| = \frac{\sqrt{3}}{2}$.
- d. The vector $\mathbf{u} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_{\mathbf{u}}f(1, 1, -1) = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle \cdot \left\langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{5}{6}$.

15.5.66

- a. We have $\nabla f = \frac{1}{(y-z)^2}((y-z)\mathbf{i} + (z-x)\mathbf{j} + (x-y)\mathbf{k})$; $\nabla f(3, 2, -1) = \frac{1}{3}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{1}{9}\mathbf{k}$.
- b. The vector $\frac{1}{3}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{1}{9}\mathbf{k}$ has length $\frac{\sqrt{26}}{9}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{26}}(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$.

- c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(3, 2, -1)| = \frac{\sqrt{26}}{9}$.
- d. The vector $\mathbf{u} = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is
- $$D_{\mathbf{u}}f(3, 2, -1) = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \cdot \left\langle \frac{1}{3}, -\frac{4}{9}, \frac{1}{9} \right\rangle = -\frac{7}{27}.$$

15.5.67

- a. False. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$.
- b. False. The gradient is a vector, so it does not make sense to say that it is positive.
- c. False. f is a function of three variables, so ∇f has three components.
- d. True. This is because $f_x = f_y = f_z = 0$.

15.5.68

- a. We can express $F(x, y, z) = f(g(x, y, z))$ where $g(x, y, z) = xyz$ and $f(w) = e^w$.
- b. Observe that $\nabla F(x, y, z) = e^{xyz}(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = e^{xyz}\nabla g = f'(g(x, y, z))\nabla g(x, y, z)$.

15.5.69 Observe that $\nabla f(x, y) = -8x\mathbf{i} - 2y\mathbf{j}$, so $\nabla f(1, 2) = -8\mathbf{i} - 4\mathbf{j}$. A vector perpendicular to $\nabla f(1, 2)$ is $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$, so the unit vectors perpendicular to $\nabla f(1, 2)$ are $\mathbf{u} = \pm \frac{1}{\sqrt{5}}(\mathbf{i} - 2\mathbf{j})$.

15.5.70 Observe that $\nabla f(x, y) = 2x\mathbf{i} - 8y\mathbf{j}$, so $\nabla f(4, 1) = 8\mathbf{i} - 8\mathbf{j}$. A vector perpendicular to $\nabla f(4, 1)$ is $\mathbf{v} = \mathbf{i} + \mathbf{j}$, so the unit vectors perpendicular to $\nabla f(4, 1)$ are $\mathbf{u} = \pm \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.

15.5.71 Observe that $\nabla f(x, y) = \frac{1}{\sqrt{3+2x^2+y^2}}(2x\mathbf{i} + y\mathbf{j})$, so $\nabla f(1, -2) = \frac{2}{3}(\mathbf{i} - \mathbf{j})$. A vector perpendicular to $\nabla f(1, -2)$ is $\mathbf{v} = \mathbf{i} + \mathbf{j}$, so the unit vectors perpendicular to $\nabla f(1, -2)$ are $\mathbf{u} = \pm \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.

15.5.72 Observe that $\nabla f(x, y) = -e^{1-xy}(y\mathbf{i} + x\mathbf{j})$, so $\nabla f(1, 0) = -e\mathbf{j}$, so the unit vectors perpendicular to $\nabla f(1, 0)$ are $\mathbf{u} = \pm \mathbf{i}$.

15.5.73 The function $f(x, y) = ax + by + c$ has $\nabla f(x, y) = a\mathbf{i} + b\mathbf{j}$, so the path in the xy -plane corresponding to the path of steepest ascent on the plane is given by $x = x_0 + at$, $y = y_0 + bt$.

15.5.74

- a. The function $z = f(x, y)$ satisfies the relation $z^2 = (x - a)^2 + (y - b)^2$, which is the equation of a cone with vertex at $(a, b, 0)$.
- b. We have $\nabla f(x, y) = \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}}((x-a)\mathbf{i} + (y-b)\mathbf{j})$, which implies that $|\nabla f(x, y)| = 1$ for all points $(x, y) \neq (a, b)$.
- c. At any point $(x, y) \neq (a, b)$, the greatest rate of change of f is 1, which occurs in the direction of the vector from (a, b) to (x, y) .

15.5.75

- a. We have $\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$, so $\nabla f(1, 1, 1) = \langle 2, 2, 2 \rangle$.
- b. Points (x, y, z) on the plane P satisfy $\langle x - 1, y - 1, z - 1 \rangle \cdot \langle 2, 2, 2 \rangle = 0$, so $x + y + z = 3$ is an equation for P .

15.5.76

- a. We have $\nabla f(x, y, z) = \langle -yz, -xz, -xy \rangle$, so $\nabla f(2, 2, 2) = \langle -4, -4, -4 \rangle$.
- b. Points (x, y, z) on the plane P satisfy $\langle x - 2, y - 2, z - 2 \rangle \cdot \langle -4, -4, -4 \rangle = 0$, so $x + y + z = 6$ is an equation for P .

15.5.77

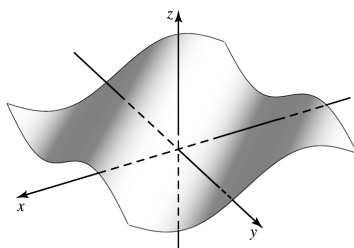
- a. We have $\nabla f(x, y, z) = e^{x+y-z} \langle 1, 1, -1 \rangle$, so $\nabla f(1, 1, 2) = \langle 1, 1, -1 \rangle$.
- b. Points (x, y, z) on the plane P satisfy $\langle x - 1, y - 1, z - 2 \rangle \cdot \langle 1, 1, -1 \rangle = 0$, so $x + y - z = 0$ is an equation for P .

15.5.78

- a. We have $\nabla f(x, y, z) = \langle y + z, x - z, x - y \rangle$, so $\nabla f(1, 1, 1) = \langle 2, 0, 0 \rangle$.
- b. Points (x, y, z) on the plane P satisfy $\langle x - 1, y - 1, z - 1 \rangle \cdot \langle 2, 0, 0 \rangle = 0$, so $x = 1$ is an equation for P .

15.5.79

a.



- b. The height function has gradient $\nabla z = \cos(x - y) \langle 1, -1 \rangle$, so the directions in which the height function has zero change are $\mathbf{v} = \pm \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.
- c. The direction $\mathbf{v}$ would be the opposite of the direction of ∇z , so $\mathbf{v} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.
- d. These answers are consistent with the graph in part (a).

15.5.80

- a. The height function has gradient $\nabla z = \cos(ax - b) \langle a, -b \rangle$, so the directions in which the height function has zero change are $\mathbf{v} = \pm \left\langle \frac{b}{\sqrt{a^2 + b^2}}, \frac{a}{\sqrt{a^2 + b^2}} \right\rangle$.
- b. The direction $\mathbf{v}$ would be the opposite of the direction of ∇z , so $\mathbf{v} = \left\langle -\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right\rangle$.

15.5.81

- a. Observe that $\varphi_x = -\frac{kQ}{r^2} r_x = -\frac{kQ}{r^2} \frac{x}{\sqrt{x^2 + y^2 + z^2}} = -\frac{kQx}{r^3}$; similarly $\varphi_y = -\frac{kQy}{r^3}$ and $\varphi_z = -\frac{kQz}{r^3}$.
Therefore $\mathbf{E}(x, y, z) = -\nabla \varphi(x, y, z) = kQ \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle$.
- b. We have $|\mathbf{E}| = \frac{kQ}{r^3} |\langle x, y, z \rangle| = \frac{kQ}{r^2}$. Therefore the magnitude of the electric field is inversely proportional to the square of the distance to the point charge.

15.5.82 Observe that

$$\varphi_x = \frac{GMm}{r^2} r_x = \frac{GMm}{r^2} \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{GMmx}{r^3}.$$

Similarly, $\varphi_y = \frac{GMmy}{r^3}$ and $\varphi_z = \frac{GMmz}{r^3}$. Therefore

$$\mathbf{F}(x, y, z) = -\nabla\varphi(x, y, z) = -\frac{GMm}{r^3} \langle x, y, z \rangle,$$

and we have $|\mathbf{F}| = \frac{GMm}{r^3} |\langle x, y, z \rangle| = \frac{GMm}{r^2}$. Therefore the magnitude of the gravitational field is inversely proportional to the square of the distance between the two objects.

15.5.83 We have $\langle u, v \rangle = \nabla\varphi = \langle \pi \cos \pi x \sin 2\pi y, 2\pi \sin \pi x \cos 2\pi y \rangle$.

15.5.84 Observe that $\nabla f(x, y) = A\mathbf{i} + B\mathbf{j}$, which is a constant vector. Therefore the direction and magnitude of the greatest directional derivative of $f(x, y, z)$ is the same at all points (x, y) .

15.5.85 We give the proofs for functions on $\mathbb{R}^2$. The proofs for functions on $\mathbb{R}^3$ are similar.

$$\text{a. } \nabla(cf) = \langle (cf)_x, (cf)_y \rangle = \langle cf_x, cf_y \rangle = c\nabla f.$$

$$\text{b. } \nabla(f+g) = \langle (f+g)_x, (f+g)_y \rangle = \langle f_x + g_x, f_y + g_y \rangle = \nabla f + \nabla g.$$

$$\text{c. } \nabla(fg) = \langle (fg)_x, (fg)_y \rangle = \langle f_x g + f g_x, f_y g + f g_y \rangle = (\nabla f)g + f\nabla g.$$

$$\text{d. } \nabla\left(\frac{f}{g}\right) = \left\langle \left(\frac{f}{g}\right)_x, \left(\frac{f}{g}\right)_y \right\rangle = \left\langle \frac{gf_x - fg_x}{g^2}, \frac{gf_y - fg_y}{g^2} \right\rangle = \frac{g(\nabla f) - f\nabla g}{g^2}.$$

$$\text{e. } \nabla(f \circ g) = \langle f'(g)g_x, f'(g)g_y \rangle = f'(g)\nabla g.$$

15.5.86 Using the product and chain rules from Exercise 85 gives $\nabla(xy \cos(xy)) = \cos(xy)\nabla(xy) + xy(-\sin(xy))\nabla(xy) = (\cos(xy) - xy \sin(xy))\langle y, x \rangle$.

15.5.87 Using the quotient rule from Exercise 85 gives

$$\begin{aligned} \nabla\left(\frac{x+y}{x^2+y^2}\right) &= \frac{(x^2+y^2)\nabla(x+y) - (x+y)\nabla(x^2+y^2)}{(x^2+y^2)^2} \\ &= \frac{1}{(x^2+y^2)^2} ((x^2+y^2)\langle 1, 1 \rangle - (x+y)\langle 2x, 2y \rangle) \\ &= \frac{1}{(x^2+y^2)^2} \langle y^2 - x^2 - 2xy, x^2 - y^2 - 2xy \rangle. \end{aligned}$$

15.5.88 Using the chain rule from Exercise 85 gives $\nabla(\ln(1+x^2+y^2)) = \frac{1}{1+x^2+y^2}\nabla(1+x^2+y^2) = \frac{1}{1+x^2+y^2}\langle 2x, 2y \rangle$.

15.5.89 Using the chain rule from Exercise 85 gives

$$\begin{aligned} \nabla\left(\sqrt{25-x^2-y^2-z^2}\right) &= \frac{1}{2\sqrt{25-x^2-y^2-z^2}}\nabla(25-x^2-y^2-z^2) \\ &= \frac{1}{2\sqrt{25-x^2-y^2-z^2}}\langle -2x, -2y, -2z \rangle \\ &= -\frac{1}{\sqrt{25-x^2-y^2-z^2}}\langle x, y, z \rangle. \end{aligned}$$

15.5.90 Using the product and chain rules from Exercise 85 gives

$$\begin{aligned}\nabla((x+y+z)e^{xyz}) &= e^{xyz}\nabla(x+y+z) + (x+y+z)\nabla e^{xyz} \\ &= e^{xyz}(\langle 1, 1, 1 \rangle + (x+y+z)\langle yz, xz, xy \rangle) \\ &= e^{xyz}\langle 1 + (x+y+z)yz, 1 + (x+y+z)xz, 1 + (x+y+z)xy \rangle.\end{aligned}$$

15.5.91 Using the quotient rule from Exercise 85 gives

$$\begin{aligned}\nabla\left(\frac{x+yz}{y+xz}\right) &= \frac{(y+xz)\nabla(x+yz) - (x+yz)\nabla(y+xz)}{(y+xz)^2} \\ &= \frac{1}{(y+xz)^2}((y+xz)\langle 1, z, y \rangle - (x+yz)\langle z, 1, x \rangle) \\ &= \frac{1}{(y+xz)^2}\langle y(1-z^2), x(z^2-1), y^2-x^2 \rangle.\end{aligned}$$

15.6 Tangent Planes and Linear Approximation

15.6.1 The gradient of F is a multiple of n .

15.6.2 Let $F(x, y, z) = z - f(x, y) = z - xy^2 - x^2y + 10$.

15.6.3 The tangent plane has equation $F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0$.

15.6.4 The tangent plane has equation $z = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b)$.

15.6.5 Multiply the change in x by f_x and the change in y by f_y and add both terms to $f(a, b)$.

15.6.6 The change in f is approximately $f_x(a, b)\Delta x + f_y(a, b)\Delta y$.

15.6.7 In terms of differentials, $dz = f_x(a, b)dx + f_y(a, b)dy$.

15.6.8 The differential for the function $w = f(x, y, z)$ at the point (a, b, c) is

$$dw = f_x(a, b, c)dx + f_y(a, b, c)dy + f_z(a, b, c)dz.$$

15.6.9 The tangent plane is $z = 5(x-1) - 3(y-2) + 4$ or $z = 5x - 3y + 5$.

15.6.10 The linear approximation is $z = 5x - 3y + 5$. Evaluating this at $(1.01, 1.99)$ gives 4.08.

15.6.11 $3(x-0) - (y-2) + 6(z-1) = 0$, or $3x - y + 6z = 4$.

15.6.12 Let $f(x, y, z) = 3x - (y-2) + 6(z-1)$. Then $f(0.1, 2, 0.99) = 0.24$.

15.6.13 $\nabla F = \langle 2x, 1, 1 \rangle$, so $\nabla F(1, 1, 1) = \langle 2, 1, 1 \rangle$. The tangent plane at $(1, 1, 1)$ has equation $2(x-1) + (y-1) + (z-1) = 0$, or $2x + y + z = 4$. Also, $\nabla F(2, 0, -1) = \langle 4, 1, 1 \rangle$, so the equation of the tangent plane there is $4(x-2) + y + (z+1) = 0$, or $4x + y + z = 7$.

15.6.14 $\nabla F = \langle 2x, 3y^2, 4z^3 \rangle$, so $\nabla F(1, 0, 1) = \langle 2, 0, 4 \rangle$. The tangent plane at $(1, 0, 1)$ is given by $2(x-1) + 4(z-1) = 0$, or $x + 2z = 3$. Also, $\nabla F(-1, 0, 1) = \langle -2, 0, 4 \rangle$, so the tangent plane there is given by $-2(x+1) + 4(z-1) = 0$, or $x - 2z = -3$.

15.6.15 $\nabla F = \langle y+z, x+z, x+y \rangle$, so $\nabla F(2, 2, 2) = \langle 4, 4, 4 \rangle$. The tangent plane at $(2, 2, 2)$ is given by $4(x-2) + 4(y-2) + 4(z-2) = 0$, or $x + y + z = 6$. At the point $(2, 0, 6)$ we have $\nabla F(2, 0, 6) = \langle 6, 8, 2 \rangle$, so the equation of the tangent plane is $6(x-2) + 8y + 2(z-6) = 0$, or $3x + 4y + z = 12$.

15.6.16 $\nabla F = \langle 2x, 2y, -2z \rangle$, so $\nabla F(3, 4, 5) = \langle 6, 8, -10 \rangle$ and the tangent plane at $(3, 4, 5)$ has equation $8(x-3) + 8(y-4) - 10(z-5) = 0$, or $3x + 4y - 5z = 0$. $\nabla F(-4, -3, 5) = \langle -8, -6, -10 \rangle$ so the tangent plane at $(-4, -3, 5)$ has equation $-8(x+4) - 6(y+3) - 10(z+5) = 0$ or $4x + 3y + 5z = 0$.

15.6.17 We have $f_x = -4x$, $f_y = -2y$, so the tangent plane at $(2, 2, -8)$ has equation $z = f(2, 2) + f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) = -8 - 8(x - 2) - 4(y - 2)$, or $z = -8x - 4y + 16$, and the equation of the tangent plane at $(-1, -1, 1)$ is $z = f(-1, -1) + f_x(-1, -1)(x + 1) + f_y(-1, -1)(y + 1) = 1 + 4(x + 1) + 2(y + 1)$, or $z = 4x + 2y + 7$.

15.6.18 We have $f_x = 4x$, $f_y = y$, so the tangent plane at $\left(-\frac{1}{2}, 1, 3\right)$ has equation $z = f\left(-\frac{1}{2}, 1\right) + f_x\left(-\frac{1}{2}, 1\right)\left(x + \frac{1}{2}\right) + f_y\left(-\frac{1}{2}, 1\right)(y - 1) = 3 - 2\left(x + \frac{1}{2}\right) + (y - 1)$, or $z = -2x + y + 1$, and the equation of the tangent plane at $(3, -2, 22)$ is $z = f(3, -2) + f_x(3, -2)(x - 3) + f_y(3, -2)(y + 2) = 22 + 12(x - 3) - 2(y + 2)$, or $z = 12x - 2y - 18$.

15.6.19 We have $f_x = ye^{xy}$, $f_y = xe^{xy}$, so the tangent plane at $(1, 0, 1)$ has equation $z = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y) = 1 + 0 + 1(y)$, or $z = 1 + y$. The equation of the tangent plane at $(0, 1, 1)$ is $z = 1 + 1(x) + 0(y - 1)$, or $z = 1 + x$.

15.6.20 We have $f_x = y \cos(xy)$ and $f_y = x \cos(xy)$, so the tangent plane at $(1, 0, 2)$ has equation $z = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y) = 2 + 0(x - 1) + 1(y)$, or $z = 2 + y$. The equation of the tangent plane at $(0, 5, 2)$ is $z = 2 + 5(x) + 0(y - 5)$, or $z = 2 + 5x$.

15.6.21 $\nabla F = \langle y \sin z, x \sin z, yz \cos z \rangle$, so $\nabla F\left(1, 2, \frac{\pi}{6}\right) = \left\langle 1, \frac{1}{2}, \sqrt{3} \right\rangle$ and the tangent plane at $\left(1, 2, \frac{\pi}{6}\right)$ has equation $(x - 1) + \frac{1}{2}(y - 2) + \sqrt{3}\left(z - \frac{\pi}{6}\right) = 0$, or $x + \frac{1}{2}y + \sqrt{3}z = 2 + \frac{\sqrt{3}\pi}{6}$. Also, $\nabla F\left(-2, -1, \frac{5\pi}{6}\right) = \left\langle -\frac{1}{2}, -1, -\sqrt{3} \right\rangle$, so the tangent plane at $\left(-2, -1, \frac{5\pi}{6}\right)$ has equation $-\frac{1}{2}(x + 2) - (y + 1) - \sqrt{3}\left(z - \frac{5\pi}{6}\right) = 0$ or $\frac{1}{2}x + y + \sqrt{3}z = \frac{5\sqrt{3}\pi}{6} - 2$.

15.6.22 $\nabla F = e^{xz} \langle yz^2, z, y(1 + x)z \rangle$, so $\nabla F(0, 2, 4) = \langle 32, 4, 8 \rangle$ and the tangent plane at $(0, 2, 4)$ has equation $32x + 4(y - 2) + 8(z - 4) = 0$, or $8x + y + 2z = 10$. Also, $\nabla F(0, -8, -1) = \langle -8, -1, 8 \rangle$, so the tangent plane at $(0, -8, -1)$ has equation $-8x - (y + 8) + 8(z + 1) = 0$ or $8x + y - 8z = 0$.

15.6.23 $\nabla F = \left\langle -\frac{x}{8}, -\frac{2y}{9}, 2z \right\rangle$, so $\nabla F(4, 3, -\sqrt{3}) = \left\langle -\frac{1}{2}, -\frac{2}{3}, -2\sqrt{3} \right\rangle$ and the tangent plane at $(4, 3, -\sqrt{3})$ has equation $-\frac{1}{2}(x - 4) - \frac{2}{3}(y - 3) - 2\sqrt{3}(z + \sqrt{3}) = 0$, or $\frac{1}{2}x + \frac{2}{3}y + 2\sqrt{3}z = -2$. Also, $\nabla F(-8, 9, \sqrt{14}) = \left\langle 1, -2, 2\sqrt{14} \right\rangle$, so the tangent plane at $(-8, 9, \sqrt{14})$ has equation $(x + 8) - 2(y - 9) + 2\sqrt{14}(z - \sqrt{14}) = 0$ or $x - 2y + 2\sqrt{14}z = 2$.

15.6.24 $\nabla F = \langle 2, 2y, -2z \rangle$, so $\nabla F(0, 1, 1) = \langle 2, 2, -2 \rangle$ and the tangent plane at $(0, 1, 1)$ has equation $2x + 2(y - 1) - 2(z - 1) = 0$, or $x + y - z = 0$. Also, $\nabla F(4, 1, -3) = \langle 2, 2, 6 \rangle$, so the tangent plane at $(4, 1, -3)$ has equation $2(x - 4) + 2(y - 1) + 6(z + 3) = 0$ or $x + y + 3z = -4$.

15.6.25 We have $f_x = (2x + x^2)e^{x-y}$, $f_y = -x^2e^{x-y}$, so the tangent plane at $(2, 2, 4)$ has equation $z = f(2, 2) + f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) = 4 + 8(x - 2) - 4(y - 2)$, or $z = 8x - 4y - 4$, and the equation of the tangent plane at $(-1, -1, 1)$ is $z = f(-1, -1) + f_x(-1, -1)(x + 1) + f_y(-1, -1)(y + 1) = 1 - (x + 1) - (y + 1)$, or $z = -x - y - 1$.

15.6.26 We have $f_x = \frac{y}{1 + xy}$, $f_y = \frac{x}{1 + xy}$, so the tangent plane at $(1, 2, \ln 3)$ has equation $z = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) = \ln 3 + \frac{2}{3}(x - 1) + \frac{1}{3}(y - 2)$, or $z = \frac{2}{3}x + \frac{1}{3}y + \ln 3 - \frac{4}{3}$, and the equation of the tangent plane at $(-2, -1, \ln 3)$ is $z = f(-2, -1) + f_x(-2, -1)(x + 2) + f_y(-2, -1)(y + 1) = \ln 3 - \frac{1}{3}(x + 2) - \frac{2}{3}(y + 1)$, or $z = -\frac{1}{3}x - \frac{2}{3}y + \ln 3 - \frac{4}{3}$.

15.6.27 We have $f_x = \frac{y^2 + 2xy - x^2}{(x^2 + y^2)^2}$, $f_y = \frac{y^2 - 2xy - x^2}{(x^2 + y^2)^2}$, so the tangent plane at $\left(1, 2, -\frac{1}{5}\right)$ has equation $z = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) = -\frac{1}{5} + \frac{7}{25}(x - 1) - \frac{1}{25}(y - 2)$, or $z = \frac{7}{25}x - \frac{1}{25}y - \frac{2}{5}$, and the equation of the tangent plane at $\left(2, -1, \frac{3}{5}\right)$ is $z = f(2, -1) + f_x(2, -1)(x - 2) + f_y(2, -1)(y + 1) = \frac{3}{5} - \frac{7}{25}(x - 2) + \frac{1}{25}(y + 1)$, or $z = -\frac{7}{25}x + \frac{1}{25}y + \frac{6}{5}$.

15.6.28 We have $f_x = -2\sin(x - y)$, $f_y = 2\sin(x - y)$, so the tangent plane at $\left(\frac{\pi}{6}, -\frac{\pi}{6}, 3\right)$ has equation $z = f\left(\frac{\pi}{6}, -\frac{\pi}{6}\right) + f_x\left(\frac{\pi}{6}, -\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + f_y\left(\frac{\pi}{6}, -\frac{\pi}{6}\right)\left(y + \frac{\pi}{6}\right) = 3 - \sqrt{3}\left(x - \frac{\pi}{6}\right) + \sqrt{3}\left(y + \frac{\pi}{6}\right)$, or $z = \sqrt{3}(y - x) + 3 + \frac{\sqrt{3}\pi}{3}$, and the equation of the tangent plane at $\left(\frac{\pi}{3}, \frac{\pi}{3}, 4\right)$ is $z = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) + f_x\left(\frac{\pi}{3}, \frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f_y\left(\frac{\pi}{3}, \frac{\pi}{3}\right)\left(y - \frac{\pi}{3}\right) = 4$, or $z = 4$.

15.6.29 Let $f(x, y) = \tan^{-1}(xy)$; then $f_x(x, y) = \frac{y}{1 + (xy)^2}$, $f_y(x, y) = \frac{x}{1 + (xy)^2}$ and the tangent plane at $\left(1, 1, \frac{\pi}{4}\right)$ has equation $z = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) + f(1, 1) = \frac{1}{2}x + \frac{1}{2}y + \frac{\pi}{4} - 1$.

15.6.30 Let $f(x, y) = \tan^{-1}(x + y)$; then $f_x(x, y) = f_y(x, y) = \frac{1}{1 + (x + y)^2}$ and the tangent plane at $(0, 0, 0)$ has equation $z = f_x(0, 0)x + f_y(0, 0)y + f(0, 0) = x + y$.

15.6.31 The branch of the surface $\sin(xyz) = \frac{1}{2}$ at the point $\left(\pi, 1, \frac{1}{6}\right)$ can be described more simply by the equation $F(x, y, z) = xyz = \frac{\pi}{6}$. We have $F_x(x, y, z) = yz$, $F_y(x, y, z) = xz$, $F_z(x, y, z) = xy$ so the tangent plane at $\left(\pi, 1, \frac{1}{6}\right)$ has equation $F_x\left(\pi, 1, \frac{1}{6}\right)(x - \pi) + F_y\left(\pi, 1, \frac{1}{6}\right)(y - 1) + F_z\left(\pi, 1, \frac{1}{6}\right)\left(z - \frac{1}{6}\right) = 0$, or $\frac{1}{6}(x - \pi) + \frac{\pi}{6}(y - 1) + \pi\left(z - \frac{1}{6}\right) = 0$.

15.6.32 Rewrite this equation as $x + z = 2(y - z)$ or equivalently, $x - 2y + 3z = 0$; therefore the surface is a plane and hence is identical to its tangent plane at any point.

15.6.33

- We have $f_x = y + 1$, $f_y = x - 1$, so the linear approximation for f at $(2, 3)$ is $L(x, y) = f(2, 3) + f_x(2, 3)(x - 2) + f_y(2, 3)(y - 3) = 5 + 4(x - 2) + (y - 3)$, or $L(x, y) = 4x + y - 6$.
- We have $L(2.1, 2.99) = 5.39$.

15.6.34

- We have $f_x = -8x$, $f_y = -16y$, so the linear approximation for f at $(-1, 4)$ is $L(x, y) = f(-1, 4) + f_x(-1, 4)(x + 1) + f_y(-1, 4)(y - 4) = -120 + 8(x + 1) - 64(y - 4)$ or $L(x, y) = 8x - 64y + 144$.
- We have $L(-1.05, 3.95) = -117.20$.

15.6.35

- We have $f_x = -2x$, $f_y = 4y$, so the linear approximation for f at $(3, -1)$ is $L(x, y) = f(3, -1) + f_x(3, -1)(x - 3) + f_y(3, -1)(y + 1) = -7 - 6(x - 3) - 4(y + 1)$ or $L(x, y) = -6x - 4y + 7$.
- We have $L(3.1, -1.04) = -7.44$.

15.6.36

- a. We have $f_x = \frac{x}{\sqrt{x^2 + y^2}}$, $f_y = \frac{y}{\sqrt{x^2 + y^2}}$, so the linear approximation for f at $(3, -4)$ is $L(x, y) = f(3, -4) + f_x(3, -4)(x - 3) + f_y(3, -4)(y + 4) = 5 + \frac{3}{5}(x - 3) + \frac{4}{5}(y + 4)$, or $L(x, y) = \frac{3}{5}x + \frac{4}{5}y$.
- b. We have $L(3.06, -3.92) = 4.972$.

15.6.37

- a. $f_x = \frac{1}{1 + x + y + 2z}$, $f_y = \frac{1}{1 + x + y + 2z}$, $f_z = \frac{2}{1 + x + y + 2z}$, so at $(0, 0, 0)$ we have $f_x = 1$, $f_y = 1$, and $f_z = 2$. So $L(x, y, z) = x + y + 2z$.
- b. $L(0.1, -0.2, 0.2) = 0.1 - 0.2 + 0.4 = 0.3$.

15.6.38

- a. $f_x = \frac{x - z - (x + y)}{(x - z)^2} = \frac{-z - y}{(x - z)^2}$, $f_y = \frac{1}{x - z}$, and $f_z = \frac{x + y}{(x - z)^2}$, so at $(3, 2, 4)$ we have $f_x = -6$, $f_y = -1$, and $f_z = 5$. So $L(x, y, z) = -6(x - 3) - 1(y - 2) + 5(z - 4) - 5 = -6x - y + 5z - 5$.
- b. $L(2.95, 2.05, 4.02) = -6(2.95) - (2.5) + 5(4.02) - 5 = -5.1$.

15.6.39 We have $f_x = 2 - 2y$, $f_y = -3 - 2x$, and $dx = 0.1$, $dy = -0.1$, so $dz = f_x(1, 4)dx + f_y(1, 4)dy = -6dx - 5dy = -0.1$.

15.6.40 We have $f_x = -2x$, $f_y = 6y$, and $dx = -0.05$, $dy = -0.1$, so $dz = f_x(-1, 2)dx + f_y(-1, 2)dy = 2dx + 12dy = -1.30$.

15.6.41 We have $f_x = e^{x+y}$, $f_y = e^{x+y}$, and $dx = 0.1$, $dy = -0.05$, so $dz = f_x(0, 0)dx + f_y(0, 0)dy = dx + dy = 0.05$.

15.6.42 We have $f_x = \frac{1}{1 + x + y}$, $f_y = \frac{1}{1 + x + y}$, and $dx = -0.1$, $dy = 0.03$, so $dz = f_x(0, 0)dx + f_y(0, 0)dy = dx + dy = -0.07$.

15.6.43

- a. If r increases and R decreases then $R^2 - r^2$ decreases, so S decreases.
- b. If both R and r increase, then it is impossible to say whether $R^2 - r^2$ increases or decreases.
- c. We have $dS = 8\pi^2(RdR - rdr) = 8\pi^2(5.50 \cdot 0.15 - 3 \cdot 0.05) = 5.4\pi^2 \approx 53.296$.
- d. We have $dS = 8\pi^2(RdR - rdr) = 8\pi^2(7 \cdot 0.04 - 3 \cdot (-0.05)) = 3.44\pi^2 \approx 33.951$.
- e. The surface area is approximately unchanged when $RdR = rdr$.

15.6.44

- a. We have $dV = \frac{\pi}{3}(2rhdr + r^2dh) = \frac{\pi}{3}(2 \cdot 6.5 \cdot 4.2 \cdot 0.1 + 6.5^2(-0.05)) \approx 3.505$.
- b. We have $dV = \frac{\pi}{3}(2rhdr + r^2dh) = \frac{\pi}{3}(2 \cdot 5.4 \cdot 12(-0.03) + 6.5^2(-0.04)) \approx -5.293$.

15.6.45 Observe that $dA = \pi(bda + adb)$ so $\frac{dA}{A} = \frac{da}{a} + \frac{db}{b}$, and hence the percentage increase in the area is approximately $2\% + 1.5\% = 3.5\%$.

15.6.46 Observe that $dV = (\pi/2)(2rhdr + r^2dh)$ so $\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$, and hence the percentage decrease in the area is approximately $2(-1.5\%) + 2.2\% = -0.8\%$.

15.6.47 We have $dw = (y^2 + 2xz) dx + (2xy + z^2) dy + (x^2 + 2yz) dz$.

15.6.48 We have $dw = \cos(x + y - z)(dx + dy - dz)$.

15.6.49 We have $dw = \frac{dx}{y+z} - \frac{u+x}{(y+z)^2} dy - \frac{u+x}{(y+z)^2} dz + \frac{du}{y+z}$.

15.6.50 We have $dw = \frac{q}{rs} dp + \frac{p}{rs} dq - \frac{pq}{r^2 s} dr - \frac{pq}{rs^2} ds$.

15.6.51

a. Observe that

$$2cdc = (2a - 2b \cos \theta) da + (2b - 2a \cos \theta) db + 2ab \sin \theta d\theta$$

so $dc = \frac{a - b \cos \theta}{c} da + \frac{b - a \cos \theta}{c} db + \frac{ab \sin \theta}{c} d\theta$. We have $a = 2$, $b = 4$, $\theta = \frac{\pi}{3}$ which gives $c = \sqrt{12}$, and $da = 0.03$, $db = -0.04$, $d\theta = \frac{\pi}{90}$; substituting in the equation above gives $dc \approx 0.035$.

b. We have $a = 2$, $b = 4$, $da = 0.03$, $db = -0.04$, $d\theta = 0$, so $dc = -\frac{0.01 + 0.04 \cos \theta}{c}$; comparing the cases $\theta = \frac{\pi}{20}$ and $\theta = \frac{9\pi}{20}$ we see that c is smaller and $\cos \theta$ is larger in the first case; therefore the change in c is greater when $\theta = \frac{\pi}{20}$.

15.6.52

a. The cost function is obtained by multiplying the distance L by the cost per mile, which is $\frac{p}{m}$.

b. We have $C_L = \frac{p}{m}$, $C_m = -\frac{Lp}{m^2}$, $C_p = \frac{L}{m}$. This shows that C is an increasing function of L and p and a decreasing function of m , which makes sense.

c. We have $dC = \frac{p}{m} dL - \frac{Lp}{m^2} dm + \frac{L}{m} dp \approx \10.29 .

d. Observe that $\frac{dC}{C} = \frac{dL}{L} - \frac{dm}{m} + \frac{dp}{p}$; therefore the cost is equally sensitive to a 1% change in any of the variables.

15.6.53

a. True. This is because the function $F(x, y, z) = x^2 + z^2$ has $F_y = 0$.

b. True. As $z > 0$ decreases, $\frac{1}{z}$ increases.

c. False. The gradient $\nabla F(a, b, c)$ is perpendicular to the tangent plane for the surface $F(x, y, z) = 0$ at (a, b, c) .

15.6.54 Let $F(x, y, z) = x^2 + 2y^2 + z^2 - 2x - 2z - 2$; then $F_x(x, y, z) = 2x - 2$, $F_y(x, y, z) = 4y$. The tangent plane to the surface $F(x, y, z) = 0$ at (a, b, c) is horizontal if and only if $F_x(a, b, c) = F_y(a, b, c) = 0$, which gives $a = 1$, $b = 0$. This implies $c^2 - 2c - 3 = 0$, which gives $c = 3, -1$ so the points are $(1, 0, 3)$ and $(1, 0, -1)$.

15.6.55 Let $F(x, y, z) = x^2 + y^2 - z^2 - 2x + 2y + 3$; then $F_x(x, y, z) = 2x - 2$, $F_y(x, y, z) = 2y + 2$. The tangent plane to the surface $F(x, y, z) = 0$ at (a, b, c) is horizontal if and only if $F_x(a, b, c) = F_y(a, b, c) = 0$, which gives $a = 1$, $b = -1$. This implies $c^2 = 1$, so the points are $(1, -1, 1)$ and $(1, -1, -1)$.

15.6.56 Let $f(x, y) = \sin(x - y)$; then $f_x(x, y) = \cos(x - y)$, $f_y(x, y) = -\cos(x - y)$ so the tangent plane at $(a, b, \sin(a - b))$ is horizontal at all points where $b - a = \frac{\pi}{2} + k\pi$ for some integer k .

15.6.57 Let $f(x, y) = \cos 2x \sin y$; then $f_x(x, y) = -2 \sin 2x \sin y$, $f_y(x, y) = \cos 2x \cos y$; so the tangent plane at $(a, b, \cos 2a \sin b)$ is horizontal at all points where $\sin 2a = \cos b = 0$ or $\sin b = \cos 2a = 0$. In the region $-\pi \leq x \leq \pi$, $-\pi \leq y \leq \pi$ the points are $a = 0, \pm \frac{\pi}{2}, \pm \pi$ and $b = \pm \frac{\pi}{2}$, or $a = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$ and $b = 0, \pm \pi$.

15.6.58

- a. Apply logarithmic differentiation to $A^2 = s(s-a)(s-b)(s-c)$: $\frac{2A_a}{A} = \frac{1}{s} + \frac{-1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}$, so $A_a = \frac{A}{4} \left(\frac{1}{s} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s-a} \right)$, and similarly $A_b = \frac{A}{4} \left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-c} - \frac{1}{s-b} \right)$, and $A_c = \frac{A}{4} \left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c} \right)$.
- b. The change is approximately $dA = A_a da + A_b db + A_c dc \approx -0.5573$.
- c. In this case $s = \frac{3}{2}a$, $s-a = s-b = s-c = \frac{1}{2}a$, and $A_a = A_b = A_c = \frac{2A}{3a}$ so if all sides change by the same amount da , then $\frac{dA}{A} = 3 \cdot \frac{2}{3a} da = 2 \frac{da}{a}$. Therefore if all sides increase by $p\%$, the area will increase by approximately $2p\%$. (This result can also be obtained more directly using the formula $A = \frac{\sqrt{3}a^2}{4}$ for the area of an equilateral triangle with side a .)

15.6.59

- a. We have $S_r = \pi \sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}$, $S_h = \frac{\pi r h}{\sqrt{r^2 + h^2}}$; using the values $r = 2.5$, $h = 0.6$, $dr = 0.05$, $dh = -0.02$ gives $dS = S_r dr + S_h dh = \frac{\pi}{\sqrt{r^2 + h^2}} ((2r^2 + h^2) dr + r h dh) \approx 0.749$.
- b. If $r = 100$, $h = 200$ then $dS = 40\sqrt{5}\pi(3dr + dh)$, so the surface area is more sensitive to small changes in r .

15.6.60

- a. The vector $\mathbf{n}_1 = \langle 1, 1, -1 \rangle$ is normal to this plane.
- b. Rewrite the equation of the paraboloid as $F(x, y, z) = z - x^2 - 3y^2 = 0$; then $\nabla F(2, 1, 7) = \langle 4, 6, -1 \rangle$, so $\mathbf{n}_2 = \langle 4, 6, -1 \rangle$ is normal to the tangent plane.
- c. The line tangent to C lies in both of these planes and hence must be orthogonal to both vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Therefore a direction vector for this line is given by $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 4 & 6 & -1 \end{vmatrix} = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.
- d. The tangent line has parametric equation $\mathbf{r}(t) = \langle 2, 1, 7 \rangle + t\langle 5, -3, 2 \rangle = \langle 2 + 5t, 1 - 3t, 7 + 2t \rangle$, or $x = 2 + 5t$, $y = 1 - 3t$, $z = 7 + 2t$.

15.6.61

- a. The differential of A is given by $dA = \frac{dx}{y} - \frac{xdy}{y^2}$; substituting $x = 60$, $y = 175$, $dx = 2$ and $dy = 5$ gives $dA = \frac{2}{1225} \approx 0.00163$.
- b. If the batter fails to get a hit, the average decreases by $\frac{x}{y} - \frac{x}{y+1} = \frac{x}{y(y+1)} = \frac{A}{y+1}$, whereas if the batter gets a hit, the average increases by $\frac{x+1}{y+1} - \frac{x}{y} = \frac{y-x}{y(y+1)} = \frac{1-A}{y+1}$. If $A = 0.350$ the second of these quantities is larger so the answer is no; the batting average changes more if the batter gets a hit than if he fails to get a hit.

- c. The answer depends on whether A is less than or greater than 0.500.

15.6.62 The volume of the tank is $V = \frac{\pi r^2 h}{3}$, so $dV = \frac{2\pi}{3} r h dr + \frac{\pi}{3} r^2 dh$ or equivalently $\frac{dV}{V} = 2 \frac{dr}{r} + \frac{dh}{h}$. If the water level drops from 2.00 to 1.95, then the radius changes proportionately so $\frac{dV}{V} = 3 \frac{dh}{h} = -0.075$, which gives $dV \approx -0.039 \text{ m}^3$.

15.6.63

- a. The centerline velocity is given by $V = \frac{R^2}{L}$ so $dV = \frac{2R}{L} dR - \frac{R^2}{L^2} dL$; evaluate this with $R = 3$, $dR = 0.05$, $L = 50$, $dL = 0.5$ to obtain $dV = \frac{21}{5000} = 0.0042 \text{ cm}^3$.
- b. Rewrite the formula for dV as $\frac{dV}{V} = \frac{2dR}{R} - \frac{dL}{L}$; hence if R decreases 1% and L increases 2%, then V will decrease by approximately 4%.
- c. If the radius of the cylinder increases by $p\%$, then the length of the cylinder must decrease by approximately $2p\%$ in order for the velocity to remain constant.

15.6.64

- a. Let $z = xy$; then $dz = ydx + xdy$, so the absolute error is at most $(|x| + |y|) \cdot 10^{-16}$ and the percentage error at most $\frac{100 \cdot (|x| + |y|) \cdot 10^{-16}}{|xy|} \% = \left(\frac{1}{|x|} + \frac{1}{|y|} \right) \cdot 10^{-14} \%$.
- b. Let $z = \frac{x}{y}$; then $dz = \frac{ydx - xdy}{y^2}$, so the absolute error is at most $\left(\frac{|x| + |y|}{y^2} \right) \cdot 10^{-16}$ and the percentage error at most $100 \cdot \left(\frac{|x| + |y|}{y^2} \right) \cdot \frac{|y|}{|x|} \cdot 10^{-16} \% = \left(\frac{1}{|x|} + \frac{1}{|y|} \right) \cdot 10^{-14} \%$.
- c. Let $w = xyz$; then $dw = yzdx + xzdy + xydz$, so the absolute error is at most $(|xy| + |yz| + |xz|) \cdot 10^{-16}$ and the percentage error at most

$$\frac{100 \cdot (|xy| + |yz| + |xz|) \cdot 10^{-16}}{|xyz|} \% = \left(\frac{1}{|x|} + \frac{1}{|y|} + \frac{1}{|z|} \right) \cdot 10^{-14} \%.$$

- d. Let $z = \frac{x/y}{z} = \frac{x}{yz}$; then $dz = \frac{yzdx - xzdy - xydz}{y^2 z^2}$, so the absolute error is at most
- $$\left(\frac{|xy| + |yz| + |xz|}{y^2 z^2} \right) \cdot 10^{-16}$$

and the percentage error at most

$$100 \cdot \left(\frac{|xy| + |yz| + |xz|}{y^2 z^2} \right) \cdot \frac{|yz|}{|x|} \cdot 10^{-16} \% = \left(\frac{1}{|x|} + \frac{1}{|y|} + \frac{1}{|z|} \right) \cdot 10^{-14} \%.$$

15.6.65

- a. We have $f_r = n(1-r)^{n-1}$ and $f_n = -(1-r)^n \ln(1-r)$.
- b. We have $\Delta P \approx f(20, 0.1) \cdot 0.01 = 20 \cdot (0.9)^{19} (0.01) \approx 0.027$.
- c. We have $\Delta P \approx f(20, 0.9) \cdot 0.01 = 20 \cdot (0.1)^{19} (0.01) \approx 2 \times 10^{-20}$.
- d. Small changes in the flu rate have a greater effect on the probability of catching the flu when the flu rate is small compared to when the flu rate is large.

15.6.66

- a. We have $R = \frac{R_1 R_2}{R_1 + R_2}$ so $dR = \frac{R_2 dR_1 + R_1 dR_2}{(R_1 + R_2)^2} = R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2} \right)$. Substituting $R_1 = 2$, $dR_1 = 0.05$, $R_2 = 3$, $dR_2 = -0.05$ gives $dR = 0.01$ ohms.
- b. True; if $R_1 = R_2$ and $dR_1 = -dR_2$ then $dR = 0$.
- c. Yes, because in this case $dR > 0$.
- d. The formula in (a) shows that if $R_1 > R_2$ then R is more sensitive to changes in R_2 , so if R_1 increases by the same small amount that R_2 decreases, then R will decrease.

15.6.67 From the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ we obtain $-\frac{1}{R^2} \frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}$, which implies that $\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}$, with similar formulas for the other partials. Hence $dR = R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2} + \frac{dR_3}{R_3^2} \right)$. Substituting the values $R_1 = 2$, $dR_1 = 0.05$, $R_2 = 3$, $dR_2 = -0.05$, $R_3 = 1.5$, $dR_3 = 0.05$ gives $R = \frac{2}{3}$ and $dR = \frac{7}{540} \approx 0.0130$ ohms.

15.6.68 Observe that $f_x = ax^{a-1}y^b$ and $f_y = bx^ay^{b-1}$, so $\frac{dz}{z} = \frac{ax^{a-1}y^b dx + bx^ay^{b-1} dy}{x^a y^b} = a \frac{dx}{x} + b \frac{dy}{y}$.

15.6.69

- a. Suppose f is a function of x and y ; then $d(\ln f) = (\ln f)_x dx + (\ln f)_y dy = \frac{f_x}{f} dx + \frac{f_y}{f} dy = \frac{df}{f}$.
- b. The absolute change in $\ln f$ is approximately $d(\ln f)$ and the relative change in f is approximately $\frac{df}{f}$; from part (a) these agree.
- c. Observe that $df = ydx + xdy$, so $\frac{df}{f} = \frac{dx}{x} + \frac{dy}{y}$.
- d. Observe that $df = \frac{ydx - xdy}{y^2}$, so $\frac{df}{f} = \frac{dx}{x} - \frac{dy}{y}$.
- e. If $f = x_1 x_2 \cdots x_n$ then $\ln f = \ln x_1 + \ln x_2 + \cdots + \ln x_n$ and therefore $\frac{df}{f} = \frac{dx_1}{x_1} + \frac{dx_2}{x_2} + \cdots + \frac{dx_n}{x_n}$.

15.6.70

- a. Let $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$; then $\nabla F = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \rangle$ so the tangent plane at (p, q, r) has equation $\frac{p}{a^2}(x - p) + \frac{q}{b^2}(y - q) + \frac{r}{c^2}(z - r) = 0$, which simplifies to $\frac{px}{a^2} + \frac{qy}{b^2} + \frac{rz}{c^2} = 1$.
- b. The vectors $\langle A, B, C \rangle$ and $\langle \frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2} \rangle$ must be proportional; therefore we can express $\langle p, q, r \rangle = \lambda \langle Aa^2, Bb^2, Cc^2 \rangle$ for some scalar λ . Substituting in the equation of the ellipsoid gives $\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = \lambda^2 (A^2 a^2 + B^2 b^2 + C^2 c^2) = 1$, so $\lambda = \pm \frac{1}{m}$ and $(p, q, r) = \pm (Aa^2, Bb^2, Cc^2)$ where $m = \sqrt{A^2 a^2 + B^2 b^2 + C^2 c^2}$.
- c. The distance of the plane $Ax + By + Cz = 1$ to the origin is $d = \frac{1}{\sqrt{A^2 + B^2 + C^2}} = h$ (this formula is derived in a previous exercise).
- d. The tangent plane at $(p, q, r) = \pm (Aa^2, Bb^2, Cc^2)$ has equation $Ax + By + Cz = \pm m$, which has distance to the origin $d = \frac{m}{\sqrt{A^2 + B^2 + C^2}} = mh$ using the formula from part c.
- e. The plane P does not intersect the ellipsoid if and only if the two tangent planes parallel to P are closer to the origin than P ; this is equivalent to the condition $m < 1$.

15.7 Maximum/Minimum Problems

15.7.1 It is locally the highest point on the surface; you cannot get to a higher point in any direction.

15.7.2 The surface looks similar to the hyperbolic paraboloid $z = x^2 - y^2$ at the origin.

15.7.3 The partial derivatives are both zero or one or both do not exist.

15.7.4 No; for example f may have a saddle point at (a, b) .

15.7.5 The discriminant of f at (a, b) is the determinant given by $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$.

15.7.6 The second derivative test may be used to determine whether a critical point is a local maximum, minimum or saddle point (see Theorem 15.14).

15.7.7 The function f has an absolute minimum value at $(a, b) \in R$ if $f(x, y) \geq f(a, b)$ for all $(x, y) \in R$.

15.7.8 Determine the values of f at all critical points in the closed bounded domain R , and find the maximum/minimum values of f on the boundary of R ; the greatest/smallest of these values is the absolute maximum/minimum of f on R .

15.7.9 The discriminant is $5(3) - (-4)^2 = 15 - 16 = -1 < 0$, so this is a saddle point.

15.7.10 The discriminant is $(-6)(-3) - 4^2 = 18 - 16 = 2 > 0$, and $f_{xx}(0, 0) < 0$, so f has a local maximum at $(0, 0)$.

15.7.11 The discriminant is $8 \cdot 5 - 36 = 4 > 0$, and $f_{xx}(0, 0) > 0$, so f has a local minimum at $(0, 0)$.

15.7.12 The discriminant is $(-9)(-4) - (-6)^2 = 0$, so the test is inconclusive.

15.7.13 We have $f_x = 6x$, $f_y = -8y$ so $(0, 0)$ is the only critical point of f .

15.7.14 We have $f_x = 2x - 6$, $f_y = 2y + 8$ so $(3, -4)$ is the only critical point of f .

15.7.15 $f_x = 6x$ and $f_y = 3 - 3y^2$. So the critical points are $(0, 1)$ and $(0, -1)$.

15.7.16 $f_x = 3x^2 - 12 = 3(x^2 - 4)$ and $f_y = 12y$, so the critical points are $(2, 0)$ and $(-2, 0)$.

15.7.17 We have $f_x = 4x^3 - 16y$, $f_y = 4y^3 - 16x$; solving $f_x = f_y = 0$ gives $y = \frac{x^3}{4}$ and $x = \frac{y^3}{4}$; therefore $x = \frac{x^9}{4^4}$ which gives $x = 0, \pm 2$, and the critical points are $(0, 0)$, $(2, 2)$ and $(-2, -2)$.

15.7.18 We have $f_x = x^2 + 3y$, $f_y = -y^2 + 3x$; solving $f_x = f_y = 0$ gives $y = -\frac{x^2}{3}$ and $x = \frac{y^2}{3}$; therefore $x = \frac{x^4}{3^3}$ which gives $x = 0, 3$, and the critical points are $(0, 0)$ and $(3, -3)$.

15.7.19 We have $f_x = 4x^3 - 4x$ and $f_y = 2y - 4$; solving $f_x = f_y = 0$ gives $x = 0, \pm 1$ and $y = 2$; and the critical points are $(0, 2)$ and $(\pm 1, 2)$.

15.7.20 $f_x = 3x^2 + 6y - 6$ and $f_y = 6x + 2y - 2$. Setting these both equal to 0 and subtracting $f_x - 3f_y$ gives $3x^2 - 18x = 0$. So $x = 0$ or $x = 6$. When $x = 0$ we have $6y - 6 = 0$ so $y = 1$, and when $x = 6$ we have $36 + 2y - 2 = 0$, so $y = -17$. The critical points are $(0, 1)$ and $(6, -17)$.

15.7.21 $f_x = 6y + 2x - 6$ and $f_y = 3y^2 + 6x - 18$. Setting these both equal to 0 and subtracting $f_y - 3f_x$ gives $3y^2 - 18y = 0$. So $y = 0$ or $y = 6$. When $y = 0$ we have $2x - 6 = 0$ so $x = 3$, and when $y = 6$ we have $36 + 2x - 6 = 0$, so $x = -15$. The critical points are $(3, 0)$ and $(-15, 6)$.

15.7.22 $f_x = e^{8x^2y^2-24x^2-8xy^4}(16xy^2 - 48x - 8y^4)$, and $f_y = e^{8x^2y^2-24x^2-8xy^4}(16x^2y - 32xy^3)$. Other than at the origin, $f_y = 0$ for $x - 2y^2 = 0$. But if $x = 2y^2$, then $f_x = 0$ for $16xy^2 - 48x - 8y^4 = 0$, or $4y^4 - 12y^2 - y^4 = 0$, or $3y^2(y^2 - 4) = 0$. So $y = \pm 2$, and the critical points are $(0, 0)$, $(8, 2)$, and $(8, -2)$.

15.7.23 We have $f_x = -8x$, $f_y = 16y$; therefore, $(0, 0)$ is the only critical point. We also have $f_{xx} = -8$, $f_{yy} = 16$ and $f_{xy} = 0$; hence $D(0, 0) = -8 \cdot 16 < 0$, which by the Second Derivative Test implies that f has a saddle point at $(0, 0)$.

15.7.24 We have $f_x = 4x^3 - 4$, $f_y = 4y^3 - 32$; therefore, $(1, 2)$ is the only critical point. We also have $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = 0$; hence $D(1, 2) = 12 \cdot 48 > 0$ and $f_{xx}(1, 2) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(1, 2)$.

15.7.25 We have $f_x = 4x$, $f_y = 6y$; therefore $(0, 0)$ is the only critical point. We also have $f_{xx} = 4$, $f_{yy} = 6$ and $f_{xy} = 0$; hence $D(0, 0) = 4 \cdot 6 - 0^2 = 24 > 0$ and $f_{xx}(0, 0) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(0, 0)$.

15.7.26 We have $f_x = (1 - x)ye^{-x-y}$, $f_y = (1 - y)xe^{-x-y}$; therefore, the critical points are $(0, 0)$ and $(1, 1)$. We also have $f_{xx} = (x - 2)ye^{-x-y}$, $f_{yy} = (y - 2)xe^{-x-y}$ and $f_{xy} = (1 - x)(1 - y)e^{-x-y}$; hence $D(0, 0) = -1 < 0$ so f has a saddle point at $(0, 0)$; and $D(1, 1) = e^{-2} > 0$, $f_{xx}(1, 1) = -e^{-2} < 0$ so f has a local maximum at $(1, 1)$.

15.7.27 We have $f_x = 4x^3 - 4y$, $f_y = 4y - 4x$; therefore, the critical points satisfy $y = x$ and $y = x^3$, which gives $x^3 = x$ and therefore $x = 0, \pm 1$, so the critical points are $(0, 0)$, $(1, 1)$ and $(-1, -1)$. We also have $f_{xx} = 12x^2$, $f_{yy} = 4$ and $f_{xy} = -4$; hence $D(x, y) = 48x^2 - 16$. Thus $D(0, 0) = -16 < 0$, so f has a saddle point at $(0, 0)$; and $D(1, 1) = D(-1, -1) = 32 > 0$, $f_{xx}(1, 1) = f_{xx}(-1, -1) = 12 > 0$ so f has a local minimum at $(1, 1)$ and $(-1, -1)$.

15.7.28 We have $f_x = 8(4x - 1)$, $f_y = 4(2y + 4)$; therefore, $\left(\frac{1}{4}, -2\right)$ is the only critical point. We also have $f_{xx} = 32$, $f_{yy} = 8$ and $f_{xy} = 0$; hence $D\left(\frac{1}{4}, -2\right) = 32 \cdot 8 > 0$ and $f_{xx}\left(\frac{1}{4}, -2\right) > 0$, which by the Second Derivative Test implies that f has a local minimum at $\left(\frac{1}{4}, -2\right)$.

15.7.29 Observe that $f_x = 4x^3$, $f_y = 12y^3$ and $f_{xx} = 12x^2$, $f_{yy} = 36y^2$ and $f_{xy} = 0$; therefore $D(0, 0) = 0$ and the Second Derivative Test is inconclusive. Observe that both x^4 and $3y^4$ have absolute minima at 0; therefore, f has an absolute minimum of 4 at $(0, 0)$.

15.7.30 Observe that $f_x = 4x^3y^2$, $f_y = 2x^4y$ and $f_{xx} = 12x^2y^2$, $f_{yy} = 2x^4$ and $f_{xy} = 8x^3y$. Note that every point on the x -axis and on the y -axis is a critical point, and that $D = 24x^6y^2 - 64x^6y^2 = -40x^6y^2$, which has value 0 at every point on the axes, so the Second Derivative Test is inconclusive. Note that the function is always nonnegative, so the value of 0 along the coordinate axes represents the minimum value.

15.7.31 Note that $f(x, y)$ has the same critical points as the simpler function

$$g(x, y) = x^2 + y^2 - 4x + 5 = (x - 2)^2 + y^2 + 1.$$

We have $g_x = 2(x - 2)$, $g_y = 2y$; therefore, $(2, 0)$ is the only critical point. We also have $g_{xx} = 2$, $g_{yy} = 2$ and $g_{xy} = 0$; hence $D(2, 0) = 4 > 0$ and $g_{xx}(2, 0) > 0$, which by the Second Derivative Test implies that g (and hence f) has a local minimum at $(2, 0)$.

15.7.32 We have $f_x = \frac{y}{1 + x^2y^2}$, $f_y = \frac{x}{1 + x^2y^2}$; therefore, $(0, 0)$ is the only critical point. We also have $f_{xx} = -\frac{2xy^3}{(1 + x^2y^2)^2}$, $f_{yy} = -\frac{2x^3y}{(1 + x^2y^2)^2}$ and $f_{xy} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2}$; hence $D(0, 0) = -1 < 0$, which by the Second Derivative Test implies that f has a saddle point at $(0, 0)$.

15.7.33 We have $f_x = 2(1 - 2x^2)ye^{-x^2-y^2}$, $f_y = 2(1 - 2y^2)xe^{-x^2-y^2}$. Therefore, the critical points are $(0, 0)$, $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Also, $f_{xx} = 4(2x^2 - 3)xye^{-x^2-y^2}$, $f_{yy} = 4(2y^2 - 3)xye^{-x^2-y^2}$ and $f_{xy} = 2(1 - 2x^2)(1 - 2y^2)e^{-x^2-y^2}$. Hence $D(0, 0) = -4 < 0$, so f has a saddle point at $(0, 0)$ by the Second Derivative Test. We also see that $D(x, y) > 0$ at the four other critical points; also $f_{xx}\left(\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) < 0$ so f has a local maximum at $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $f_{xx}\left(\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right) > 0$ so f has a local minimum at $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

15.7.34 We have $f_x = 2x + y^2 - 2$, $f_y = 2xy$. Solving $f_x = f_y = 0$ gives critical points at $(1, 0)$ and $(0, \pm\sqrt{2})$. We have $f_{xx} = 2$, $f_{yy} = 2x$ and $f_{xy} = 2y$. Hence, $D(1, 0) = 4$, so f has a local minimum at $(1, 0)$. Also, $D(0, \pm\sqrt{2}) = 0 - (2\sqrt{2}) < 0$, so there are saddle points at $(0, \pm\sqrt{2})$.

15.7.35 We have $f_x = \frac{1+y^2-x^2}{(x^2+y^2+1)^2}$ and $f_y = \frac{-2xy}{(x^2+y^2+1)^2}$, so the critical points are $(\pm 1, 0)$. We have $f_{xx} = \frac{2x(x^2 - 3(y^2 + 1))}{(x^2 + y^2 + 1)^3}$, so $f_{xx}(\pm 1, 0) = \mp \frac{1}{2}$. We have $f_{yy} = -\frac{2x(x^2 - 3y^2 + 1)}{(x^2 + y^2 + 1)^3}$, so $f_{yy}(\pm 1, 0) = \mp \frac{1}{2}$. Also, $f_{xy} = -\frac{2y(-3x^2 + y^2 + 1)}{(x^2 + y^2 + 1)^3}$, so $f_{xy}(\pm 1, 0) = 0$. Thus, $D(\pm 1, 0) = \frac{1}{4}$, so there is a local maximum at $(1, 0)$ and a local minimum at $(-1, 0)$.

15.7.36 We have $f_x = \frac{-x^2 + 2x + y^2}{(x^2 + y^2)^2}$ and $f_y = -\frac{2(x-1)y}{(x^2 + y^2)^2}$. The only critical point in the domain of f is $(2, 0)$. Note that $f_{xx} = \frac{2(x^3 - 3x^2 - 3xy^2 + y^2)}{(x^2 + y^2)^3}$, so $f_{xx}(2, 0) = -1/8$. Also, $f_{yy} = -\frac{2(x-1)(x^2 - 3y^2)}{(x^2 + y^2)^3}$ and $f_{yy}(2, 0) = -1/8$. Also, $f_{xy} = -\frac{2y(-3x^2 + 4x + y^2)}{(x^2 + y^2)^3}$ and $f_{xy}(2, 0) = 0$. Thus $D(2, 0) > 0$, so there is a local maximum at $(2, 0)$.

15.7.37 We have $f_x = 4x^3 + 8x(y - 2)$ and $f_y = 4x^2 + 16(y - 1)$. Solving $f_x = f_y = 0$ yields critical points $(0, 1)$ and $(\pm 2, 0)$. We have $f_{xx} = 12x^2 + 8(y - 2)$, $f_{yy} = 16$, and $f_{xy} = 8x$. Thus $D(0, 1) < 0$, $D(\pm 2, 0) > 0$. There is a saddle point at $(0, 1)$ and local minimums at $(\pm 2, 0)$.

15.7.38 We have $f_x = e^{(-x-y)}(\sin y) - xe^{(-x-y)}(\sin y) = e^{(-x-y)}(\sin y)(1 - x)$ and $f_y = xe^{(-x-y)}\cos y - xe^{(-x-y)}(\sin y) = xe^{(-x-y)}(\cos y - \sin y)$. We have critical points at $(1, \pi/4)$ and $(0, 0)$. $f_{xx} = (x - 2)e^{(-x-y)}\sin y$, $f_{yy} = -2xe^{(-x-y)}\cos y$ and $f_{xy} = (x - 1)(-e^{(-x-y)})(\cos y - \sin y)$. We have $D(0, 0) = -1 < 0$ so there is a saddle point at $(0, 0)$. $D(1, \pi/4) > 0$ and $f_{xx}(1, \pi/4) < 0$, so there is a local maximum at $(1, \pi/4)$.

15.7.39 We have $f_x = ye^x$, $f_y = e^x - e^y$; therefore, the critical points must satisfy $y = 0$ and $y = x$, so $(0, 0)$ is the only critical point. We also have $f_{xx} = ye^x$, $f_{yy} = -e^y$ and $f_{xy} = e^x$; hence $D(0, 0) = -1 < 0$ so f has a saddle point at $(0, 0)$.

15.7.40 We have $f_x = 2\pi \cos(2\pi x) \cos(\pi y)$, $f_y = -\pi \sin(2\pi x) \sin(\pi y)$, so the critical points must satisfy $\cos(2\pi x) = \sin(\pi y) = 0$, or $\sin(2\pi x) = \cos(\pi y) = 0$; hence, the only critical points in the interior of the domain of f are $\pm\left(\frac{1}{4}, 0\right)$. We also have $f_{xx} = -4\pi^2 \sin(2\pi x) \cos(\pi y)$, $f_{yy} = -\pi^2 \sin(2\pi x) \cos(\pi y)$, $f_{xy} = -2\pi^2 \cos(2\pi x) \sin(\pi y)$; hence $D\left(\pm\left(\frac{1}{4}, 0\right)\right) > 0$. We also have $f_{xx}\left(\frac{1}{4}, 0\right) < 0$ and $f_{xx}\left(-\frac{1}{4}, 0\right) > 0$, so f has a local maximum at $\left(\frac{1}{4}, 0\right)$ and a local minimum at $\left(-\frac{1}{4}, 0\right)$.

15.7.41 Observe that $f_x = 2xy$, $f_y = x^2$ and $f_{xx} = 2y$, $f_{yy} = 0$ and $f_{xy} = 2x$; therefore $D(0, 0) = 0$ and the Second Derivative Test is inconclusive. Observe that x^2y takes on both negative and positive values in any disc centered at $(0, 0)$; therefore, f has a saddle at $(0, 0)$.

15.7.42 Observe that $f_x = 2xy^2 \cos(x^2y^2)$, $f_y = 2x^2y \cos(x^2y^2)$, $f_{xx} = 2y^2 \cos(x^2y^2) - 4x^2y^4 \sin(x^2y^2)$, $f_{yy} = 2x^2 \cos(x^2y^2) - 4x^4y^2 \sin(x^2y^2)$ and $f_{xy} = 4xy \cos(x^2y^2) - 4x^3y^3 \sin(x^2y^2)$. Therefore, $D(0,0) = 0$ and the Second Derivative Test is inconclusive. Observe that both $x^2, y^2 \geq 0$ so $\sin(x^2y^2) \geq 0$ in sufficiently small discs centered at $(0,0)$; therefore, f has an absolute minimum at $(0,0)$ (and in fact along both coordinate axes).

15.7.43 Let $x, y \geq 0$ be the dimensions of the base of the box; then for any given x, y the maximum allowable height is given by $h = 96 - 2x - 2y$, and we must have $h \geq 0$ which implies $x + y \leq 48$. The volume of the box is given by $V = 2xy(48 - x - y)$, which we must maximize over the domain R given by $x, y \geq 0$ and $x + y \leq 48$. The critical points of V satisfy $V_x = 2(48 - 2x - y)y = 0$, $V_y = 2(48 - x - 2y)x = 0$; hence the critical points in the interior of R satisfy $2x + y = 2y + x = 48$, which gives $x = y = 16$. Furthermore $V(x, y) = 0$ on the boundary of R , so the maximum volume must occur at the point $(16, 16)$. Therefore the box with largest volume has height 32 in and base 16 in $\times$ 16 in with a volume of 8192 in^3 .

15.7.44 Let $x, y \geq 0$ be the dimensions of the base and $h \geq 0$ be the height of the box; then the four sides of the box plus the base have total area $2xh + 2yh + xy = 2$, so $h = \frac{2 - xy}{2x + 2y}$. The volume of the box is given by $V = xyh = \frac{1}{2} \frac{(2 - xy)xy}{x + y}$, which we must maximize over the domain R given by $x, y \geq 0$ and $xy \leq 2$. The critical points of V satisfy $V_x = \frac{1}{2} \frac{(2 - x^2 - 2xy)y^2}{(x + y)^2} = 0$, $V_y = \frac{1}{2} \frac{(2 - y^2 - 2xy)x^2}{(x + y)^2} = 0$; hence the critical points in R satisfy $x = y$ and $3x^2 = 2$, which gives $x = y = \frac{\sqrt{6}}{3}$, and the corresponding height is $\frac{\sqrt{6}}{6}$. Furthermore $V(x, y) \rightarrow 0$ as (x, y) approaches the boundary of R , so the maximum volume must occur at the point $\left(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}\right)$. Therefore, the box with largest volume has height $\frac{\sqrt{6}}{6}$ m and base $\frac{\sqrt{6}}{3} \times \frac{\sqrt{6}}{3}$ m.

15.7.45 Let $x, y \geq 0$ be the dimensions of the base and $h \geq 0$ be the height of the box; then the volume of the box is $xyh = 4$, so $h = \frac{4}{xy}$. The four sides of the box plus the base have total area $A = 2xh + 2yh + xy = \frac{8}{x} + \frac{8}{y} + xy$, which we must maximize over the domain R given by $x, y \geq 0$. The critical points of A satisfy $A_x = y - \frac{8}{x^2} = 0$, $A_y = x - \frac{8}{y^2} = 0$; hence the critical points in R satisfy $x^2y = xy^2 = 8$, which gives $x = y$ and hence $x, y = 2$; the corresponding height is 1. Furthermore $V(x, y) \rightarrow \infty$ as (x, y) approaches the boundary of R or as either $x, y \rightarrow \infty$, so the minimum area must occur at the point $(2, 2)$. Therefore the box with smallest area has dimensions $2\text{m} \times 2\text{m} \times 1\text{m}$.

15.7.46 Let $x, y \geq 0$ be the dimensions of the base and $z \geq 0$ be the height of the box; then $z = 2 - \frac{x}{3} - \frac{2y}{3}$. The box has volume $V = xyz = xy \left(2 - \frac{x}{3} - \frac{2y}{3}\right)$, which we must maximize over the domain R given by $x, y \geq 0$ and $x + 2y \leq 6$. The critical points of V satisfy $V_x = 2y - \frac{2xy}{3} - \frac{2y^2}{3} = 0$, $V_y = 2x - \frac{x^2}{3} - \frac{4xy}{3} = 0$; hence the critical points in the interior of R satisfy $x + y = 3$ and $x + 4y = 6$, which gives $x = 2$ and $y = 1$; the corresponding height is $z = \frac{2}{3}$. Furthermore $V(x, y) = 0$ on the boundary of R , so the maximum area must occur at the point $(2, 1)$. Therefore the box with largest volume has dimensions $x = 2, y = 1, z = \frac{2}{3}$.

15.7.47 First find the values of f at all critical points in the interior of $R = \{x^2 + y^2 \leq 4\}$; we have $f_x = 2x$, $f_y = 2y - 2$, so $(0, 1)$ is the only critical point in R , and $f(0, 1) = 0$. Next, find the minimum and maximum values of f on the boundary of R , which we can parameterize by $x = 2 \cos \theta$, $y = 2 \sin \theta$ for $0 \leq \theta \leq 2\pi$. Then $f(2 \cos \theta, 2 \sin \theta) = 5 - 4 \sin \theta$, which has maximum value 9 at $\theta = \frac{3\pi}{2}$ and minimum value 1 at $\theta = \frac{\pi}{2}$. Therefore, the maximum value of f on R is $f(0, -2) = 9$ and the minimum value is $f(0, 1) = 0$.

15.7.48 First, find the values of f at all critical points in the interior of R . We have $f_x = 4x$ and $f_y = 2y$, so the only critical point in the interior of R is $(0, 0)$ and the value of f there is 0 (which is clearly the absolute minimum for f). We parameterize the boundary by letting $x = 4\cos t$ and $y = 4\sin t$ for $0 \leq t \leq 2\pi$. On the boundary we have $f(x, y) = f(t) = 32\cos^2 t + 16\sin^2 t = 16\cos^2 t + 16$. Note that $f(t)$ has a maximum value of 32 at $t = 0$ and $t = \pi$, which corresponds to the points $(\pm 4, 0)$ in R . Thus, the absolute minimum of f is 0 at $(0, 0)$ and the absolute maximum is 32 at $(\pm 4, 0)$.

15.7.49 First find the values of f at all critical points in the interior of R ; we have $f_x = 4x$, $f_y = 2y$, so $(0, 0)$ is the only critical point in R , and $f(0, 0) = 4$. Next, find the minimum and maximum values of f on the boundary of R , which is a square. On the sides $y = \pm 1$, $-1 \leq x \leq 1$ we have $f(x, \pm 1) = 4 + 2x^2 + 1 = 2x^2 + 5$, which has extreme values 5 and 7 on $[-1, 1]$. On the sides $x = \pm 1$, $-1 \leq y \leq 1$ we have $f(\pm 1, y) = 4 + 2 + y^2 = y^2 + 6$, which has extreme values 6 and 7 on $[-1, 1]$. Therefore, the maximum value of f on R is 7 and the minimum value is 4.

15.7.50 First find the values of f at all critical points in the interior of R ; we have $f_x = -2x$, $f_y = -8y$, so $(0, 0)$ is the only critical point in R , and $f(0, 0) = 6$. Next, find the minimum and maximum values of f on the boundary of R , which is a rectangle. On the sides $y = \pm 1$, $-2 \leq x \leq 2$ we have $f(x, \pm 1) = 6 - x^2 - 4 = 2 - x^2$, which has extreme values ± 2 on $[-2, 2]$. On the sides $x = \pm 2$, $-1 \leq y \leq 1$ we have $f(\pm 2, y) = 6 - 4 - 4y^2 = 2 - 4y^2$, which has extreme values ± 2 on $[-1, 1]$. Therefore, the maximum value of f on R is 6 and the minimum value is -2 .

15.7.51 First find the values of f at all critical points in the interior of R ; we have $f_x = 4x - 4$ and $f_y = 6y$, so the only critical point is $(1, 0)$. Note that $f(1, 0) = 0$. Parameterize the boundary of R by letting $x = \cos t + 1$ and $y = \sin t$ for $0 \leq t \leq 2\pi$. Note that $f(x, y) = (2x^2 - 4x + 2) + 3y^2 = 2(x - 1)^2 + 3y^2$, so $f(t) = 2\cos^2 t + 3\sin^2 t = 2 + \sin^2 t$, and f has a maximum of 3 on the boundary of R at $(1, \pm 1)$ and an absolute minimum of 2 on the boundary of R . Thus, the absolute maximum of f on all of R is 3 and the absolute minimum is 0.

15.7.52 First find the values of f at all critical points in the interior of R ; we have $f_x = 2x - 2$, $f_y = 2y - 2$, so $(1, 1)$ is the only critical point for f , and hence there are no critical points in the interior of R (the point $(1, 1)$ is on the boundary of R). Next, find the minimum and maximum values of f on the boundary of R , which is a triangle. On the side $y = 0$, $0 \leq x \leq 2$ we have $f(x, 0) = x^2 - 2x = (x - 1)^2 - 1$ which has extreme values $-1, 0$ on $[0, 2]$. Similarly on the side $x = 0$, $0 \leq y \leq 2$ we have $f(0, y) = y^2 - 2y = (y - 1)^2 - 1$, which is the same function as above. The third side is given by $y = 2 - x$ where $0 \leq x \leq 2$; we have $f(x, 2 - x) = x^2 + (2 - x)^2 - 2x - 2(2 - x) = 2(x - 1)^2 - 2$, which has extreme values $-2, 0$ on $[0, 2]$. Therefore, the maximum value of f on R is 0 and the minimum value is -2 .

15.7.53 First find the values of f at all critical points in the interior of R ; we have $f_x = -4x + 4$ and $f_y = -6y - 6$, so the only critical point in the interior of R is $(1, -1)$. Note that $f(1, -1) = 4$. We parameterize the boundary of R by letting $x = 1 + \cos t$ and $y = -1 + \sin t$ for $0 \leq t \leq 2\pi$. Note that $f(x, y) = -2(x^2 - 2x + 1) - 3(y^2 + 2y + 1) - 1 + 2 + 3 = -2(x - 1)^2 - 3(y + 1)^2 + 4$. Thus $f(t) = -2\cos^2 t - 3\sin^2 t + 3\cos^2 t + 3\sin^2 t + 1 = \cos^2 t + 1$. This has a maximum of 2 at $t = 0$ and $t = \pi$ and a minimum of 1 at $t = \pi/2$ and $t = 3\pi/2$. The original function therefore has an absolute maximum of 4 at $(1, -1)$ and an absolute minimum of 1 at $(1, 0)$ and at $(1, -2)$.

15.7.54 Observe that it is easier to find the extreme values of the function $g(x, y) = x^2 + y^2 - 2x + 2$; then the extreme values of f on R will be the square roots of the extreme values of g on R . We first find the values of g at all critical points in the interior of R ; we have $g_x = 2x - 2$, $g_y = 2y$, so $(1, 0)$ is the only critical point for g , and hence there are no critical points in the interior of R (the point $(1, 0)$ is on the boundary of R). Next, find the minimum and maximum values of g on the boundary of R . On the side $y = 0$, $-2 \leq x \leq 2$ we have $g(x, 0) = x^2 - 2x + 2 = (x - 1)^2 + 1$, which has extreme values 1, 10 on $[-2, 2]$. We can parameterize the semicircular part of the boundary by $x = 2\cos \theta$, $y = 2\sin \theta$ for $0 \leq \theta \leq \pi$. Then $g(2\cos \theta, 2\sin \theta) = 4 - 4\cos \theta + 2 = 6 - 4\cos \theta$, which has extreme values 2, 10 on $[0, \pi]$. Therefore, the maximum value of g on R is 10 and the minimum value is 1 hence, the maximum and minimum values of f are $\sqrt{10}$ and 1.

15.7.55 $f_x = -\frac{x(2y^4+1)}{(x^2y^2+1)^2}$, which is 0 for $x = 0$, and $f_y = \frac{(x^4+2)y}{(x^2y^2+1)^2}$, which is 0 for $y = 0$. On the boundary $y = 2$ we have $f(x, y) = \frac{8-x^2}{2+8x^2}$ which is maximized on $[1, 2]$ at $x = 1$ (with value $f(1, 2) = \frac{7}{9}$) and minimized at $x = 2$ (with value $f(2, 2) = \frac{1}{3}$). On the boundary $y = x$ we have $f(x, y) = \frac{x^2}{2+2x^2}$ which has minimum value 0 at $(0, 0)$ and maximum value $\frac{1}{4}$ at $(1, 1)$. On the boundary $y = 2x$ we have $f(x, y) = \frac{7x^2}{2+8x^4}$ which has derivative $\frac{7x-28x^5}{(4x^4+1)^2}$. This is zero for $x = 0$ and for $x = \frac{1}{\sqrt{2}}$. These points lead to a minimum of 0 at $(0, 0)$ and a maximum of $\frac{7}{8}$ at $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$. Therefore the global minimum of f on the given set is $0 = f(0, 0)$ and the global maximum is $\frac{7}{8} = f\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$.

15.7.56 $f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$, so the critical point in the interior of the ellipse is $(0, 0)$ which yields the value $f(0, 0) = 0$ which is clearly the minimum for the function. We can parameterize the boundary with $x = 2\cos\theta$, $y = \sin\theta$ for $0 \leq \theta \leq 2\pi$. Then $f(x, y) = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{4\cos^2\theta + \sin^2\theta} = \sqrt{3\cos^2\theta + 1}$. This has derivative $\frac{-3\cos\theta\sin\theta}{\sqrt{3\cos^2\theta + 1}}$ which is 0 for $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$. Checking these points, the maximum for f is 2 which occurs for $\theta = 0$ and $\theta = \pi$, which correspond with the points $(-2, 0)$ and $(2, 0)$ on the ellipse.

15.7.57

a. $F_H = 58.729 - 22.438H - 0.213T$ and $F_T = 7.327 - 0.213H - 0.084T$. Setting these equal to 0 and solving the resulting system of linear equations gives $H \approx 1.83$ and $T \approx 82.6$. So the only critical point on the interior is $(1.83, 82.6)$.

b. $F(1.83, 82.6) \approx 13.67 \approx 14\%$.

15.7.58 Because $-1 < x < 1$ and $-2 < y < 2$ on R , the function $f(x, y) = x + 3y$ takes on all values between $-1 + 3(-2) = -7$ and $1 + 3 \cdot 2 = 7$; in other words, the range of f on R is the interval $(-7, 7)$, and we see that f has neither an absolute minimum or maximum on R .

15.7.59 Observe that $f(0, 0) = -4$ and $f(x, y) \geq -4$ for all points $(x, y) \in R$; hence the absolute minimum value of f on R is -4 . The function f on R takes on all values in the interval $[-4, 0]$; therefore f has no absolute maximum on R .

15.7.60 Because $-1 < x, y < 1$ on R , the function $f(x, y) = x^2 - y^2$ takes on all values between -1 and 1 ; in other words, the range of f on R is the interval $(-1, 1)$, and we see that f has neither an absolute minimum or maximum on R .

15.7.61 Observe that $f(0, 0) = 2$ and $f(x, y) \leq 2$ for all points $(x, y) \in R$; hence, the absolute maximum value of f on R is 2 . The function f on R takes on all values in the interval $(0, 2]$; therefore, f has no absolute minimum on R .

15.7.62 The equation of the plane can be written as $z = 4 - x - y$; so it suffices to minimize the square of the distance from $(x, y, 4 - x - y)$ to $(5, 4, 4)$, which is given by $w = (x - 5)^2 + (y - 4)^2 + (x + y)^2$. We have $w_x = 2x - 10 + 2x + 2y$, $w_y = 2y - 8 + 2x + 2y$, so the critical points of w satisfy $4x + 2y = 10$, $4y + 2x = 8$ which gives $(x, y, z) = (2, 1, 1)$. Because we know that there is some point on the plane closest to $(2, 1, 1)$, this critical point must be that point.

15.7.63 The plane has equation $z = 2 - x + y$; the distance from a point $(x, y, 2 - x + y)$ on the plane to the point $(1, 1, 1)$ is given by $d^2 = (x - 1)^2 + (y - 1)^2 + (x - y - 1)^2$. It suffices to minimize the function

$f(x, y) = (x-1)^2 + (y-1)^2 + (x-y-1)^2$ on R^2 . We have $f_x = 2(x-1+x-y-1) = 2(2x-y-2)$, $f_y = 2(y-1+y-x+1) = 2(-x+2y)$, so the critical point of f satisfies $2x-y=2$, $x-2y=0$ which gives $x = \frac{4}{3}$, $y = \frac{2}{3}$. The corresponding point on the plane is $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$. Because there is a point on the plane closest to the point $(1, 1, 1)$, this must be the point we found.

15.7.64 We need to minimize the square of the distance from $(x, y, x^2 + y^2)$ to $(3, 3, 1)$, which is

$$w = (x-3)^2 + (y-3)^2 + (x^2 + y^2 - 1)^2.$$

We have $w_x = 2x - 6 + (2x^2 + 2y^2 - 2)(2x)$ and $w_y = 2y - 6 + (2x^2 + 2y^2 - 2)(2y)$. Setting these equal to 0 and solving gives $x = 1$ and $y = 1$. The closest point is $(1, 1, 2)$.

15.7.65 It suffices to minimize the square of the distance from points (x, y, z) on the cone to $(6, 8, 0)$, which is given by $w = (x-6)^2 + (y-8)^2 + z^2 = (x-6)^2 + (y-8)^2 + x^2 + y^2$. We have $w_x = 2x - 12 + 2x$, $w_y = 2y - 16 + 2y$, so $(3, 4)$ is the only critical point of w ; the corresponding points on the cone are $(3, 4, 5)$ and $(3, 4, -5)$. These two points are tied for being the closest point on the cone to $(6, 8, 0)$.

15.7.66 Let $x, y > 0$ be the dimensions of the base and $z > 0$ be the height of the box; then $xyz = 10$ so $z = \frac{10}{xy}$. The cost to produce the box is $C = 2 \cdot 10 \cdot xy + 2 \cdot 1 \cdot xz + 2 \cdot 1 \cdot yz = 20xy + \frac{20}{x} + \frac{20}{y}$, which we must minimize over the domain R given by $x, y > 0$. The critical points of C satisfy $C_x = 20y - \frac{20}{x^2} = 0$, $C_y = 20x - \frac{20}{y^2} = 0$; hence the critical points satisfy $x^2y = xy^2 = 1$, which gives $x = y = 1$; the corresponding height is $z = 10$. Furthermore $C(x, y) \rightarrow 0$ as (x, y) approaches the boundary of R or as either x or y approach ∞ , so the minimum cost must occur at the critical point. Therefore, the box with least cost has dimensions $1\text{m} \times 1\text{m} \times 10\text{m}$.

15.7.67

- True. This is because our definition of saddle point is a critical point which is neither a local maximum or local minimum.
- False. A necessary condition for a local maximum at (a, b) is that both $f_x = f_y = 0$ at (a, b) , assuming both partials exist.
- True. This is because f may take on its absolute maximum or minimum at a point on the boundary of its domain.
- True. The equation of the tangent plane at a critical point (a, b) is $z = f(a, b)$.

15.7.68 This function has a local maximum at $\left(\frac{1}{2}, -1\right)$ and saddle points at $(0, 0)$, $(1, 0)$, $(-2, 0)$ and $(1, -2)$.

15.7.69 This function has a local minimum near $(0.3, -0.3)$ and a saddle point at $(0, 0)$.

15.7.70 Let (x, y, z) be the point on the ellipsoid; then the box has volume $V = xyz$. We have the relation $36x^2 + 4y^2 + 9z^2 = 36$ which implies that $z^2 = 4\left(1 - x^2 - \frac{y^2}{9}\right)$, and it suffices to maximize the square of the volume, which is given by $w = 4x^2y^2\left(1 - x^2 - \frac{y^2}{9}\right)$ over the region R given by $x, y \geq 0$ and $x^2 - \frac{y^2}{9} \leq 1$. The critical points in the interior of R satisfy $w_x = 8xy^2\left(1 - 2x^2 - \frac{y^2}{9}\right) = 0$, $w_y = 8x^2y\left(1 - x^2 - \frac{2y^2}{9}\right) = 0$, which gives $x^2 = \frac{1}{3}$, $y^2 = 3$ so $x = \frac{\sqrt{3}}{3}$, $y = \sqrt{3}$; The corresponding value of $z = \frac{2\sqrt{3}}{3}$. Notice that $w = 0$ on the boundary of the closed and bounded region R ; therefore the critical point we found must give the absolute maximum value of w , so the box with maximum volume has dimensions $\frac{\sqrt{3}}{3} \times \sqrt{3} \times \frac{2\sqrt{3}}{3}$.

15.7.71

- a. Using the relation $z = 200 - x - y$, we see that it suffices to minimize the function $f(x, y) = x^2 + y^2 + (200 - x - y)^2$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 200$. We have $f_x = 2(x + y - 200) = 2(2x + y - 200)$, $f_y = 2(y + x - 200) = 2(x + 2y - 200)$; therefore, the critical point of f satisfies $2x + y = 200$, $x + 2y = 200$ which gives $x = y = \frac{200}{3}$ (the corresponding value of $z = \frac{200}{3}$ as well), and we have $f\left(\frac{200}{3}, \frac{200}{3}\right) = \frac{40,000}{3}$. We must also find the extreme values of f on the boundary of R . Along the segment $y = 0$, $0 \leq x \leq 200$ we have $f(x, 0) = x^2 + (200 - x)^2$ which has range $[20,000, 40,000]$, and we get the same result along the other two segments that make up the boundary of R . Therefore, the minimum value of $x^2 + y^2 + z^2$ is given by $x = y = z = \frac{200}{3}$.
- b. The function $\sqrt{x^2 + y^2 + z^2}$ takes its minimum at the same point that minimizes $x^2 + y^2 + z^2$, which we saw in part (a) is $x = y = z = \frac{200}{3}$.
- c. Using the relation $z = 200 - x - y$, we see that it suffices to minimize the function $f(x, y) = xy(200 - x - y) = 200xy - x^2y - xy^2$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 200$. We have $f_x = y(200 - 2x - y)$, $f_y = x(200 - x - 2y)$; therefore, the critical points of f in the interior of R satisfy $2x + y = 200$, $x + 2y = 200$ which gives $x = y = \frac{200}{3}$ (the corresponding value of $z = \frac{200}{3}$ as well), and we have $f\left(\frac{200}{3}, \frac{200}{3}\right) = \left(\frac{200}{3}\right)^2$. We also observe that $f(x, y) = 0$ on the boundary of R ; therefore, the maximum value of xyz is given by $x = y = z = \frac{200}{3}$.
- d. The function $x^2y^2z^2$ takes its maximum at the same point that maximizes xyz , which we saw in part (c) is $x = y = z = \frac{200}{3}$.

15.7.72 The level curves of a linear function $f(x, y)$ are a family of parallel lines; therefore at any interior point of a region R , one can move to a nearby level curve in such a way as to either increase or decrease the value of f . This shows that the absolute maximum or minimum of f on R cannot occur at an interior point, and hence must occur along the boundary of R .

15.7.73

- a. The function to be minimized is $f(x, y) = x^2 + y^2 + (x - 2)^2 + y^2 + (x - 1)^2 + (y - 1)^2 = 3x^2 - 6x + 3y^2 - 2y$; we have $f_x = 6(x - 1)$, $f_y = 2(3y - 1)$ so the optimal location is the unique critical point $\left(1, \frac{1}{3}\right)$.
- b. The function to be minimized is now $f(x, y) = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + (x - x_3)^2 + (y - y_3)^2 = 3(x^2 - 2\bar{x}x + y^2 - 2\bar{y}y) + \text{const}$; where $\bar{x} = \frac{x_1 + x_2 + x_3}{3}$, $\bar{y} = \frac{y_1 + y_2 + y_3}{3}$. we have $f_x = 6(x - \bar{x})$, $f_y = 6(y - \bar{y})$ so the optimal location is the unique critical point $(\bar{x}, \bar{y})$.
- c. The function to be minimized is now

$$f(x, y) = \sum_{i=1}^n (x - x_i)^2 + \sum_{i=1}^n (y - y_i)^2 = n(x^2 - 2\bar{x}x + y^2 - 2\bar{y}y) + \text{const}$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$, $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$. We have $f_x = n(x - \bar{x})$, $f_y = n(y - \bar{y})$ so the optimal location is the unique critical point $(\bar{x}, \bar{y})$.

d. The actual sum of the distances is given by the function $f(x, y) = \sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} + \sqrt{(x-1)^2 + (y-1)^2}$. We can minimize this function as follows. First, fix $y = y_0$ and consider the function $g(x) = f(x, y_0)$. Each of the three terms in this function has positive second derivative, and approaches ∞ as $x \rightarrow \pm\infty$, so the same is true for their sum; therefore $g(x)$ must have a unique absolute minimum. Observe in addition that g is symmetric in the line $x = 1$, which implies that the absolute minimum must occur at $x = 1$. So to minimize $f(x, y)$, we can set $x = 1$ and reduce to minimizing the function $h(y) = f(1, y) = 2\sqrt{1 + y^2} + |y - 1|$, which has absolute minimum at $y = \frac{1}{\sqrt{3}}$.

Therefore, the optimal location is $\left(1, \frac{1}{\sqrt{3}}\right)$, which is different from the point found in part (a).

15.7.74 The sum of the squares of the vertical distances is $E(m, b) = [(m+b) - 2]^2 + [(3m+b) - 5]^2 + [(4m+b) - 6]^2 = 26m^2 + 16mb + 3b^2 - 82m - 26b + 65$. We have $E_m = 52m + 16b - 82$, $E_b = 16m + 6b - 26$, and solving the simultaneous equations $52m + 16b = 82$, $16m + 6b = 26$ gives $m = \frac{19}{14}$, $b = \frac{10}{14}$.

15.7.75 Given the n data points $(x_1, y_1), \dots, (x_n, y_n)$, we seek to minimize the function $E(m, b) = \sum_{k=1}^n (mx_k + b - y_k)^2 = (\sum x_k^2)m^2 + 2(\sum x_k)mb + nb^2 - 2(\sum x_k y_k)m - 2(\sum y_k)b + \sum y_k^2$. We have $E_m = 2(\sum x_k^2)m + 2(\sum x_k)b - 2(\sum x_k y_k)$, $E_b = 2(\sum x_k)m + 2nb - 2(\sum y_k)$ and solving $E_m = E_b = 0$ gives $m = \frac{(\sum x_k)(\sum y_k) - n \sum x_k y_k}{(\sum x_k)^2 - n \sum x_k^2}$, $b = \frac{1}{n} \left(\sum y_k - m \sum x_k \right)$.

15.7.76 Using the result in Exercise 75, we obtain $m = \frac{6 \cdot 8 - 3 \cdot 26}{6^2 - 3 \cdot 20} = \frac{5}{4}$, $b = \frac{1}{3} \left(8 - \frac{5}{4} \cdot 6 \right) = \frac{1}{6}$ so the line has equation $y = \frac{5}{4}x + \frac{1}{6}$.

15.7.77 Using the result in Exercise 75, we obtain $m = \frac{2 \cdot 14 - 3 \cdot 24}{2^2 - 3 \cdot 10} = \frac{22}{13}$, $b = \frac{1}{3} \left(14 - \frac{22}{13} \cdot 2 \right) = \frac{46}{13}$ so the line has equation $y = \frac{22}{13}x + \frac{46}{13}$.

15.7.78 Observe that $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2 < 0$ because $f_{xx}(a, b)f_{yy}(a, b) < 0$ in both cases. Therefore f has a saddle at (a, b) .

15.7.79 Let $s = \frac{a+b+c}{2}$ be the semi-perimeter, which we assume is constant. Then we may express $c = 2s - a - b$ and therefore $A^2 = s(s-a)(s-b)(s-(2s-a-b)) = s(s-a)(s-b)(a+b-s)$, so it suffices to maximize the simpler function $f(a, b) = s(s-a)(s-b)(a+b-s)$ over the closed bounded region given by $0 \leq a, b \leq s$ and $a+b \geq s$. We have $f_a = s(s-b)(-(a+b-s) + s-a) = s(s-b)(2s-2a-b)$ and similarly $f_b = s(s-a)(2s-a-2b)$, so the critical points of f in the interior of R satisfy $2a+b=2s$, $a+2b=2s$ which gives $a=b=\frac{2}{3}s$, and therefore $c=\frac{2}{3}s$ as well, so we get an equilateral triangle. We also observe that $f(a, b) = 0$ on the boundary of R , so this critical point must give the absolute maximum of f . We conclude that of all triangles with a given perimeter, the maximum area is obtained when all three sides are equal (in the special case of perimeter 9 units, each side length is 3 units).

15.7.80 Write the equation of the plane in the form $ax+by+cz=1$, with $a, b, c > 0$; then a, b, c must satisfy the condition $3a+2b+c=1$. This plane meets the coordinate axes in the points $\left(\frac{1}{a}, 0, 0\right)$, $\left(0, \frac{1}{b}, 0\right)$, $\left(0, 0, \frac{1}{c}\right)$. The region between then plane and the coordinate planes is a tetrahedron T which has base area $A = \frac{1}{2ab}$ and height $h = \frac{1}{c}$, so its volume is $V = \frac{1}{6abc}$. Minimizing this function is equivalent to maximizing its reciprocal, or equivalently the quantity $Q = abc$. Now if we let $u = 3a$, $v = 2b$, $w = c$, then an equivalent

(but simpler) problem is to maximize $Q = uvw$ where $u, v, w \geq 0$ and $u + v + w = 1$. The maximum occurs when $u = v = w = \frac{1}{3}$ (see problem 65c), so the tetrahedron of minimal volume is given by $3a = 2b = c = \frac{1}{3}$, which gives the plane $\frac{x}{9} + \frac{y}{6} + \frac{z}{3} = 1$.

15.7.81

- We have $d_1(x, y) = \sqrt{(x - x_1)^2 + (y - y_1)^2}$, so $\nabla d_1(x, y) = \frac{x - x_1}{d_1(x, y)}\mathbf{i} + \frac{y - y_1}{d_1(x, y)}\mathbf{j}$; observe that this is a unit vector in the direction of the vector joining (x_1, y_1) to (x, y) .
- Similarly, we have $\nabla d_2(x, y) = \frac{x - x_2}{d_2(x, y)}\mathbf{i} + \frac{y - y_2}{d_2(x, y)}\mathbf{j}$ and $\nabla d_3(x, y) = \frac{x - x_3}{d_3(x, y)}\mathbf{i} + \frac{y - y_3}{d_3(x, y)}\mathbf{j}$.
- Because $\nabla f = \nabla d_1 + \nabla d_2 + \nabla d_3$, the condition $f_x = f_y = 0$ is equivalent to $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.
- If three unit vectors add to 0, they must make angles of $\pm \frac{2\pi}{3}$ with each other.
- In this case the optimal point is the vertex at the large angle.
- Solving the equations $f_x = f_y = 0$ numerically gives $(0.255457, 0.304504)$.

15.7.82

- We have $f_x = 3e^y - 3x^2$, $f_y = 3xe^y - 3e^{3y}$, so the critical points must satisfy $x = e^{2y}$, and then the equation $f_x = 0$ reduces to $e^{3y} = 1$, so the only critical point is $(1, 0)$. Next we compute $f_{xx} = -6x$, $f_{yy} = 3xe^y - 9e^{3y}$, $f_{xy} = 3e^y$, so $f_{xx}f_{yy} - f_{xy}^2 = (-6)(-6) - 3^2 = 27 > 0$ at the critical point $(1, 0)$; we also have $f_{xx} < 0$ at $(1, 0)$, so the critical point is a local maxima, and we conclude that f has a unique local maximum. However observe that if we fix $y = 0$, then $f(x, 0) = -x^3 + 3x - 1$ takes on all real values, so f does not have an absolute maximum or minimum.
- We have $f_x = (2y^2 - y^4)e^x - (2y^2 - y^4 - 1)\frac{2x}{(1+x^2)^2}$, $f_y = (4y - 4y^3)\left(e^x + \frac{1}{1+x^2}\right)$, so the critical points must satisfy $y = 0, \pm 1$ from the equation $f_y = 0$. Substituting $y = 0$ into the equation $f_x = 0$ gives $x = 0$, whereas substituting $y = \pm 1$ into $f_x = 0$ gives no solutions; therefore, $(0, 0)$ is the only critical point. We compute $f_{xx}(0, 0) = 2$, $f_{yy}(0, 0) = 8$, $f_{xy} = 0$, so $D(0, 0) > 0$, and $f_{xx}(0, 0) > 0$ implies that f has a local minimum at $(0, 0)$. However, observe that if we fix $y = 1$, then $f(x, 1) = e^x$ takes on all positive values, and if we fix $y = 2$, then $f(x, 2) = -8e^x - \frac{9}{1+x^2}$ takes on all negative values. Hence f must take on all real values on $\mathbb{R}^2$, and therefore has no absolute maximum or minimum.

15.7.83

- We have $f(x, y) = -2x^4 + 2x^2(e^y + 1) - e^{2y} - 1$; hence $f_x = -8x^3 + 4x(e^y + 1)$, $f_y = 2x^2e^y - 2e^{2y}$, so the critical points must satisfy $x^2 = e^y$, and then the equation $f_x = 0$ reduces to $e^y = 1$, so the critical points are $(\pm 1, 0)$. Next we compute $f_{xx} = -24x^2 + 4(1 + e^y)$, $f_{yy} = 2x^2e^y - 4e^{2y}$, $f_{xy} = 4e^y$ so $f_{xx}f_{yy} - f_{xy}^2 = (-16)(-2) - (4)^2 = 16$ at both critical points; we also have $f_{xx} < 0$ at both critical points, so the critical points both yield local maxima.
- We have $f_x = 8xe^y - 8x^3$, $f_y = 4x^2e^y - 4e^{4y}$, so the critical points must satisfy $x^2 = e^{3y}$, and then the equation $f_x = 0$ reduces to $e^y = 1$, so the critical points are $(\pm 1, 0)$. Next we compute $f_{xx} = 8e^y - 24x^2$, $f_{yy} = 4x^2e^y - 16e^{4y}$, $f_{xy} = 8xe^y$, so $f_{xx}f_{yy} - f_{xy}^2 = (-16)(-12) - (\pm 8)^2 = 128$ at both critical points; we also have $f_{xx} < 0$ at both critical points, so the critical points both yield local maxima.

15.7.84

- a. Using the relation $z = 1 - x - y$, we see that it suffices to minimize the function $f(x, y) = (1 + x^2)(1 + y^2)(1 + z^2) = (1 + x^2)(1 + y^2)(1 + (1 - x - y)^2)$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 1$. Using the chain rule, we have $f_x = 2(1 + y^2)(x(1 + z^2) - z(1 + x^2))$, $f_y = 2(1 + x^2)(y(1 + z^2) - z(1 + y^2))$. The critical points of f in the interior of R satisfy $f_x = f_y = 0$; the first equation gives $x(1 + z^2) - z(1 + x^2) = (x - z)(1 - xz) = 0$, and we observe that $xz = 1$ is impossible, so we conclude that $x = z$. Similarly $y = z$ so the critical point is $(\frac{1}{3}, \frac{1}{3})$, and we have $f(\frac{1}{3}, \frac{1}{3}) = (\frac{10}{9})^3$. On the boundary segment given by $y = 0$, $0 \leq x \leq 1$ we have $f(x, 0) = (1 + x^2)(1 + (1 - x)^2)$ and this function has a maximum value $f(0) = f(1) = 2$, and minimum value $f(\frac{1}{2}) = (\frac{5}{4})^2$. The range of f on the other two boundary components is the same. Therefore the maximum value of f is 2, and the minimum is $(\frac{10}{9})^3$.

- b. Using the relation $z = 1 - x - y$, we see that it suffices to minimize the function $f(x, y) = (1 + \sqrt{x})(1 + \sqrt{y})(1 + \sqrt{z}) = (1 + \sqrt{x})(1 + \sqrt{y})(1 + \sqrt{1 - x - y})$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 1$. Using the chain rule, we have

$$\begin{aligned} f_x &= (1 + \sqrt{y}) \left(\left(1 + \frac{1}{2\sqrt{x}}\right) (1 + \sqrt{z}) - (1 + \sqrt{x}) \left(1 + \frac{1}{2\sqrt{z}}\right) \right) \\ &= \left(\frac{1 + \sqrt{y}}{4\sqrt{x}\sqrt{z}} \right) ((2\sqrt{x} + 1)(\sqrt{z} + 1) - (2\sqrt{z} + 1)(\sqrt{x} + 1)) = \left(\frac{1 + \sqrt{y}}{4\sqrt{x}\sqrt{z}} \right) (\sqrt{x} - \sqrt{z}), \end{aligned}$$

and similarly $f_y = \left(\frac{1 + \sqrt{x}}{4\sqrt{y}\sqrt{z}} \right) (\sqrt{y} - \sqrt{z})$. The critical points of f in the interior of R satisfy $f_x = f_y = 0$ which gives $x = y = \frac{1}{3}$, so the critical point is $(\frac{1}{3}, \frac{1}{3})$, and we have $f(\frac{1}{3}, \frac{1}{3}) \approx 3.925$. On the boundary segment given by $y = 0$, $0 \leq x \leq 1$ we have $f(x, 0) = (1 + \sqrt{x})(1 + \sqrt{1 - x})$, and this function has a maximum value $f(0) = f(1) = 2$, and minimum value $f(\frac{1}{2}) \approx 2.914$. The range of f on the other two boundary components is the same. Therefore the minimum value of f is 2, and the maximum is 3.925.

15.7.85 Let (x_0, y_0, z_0) be a point on the ellipsoid; then the tangent plane P at this point has equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$, and this plane meets the coordinate axes in the points $(\frac{a^2}{x_0}, 0, 0)$, $(0, \frac{b^2}{y_0}, 0)$, $(0, 0, \frac{c^2}{z_0})$. Therefore, the tetrahedron T has base area $A = \frac{a^2 b^2}{2x_0 y_0}$ and height $h = \frac{c^2}{z_0}$, so its volume is $V = \frac{a^2 b^2 c^2}{6x_0 y_0 z_0}$. Minimizing this function is equivalent to maximizing its reciprocal, or equivalently the quantity $Q = \frac{x_0^2}{a^2} \cdot \frac{y_0^2}{b^2} \cdot \frac{z_0^2}{c^2}$. Now if we let $u = \frac{x_0^2}{a^2}$, $v = \frac{y_0^2}{b^2}$, $w = \frac{z_0^2}{c^2}$, then an equivalent (but simpler) problem is to maximize $Q = uvw$ where $u, v, w \geq 0$ and $u + v + w = 1$. The maximum occurs when $u = v = w = \frac{1}{3}$ (see problem 65 c), so the tetrahedron of minimal volume is given by $\frac{x_0^2}{a^2} = \frac{y_0^2}{b^2} = \frac{z_0^2}{c^2} = \frac{1}{3}$ which gives $V = \frac{abc}{6} \left(\frac{a}{x_0} \right)^3 = \frac{abc\sqrt{3}}{2}$.

15.8 Lagrange Multipliers

15.8.1 The level curves of f must be tangential to the level curves of g at the optimal point; thus, the gradients are parallel.

15.8.2 Let $g(x, y) = x^2 + y^2 - 1$. Solve the system $\nabla f = \lambda \nabla g$ and $g(x, y) = 0$. Among the solutions, find the corresponding maximum and minimum values of f .

15.8.3 $1 = 2\lambda x$ and $4 = 2\lambda y$, and $x^2 + y^2 - 1 = 0$.

15.8.4 $yz = 2\lambda x$, $xz = 4\lambda y$, $xy = 6\lambda z$, and $x^2 + 2y^2 + 3z^2 - 1 = 0$.

15.8.5 The maximum and minimum values of f along the curve $g(x, y) = 0$ occur at points where the level curves of f are tangent to the curve $g(x, y) = 0$; using this, we see that the minimum and maximum values are 1 and 8.

15.8.6 The maximum and minimum values of f along the curve $g(x, y) = 0$ occur at points where the level curves of f are tangent to the curve $g(x, y) = 0$; using this, we see that the minimum and maximum values are 2 and 6.

15.8.7 We have $\nabla f = \langle 1, 2 \rangle$, $\nabla g = \langle 2x, 2y \rangle$ so the Lagrange multiplier conditions are $1 = 2\lambda x$, $2 = 2\lambda y$, $x^2 + y^2 - 4 = 0$. Hence $\frac{1}{x} = 2\lambda = \frac{2}{y} \implies y = 2x$; substituting this in the constraint gives $5x^2 = 4$, so $x = \pm \frac{2}{\sqrt{5}}$ and the extreme values of f on the circle $x^2 + y^2 = 4$ must occur at the points $\pm \left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right)$. We see that $f\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = 2\sqrt{5}$, $f\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = -2\sqrt{5}$, so these are the maximum and minimum values.

15.8.8 We have $\nabla f = \langle y^2, 2xy \rangle$, $\nabla g = \langle 2x, 2y \rangle$ so the Lagrange multiplier conditions are $y^2 = 2\lambda x$, $2xy = 2\lambda y$, $x^2 + y^2 - 1 = 0$. One possible solution is $y = \lambda = 0$, which gives the points $(\pm 1, 0)$. All other solutions will have $x, y, \lambda \neq 0$, so we can eliminate λ and obtain $2\lambda = \frac{y^2}{x} = 2x \implies y^2 = 2x^2$; substituting this in the constraint gives $3x^2 = 1$, so $x = \pm \frac{1}{\sqrt{3}}$, hence $y = \pm \frac{\sqrt{2}}{\sqrt{3}}$ and the extreme values of f on the circle $x^2 + y^2 = 1$ must occur at the points $(\pm 1, 0)$, $\left(\frac{1}{\sqrt{3}}, \pm \frac{\sqrt{2}}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \pm \frac{\sqrt{2}}{\sqrt{3}}\right)$. We see that $f(\pm 1, 0) = 0$, $f\left(\frac{1}{\sqrt{3}}, \pm \frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9}$, $f\left(-\frac{1}{\sqrt{3}}, \pm \frac{\sqrt{2}}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$, so the maximum and minimum values are $\pm \frac{2\sqrt{3}}{9}$.

15.8.9 We have $\nabla f = \langle 1, 1 \rangle$, $\nabla g = \langle 2x - y, 2y - x \rangle$, so the Lagrange multiplier conditions are $2x - y = 1/\lambda$, $2y - x = 1/\lambda$, $x^2 - xy + y^2 = 1$. Subtracting the first two equations gives $3x - 3y = 0$, so $x = y = 1/\lambda$, which yields the points $(\pm 1, \pm 1)$. The maximum value is 2 at $(1, 1)$ and the minimum is -2 at $(-1, -1)$.

15.8.10 We have $\nabla f = \langle 2x, 2y \rangle$, $\nabla g = \langle 4x + 3y, 4y + 3x \rangle$, so the Lagrange multiplier conditions are $2x = \lambda(4x + 3y)$, $2y = \lambda(4y + 3x)$, $2x^2 + 3xy + 2y^2 = 7$. Multiplying the first equation by y and the second by x and subtracting gives $0 = \lambda(3y^2 - 3x^2)$. Note that if $\lambda = 0$, then $x = 0$ and $y = 0$ which does not satisfy the constraint. If $\lambda \neq 0$, we have $x = \pm y$. If $x = y$ we obtain the points $x = y = \pm 1$, if $x = -y$, we obtain the point $(\pm\sqrt{7}, \mp\sqrt{7})$. The minimum value of f is 2 at $(\pm 1, \pm 1)$ and the maximum is 14 at $(\pm\sqrt{7}, \mp\sqrt{7})$.

15.8.11 We have $\nabla f = \langle y, x \rangle$ and $\nabla g = \langle 2x - y, 2y - x \rangle$, so the Lagrange multiplier conditions are $y = \lambda(2x - y)$, $x = \lambda(2y - x)$, $x^2 + y^2 - xy = 9$. Multiplying the first equation by x and the second by y and subtracting leads to $\lambda(2x^2 - 2y^2) = 0$, so either $x = \pm y$ or $\lambda = 0$. If $\lambda = 0$, we have $x = y = 0$ which doesn't meet the constraint. If $x = y$, we have $x = \pm 3 = y$, if $x = -y$, we have $x = \pm\sqrt{3}$, $y = \mp\sqrt{3}$. There is a minimum value of -3 at $(\pm\sqrt{3}, \mp\sqrt{3})$ and a maximum value of 9 at $(\pm 3, \pm 3)$.

15.8.12 We have $\nabla f = \langle 1, -1 \rangle$ and $\nabla g = \langle 2x - 3y, 2y - 3x \rangle$, so the Lagrange multiplier conditions are $2x - 3y = 1/\lambda$, $2y - 3x = -1/\lambda$, $x^2 + y^2 - 3xy = 20$. Adding the first two equations gives $-y - x = 0$, so $x = -y$ and we have the points $(\pm 2, \mp 2)$. There is a maximum value of 4 at $(2, -2)$ and a minimum value of -4 at $(-2, 2)$.

15.8.13 We have $\nabla f = e^{xy} \langle y, x \rangle$ and $\nabla g = \langle 2x + y, 2y + x \rangle$. Then the conditions are $ye^{xy} = \lambda(2x + y)$, $xe^{xy} = \lambda(2y + x)$, and $x^2 + xy + y^2 = 9$. Subtracting the second from the first gives $(y - x)e^{xy} = \lambda(x - y)$, so either $y = x$ or $\lambda = -e^{xy}$. The former leads to $x = \pm\sqrt{3}$, $y = \pm\sqrt{3}$ while the second leads to $x = -y$, which in turn leads to $x = \pm 3$, $y = \mp 3$. We have an absolute maximum of e^3 at $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$ and an absolute minimum of e^{-9} at $(-3, 3)$ and at $(3, -3)$.

15.8.14 We have $\nabla f = \langle 2xy, x^2 \rangle$ and $\nabla g = \langle 2x, 2y \rangle$ so our conditions are $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ and $x^2 + y^2 = 9$. The first equation tells us that either $x = 0$ or $y = \lambda$, from which we can deduce critical points of $(0, \pm 3)$ or in the second case, we have $x^2 = 2\lambda^2$, so $y = \lambda$, and we can deduce that $\lambda = \pm\sqrt{3}$ and then $x = \pm\sqrt{6}$ and in each case, $y = \pm\sqrt{3}$. So the complete set of points to check includes $(0, \pm 3)$, $(\pm\sqrt{6}, \sqrt{3})$, $(\pm\sqrt{6}, -\sqrt{3})$. We obtain a minimum of $-6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3})$ and a maximum of $6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3})$.

15.8.15 We have $\nabla f = \langle 4x, 2y \rangle$ and $\nabla g = \langle 2x, 2 + 2y \rangle$ so our conditions are $4x = 2\lambda x$ and $2y = \lambda(2y + 2)$ and $x^2 + 2y + y^2 = 15$. The first equation tells us that either $x = 0$ or $\lambda = 2$. If $x = 0$ we can deduce that $y = -5$ or $y = 3$. From $\lambda = 2$ we can deduce that $2y = 4y + 4$, so $y = -2$ and $x = \pm\sqrt{15}$. We obtain a minimum of 9 at $(0, 3)$ and a maximum of 34 at $(\pm\sqrt{15}, -2)$.

15.8.16 We have $\nabla f = \langle 2x, 0 \rangle$ and $\nabla g = \langle 2x + y, x + 2y \rangle$, so our conditions become $2x = \lambda(2x + y)$ and $0 = \lambda(x + 2y)$, and $x^2 + xy + y^2 = 3$. From the second equation, we can deduce that either $\lambda = 0$ (which leads to $x = 0$ and $y = \pm\sqrt{3}$), or $x = -2y$ (from which we can deduce that $-4y = \lambda(-3y)$, so either $y = 0$ which leads to a contradiction or $\lambda = 4/3$ and $y = \pm 1$). The latter case results in the critical points $(2, -1)$ and $(-2, 1)$. The absolute minimum is 0 at $(0, \sqrt{3})$ and $(0, -\sqrt{3})$, and the absolute maximum is 4 at $(2, -1)$ and $(-2, 1)$.

15.8.17 We have $\nabla f = \langle 1, 3, -1 \rangle$, $\nabla g = \langle 2x, 2y, 2z \rangle$ so the Lagrange multiplier conditions are $1 = 2\lambda x$, $3 = 2\lambda y$, $-1 = 2\lambda z$, $x^2 + y^2 + z^2 - 4 = 0$. These equations imply $x, y, z \neq 0$, so we can eliminate λ and obtain $y = 3x$, $z = -x$. Then the constraint gives $11x^2 = 4$, so $x = \pm \frac{2}{\sqrt{11}}$ and the solutions are the points $\pm \left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}} \right)$. We compute $f \left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}} \right) = 2\sqrt{11}$, $f \left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right) = -2\sqrt{11}$, and hence the minimum and maximum values of f on the closed bounded set given by $x^2 + y^2 + z^2 = 4$ are $-2\sqrt{11}$ and $2\sqrt{11}$.

15.8.18 We have $\nabla f = \langle yz, xz, xy \rangle$, $\nabla g = \langle 2x, 4y, 8z \rangle$ so the Lagrange multiplier conditions are $yz = 2\lambda x$, $xz = 4\lambda y$, $xy = 8\lambda z$, $x^2 + 2y^2 + 4z^2 - 9 = 0$. Assume first that $x, y, z \neq 0$, so we can eliminate λ and obtain $\frac{xyz}{2\lambda} = x^2 = 2y^2 = 4z^2$. Then using the constraint we obtain $3x^2 = 9$ so $x = \pm\sqrt{3}$, $y = \pm\frac{\sqrt{3}}{2}$ and $z = \pm\frac{\sqrt{3}}{2}$.

The value of xyz at any of these points is $\pm\sqrt{3} \cdot \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \pm\frac{3\sqrt{6}}{4}$. We also observe that if any of $x, y, z = 0$ then $xyz = 0$. Hence, the minimum and maximum values of f on the closed bounded set given by $x^2 + 2y^2 + 4z^2 = 9$ are $\pm\frac{3\sqrt{6}}{4}$.

15.8.19 We have $\nabla f = \langle 1, 0, 0 \rangle$ and $\nabla g = \langle 2x, 2y, 2z - 1 \rangle$. The Lagrange multiplier conditions are $1 = 2x\lambda$, $0 = 2y\lambda$, $0 = (2z - 1)\lambda$, and $x^2 + y^2 + z^2 - z = 1$. Clearly $\lambda \neq 0$, so we must have $y = 0$ and $z = 1/2$. Solving the constraint for x gives $x = \pm\sqrt{5}/2$. There is a maximum of $\sqrt{5}/2$ at $(\sqrt{5}/2, 0, 1/2)$ and a minimum of $-\sqrt{5}/2$ at $(-\sqrt{5}/2, 0, 1/2)$.

15.8.20 We have $\nabla f = \langle 1, 0, -1 \rangle$ and $\nabla g = \langle 2x, 2y - 1, 2z \rangle$. The Lagrange multiplier conditions are $1 = 2x\lambda$, $0 = (2y - 1)\lambda$, and $-1 = 2z\lambda$. Clearly $\lambda \neq 0$, so we must have $y = 1/2$. Also, $\frac{1}{x} = \frac{-1}{z} = 2\lambda$, so $x = -z$. The constraint equation then gives $x = \pm 3/(2\sqrt{2})$ and $z = \mp 3/(2\sqrt{2})$. There is a minimum of $-3/\sqrt{2}$ at $(-3/(2\sqrt{2}), 1/2, 3/(2\sqrt{2}))$ and a maximum of $3/\sqrt{2}$ at $(3/(2\sqrt{2}), 1/2, -3/(2\sqrt{2}))$.

15.8.21 We have $\nabla f = \langle 1, 1, 1 \rangle$ and $\nabla g = \langle 2x - y, 2y - x, 2z \rangle$. The Lagrange multiplier conditions are $1 = \lambda(2x - y)$, $1 = \lambda(2y - x)$, and $1 = 2\lambda z$ and $x^2 + y^2 + z^2 - xy = 5$. Combining the first two equations gives us that $x = y$, from which we can deduce that $x = y = 1/\lambda$ and $z = 1/(2\lambda)$. Then the last equation gives us $5/(4\lambda^2) = 5$, so $\lambda = \pm 1/2$. Then we can deduce that $x = y = 2$ and $z = 1$, or $x = y = -2$, and $z = -1$. There is an absolute minimum of -5 at $(-2, -2, -1)$ and an absolute maximum of 5 at $(2, 2, 1)$.

15.8.22 We have $\nabla f = \langle 1, 1, 1 \rangle$ and $\nabla g = \langle 2x - 2, 2y - 2, 2z \rangle$. The Lagrange multiplier conditions are $1 = \lambda(2x - 2)$, $1 = \lambda(2y - 2)$, $1 = 2\lambda z$, and $x^2 + y^2 + z^2 - 2x - 2y = 1$. The first two equations imply that $x = y$ and the third together with the first two implies that $x = y = z + 1$. The constraint then implies that $2x^2 + (x - 1)^2 - 4x = 1$, or $3x^2 - 6x = 0$. So $x = 0$ or $x = 2$. There is a maximum of 5 at $(2, 2, 1)$ and a minimum of -1 at $(0, 0, -1)$.

15.8.23 $\nabla f = \langle 2, 0, 2z \rangle$ and $\nabla g = \langle 2x, 2y, 4z \rangle$. The Lagrange multiplier conditions are $2 = 2\lambda x$, $0 = 2\lambda y$, $2z = 4\lambda z$, and $x^2 + y^2 + 2z^2 = 25$. Note that $\lambda \neq 0$ so $y = 0$ and if $z \neq 0$ then $\lambda = 1/2$ so $x = 2$ and we have the point $(2, 0, \sqrt{21}/2)$. If $z = 0$, then we have $x = \pm 5$. The minimum is -10 at $(-5, 0, 0)$ and the maximum is 14.5 at $(2, 0, \sqrt{21}/2)$.

15.8.24 We have $\nabla f = \langle y, x, -1 \rangle$ and $\nabla g = \langle 2x - y, 2y - x, 2z \rangle$. The Lagrange multiplier conditions are $y = \lambda(2x - y)$, $x = \lambda(2y - x)$, $-1 = 2\lambda z$, and $x^2 + y^2 + z^2 - xy = 1$. Solving the first two equations for λ gives

$$\lambda = \frac{y}{2x - y} = \frac{x}{2y - x},$$

so $2y^2 - xy = 2x^2 - xy$, and $x = \pm y$. If $x = y$ we obtain the points $(0, 0, \pm 1)$, or (if $\lambda = 1$) we obtain $z = -1/2$ and then $x = y = \pm\sqrt{3}/2$. In the case where $x = -y$, we have $-y = \lambda(2y + y)$, so if $y \neq 0$, we have $\lambda = -1/3$, which leads to $z = 3/2$ and no solutions for the last constraint equation. Therefore our only critical points are $(0, 0, \pm 1)$ and $(\pm\sqrt{3}/2, \pm\sqrt{3}/2, -1/2)$. There is an absolute minimum of -1 at $(0, 0, 1)$ and an absolute maximum of $5/4$ at both $(\sqrt{3}/2, \sqrt{3}/2, -1/2)$ and $(-\sqrt{3}/2, -\sqrt{3}/2, -1/2)$.

15.8.25 We have $\nabla f = \langle 2x, 1, 1 \rangle$ and $\nabla g = \langle 4x, 4y, 2z \rangle$, so the Lagrange multiplier conditions are $2x = 4\lambda x$, $1 = 4\lambda y$, $1 = 2\lambda z$, and $2x^2 + 2y^2 + z^2 = 2$. The second and third equations tell us that $z = 2y$. The first equation tells us that either $x = 0$ or $\lambda = \frac{1}{2}$. If $\lambda = \frac{1}{2}$, we can deduce that $y = \frac{1}{2}$ and $z = 1$. If $x = 0$, we can deduce from the fact that $z = 2y$ that $y = \pm 1/\sqrt{3}$ and $z = \pm 2/\sqrt{3}$. The critical points are $(0, 1/\sqrt{3}, 2/\sqrt{3})$, $(0, -1/\sqrt{3}, 2/\sqrt{3})$ and $(1/2, 1/2, 1)$ and $(-1/2, 1/2, 1)$. The absolute minimum is $-\sqrt{3}$ at $(0, -1/\sqrt{3}, -2/\sqrt{3})$ and the absolute maximum is $\frac{7}{4}$ at both $(1/2, 1/2, 1)$ and $(-1/2, 1/2, 1)$.

15.8.26 Observe first that it is equivalent to find the extreme values of the simpler function $h(x, y, z) = f(x, y, z)^2 = xyz$ subject to the constraints $g(x, y, z) = x + y + z - 1 = 0$ and $x, y, z \geq 0$. We have $\nabla h = \langle yz, xz, xy \rangle$, $\nabla g = \langle 1, 1, 1 \rangle$, so the Lagrange multiplier conditions are $yz = \lambda$, $xz = \lambda$, $xy = \lambda$, $x + y + z - 1 = 0$. Assume first that $x, y, z \neq 0$; then first three equations give $x = y = z$, and the constraint implies $3x = 1$, so $x, y, z = \frac{1}{3}$. The domain of f (or g) is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, which is closed and bounded. We also note that $f = h = 0$ along any of the edges of the triangle. Therefore, the maximum value of f is $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3\sqrt{3}}$ and the minimum value is 0 .

15.8.27 Let $x, y, z \geq 0$ denote the lengths of the sides of the box, with z the longest side. Then the length plus girth of the box is $2x + 2y + z$ so we must maximize the volume $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2x + 2y + z - 108 = 0$. We have $\nabla f = \langle yz, xz, xy \rangle$, $\nabla g = \langle 2, 2, 1 \rangle$ so the Lagrange multiplier conditions are $yz = 2\lambda$, $xz = 2\lambda$, $xy = \lambda$, $2x + 2y + z - 108 = 0$. Assume that $x, y, z > 0$ (otherwise the volume is 0); then the first three equations give $\frac{xyz}{2\lambda} = x = y = \frac{z}{2}$, and the constraint gives $6x = 108$, so $x = 18$ and the box has dimensions $18 \text{ in} \times 18 \text{ in} \times 36 \text{ in}$. The domain of f is the triangle with vertices $(54, 0, 0)$, $(0, 54, 0)$, $(0, 0, 108)$, which is closed and bounded. We also note that $f = 0$ along any of the edges of the triangle. Therefore, the maximum value of f occurs at the point we found, and the minimum value is 0 .

15.8.28 Let $x, y, z > 0$ denote the lengths of the sides of the box; then its surface area is given by $f(x, y, z) = 2xy + 2yz + 2xz$, which we must minimize subject to the constraint $g(x, y, z) = xyz - 16 = 0$. We have $\nabla f = \langle 2(y+z), 2(x+z), 2(x+y) \rangle$, $\nabla g = \langle yz, xz, xy \rangle$ so the Lagrange multiplier conditions are $2(y+z) = \lambda yz$, $2(x+z) = \lambda xz$, $2(x+y) = \lambda xy$, $xyz - 16 = 0$. The first three equations give $\frac{\lambda}{2} = \frac{1}{y} + \frac{1}{z} = \frac{1}{x} + \frac{1}{z} = \frac{1}{x} + \frac{1}{y}$ which implies $x = y = z$; the constraint gives $x^3 = 16$, so $x = y = z = 2\sqrt[3]{2}$, and the box is a cube with side length $2\sqrt[3]{2}$ ft. The domain of f is the surface in the first quadrant given by $xyz = 16$, which is unbounded. We can rewrite $f(x, y, z) = 2xy + 2yz + 2xz = 2xyz \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = 32 \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$ on this surface. Now if, say, $x \rightarrow \infty$ on this surface either $y \rightarrow 0$ or $z \rightarrow 0$ (or both) because of the relation $xyz = 16$; therefore, $f(x, y, z) \rightarrow \infty$. This shows that the point we found above must give the absolute minimum value of f on the surface.

15.8.29 It suffices to find the extreme values of the function $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x^2 + xy + 2y^2 - 1 = 0$. We have $\nabla f = \langle 2x, 2y \rangle$, $\nabla g = \langle 2x + y, 4y + x \rangle$, so the Lagrange multiplier conditions are $2x = \lambda(2x + y)$, $2y = \lambda(4y + x)$, $x^2 + xy + 2y^2 - 1 = 0$. The first two equations give $2x(4y + x) = \lambda(2x + y)(4y + x) = 2y(2x + y) \implies x^2 + 4xy = y^2 + 2xy$, or $x^2 + 2xy - y^2 = 0$. This implies that both $x, y \neq 0$, for if, say, $x = 0$ then this condition gives $y = 0$ as well, which violates the constraint (same argument for $y = 0$). Let $r = \frac{y}{x}$, and rewrite this equation as $\frac{y^2}{x^2} - \frac{2y}{x} - 1 = r^2 - 2r - 1 = 0$, which we can solve to obtain $r = 1 \pm \sqrt{2}$. We now use the constraint and the relation $y = rx$ to obtain $x^2(1 + r + 2r^2) = 1$ or $x^2 = \frac{1}{2r^2 + r + 1}$. Then the values of the function f are given by $f(x, y) = f(x, rx) = x^2(1 + r^2) = \frac{1 + r^2}{2r^2 + r + 1} = \frac{6 \pm 2\sqrt{2}}{7} \approx 0.4531, 1.2612$, and the corresponding minimum and maximum distances are $\sqrt{\frac{6 \pm 2\sqrt{2}}{7}} \approx 0.6731, 1.1230$.

15.8.30 Let (x, y) be the vertex of the rectangle in the first quadrant; then the area of the rectangle is $4xy$, so it suffices to maximize the function $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + 4y^2 - 4 = 0$ and $x, y \geq 0$. We have $\nabla f = \langle y, x \rangle$, $\nabla g = \langle 2x, 8y \rangle$ so the Lagrange multiplier conditions are $y = 2\lambda x$, $x = 8\lambda y$, $x^2 + 4y^2 - 4 = 0$. The first two equations give $4y^2 = 8\lambda xy = x^2$, and substituting in the constraint gives $2x^2 = 4$ so $x = \sqrt{2}$ and $y = \frac{\sqrt{2}}{2}$. The domain given by the constraint and $x, y \geq 0$ is a closed and bounded arc, and we observe that $f = 0$ at the boundary points $(2, 0)$ and $(0, 2)$. Therefore, f takes its maximum at $\left(\sqrt{2}, \frac{\sqrt{2}}{2} \right)$, and the corresponding area of the rectangle is $A = 4 \cdot \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 4$.

15.8.31 Let (x, y) be the vertex of the rectangle in the first quadrant; then the perimeter of the rectangle is $4(x + y)$, so it suffices to maximize the function $f(x, y) = x + y$ subject to the constraint $g(x, y) = 2x^2 + 4y^2 - 3 = 0$ and $x, y \geq 0$. We have $\nabla f = \langle 1, 1 \rangle$, $\nabla g = \langle 4x, 8y \rangle$, so the Lagrange multiplier conditions are $1 = 4\lambda x$, $1 = 8\lambda y$, $2x^2 + 4y^2 - 3 = 0$. The first two equations give $4x = \frac{1}{\lambda} = 8y$; so $x = 2y$. Substituting in the constraint gives $8y^2 + 4y^2 = 3$, so $y^2 = 1/4$ and $y = 1/2$ and thus $x = 1$. Note that $f(1, 1/2) = 1.5$. The domain given by the constraint and $x, y \geq 0$ is a closed and bounded arc, and we observe that at the boundary points we have $f(0, \sqrt{3}/2) = \sqrt{3}/2 \approx .87$ and $f(\sqrt{3}/2, 0) = \sqrt{3}/2 \approx 1.25$. Thus, the dimensions of the rectangle of maximum perimeter is 2×1 .

15.8.32 It suffices to maximize the function $f(x, y, z) = (x + 2)^2 + (y - 5)^2 + (z - 1)^2$ subject to the constraint $g(x, y, z) = 2x + 3y + 6z - 10 = 0$. We have $\nabla f = \langle 2x + 4, 2y - 10, 2z - 2 \rangle$, $\nabla g = \langle 2, 3, 6 \rangle$, so the Lagrange multiplier conditions are $2x + 4 = 2\lambda$, $2y - 10 = 3\lambda$, $2z - 2 = 6\lambda$, $2x + 3y + 6z - 10 = 0$. The first three equations give $x + 2 = \lambda = \frac{2y - 10}{3} = \frac{z - 1}{3}$, which gives $y = \frac{3}{2}x + 8$, $z = 3x + 7$. Substituting in the constraint and solving for x gives $x = -\frac{16}{7}$, so the closest point is $\left(-\frac{16}{7}, \frac{32}{7}, \frac{1}{7} \right)$; $f\left(-\frac{16}{7}, \frac{32}{7}, \frac{1}{7} \right) = 1$ so the distance is 1.

15.8.33 It suffices to minimize the function $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z + 3)^2$ subject to the constraint $4x + y - 1 = 0$. We have $\nabla f = \langle 2(x - 1), 2(y - 2), 2(z + 3) \rangle$ and $\nabla g = \langle 4, 1, 0 \rangle$. Then the Lagrange multiplier conditions are $2x - 2 = 4\lambda$, $2y - 4 = \lambda$, $2z + 6 = 0$, $4x + y - 1 = 0$. Solving this system gives $x = -3/17$, $y = 29/17$, $z = -3$, and $\lambda = -10/17$, so the closest point on the surface to the given point is $(-3/17, 29/17, -3)$.

15.8.34 It suffices to minimize the function $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 - z^2 = 0$. We have $\nabla f = \langle 2x - 2, 2y - 4, 2z \rangle$, $\nabla g = \langle 2x, 2y, -2z \rangle$ so the Lagrange multiplier conditions are $2x - 2 = 2\lambda x$, $2y - 4 = 2\lambda y$, $2z = -2\lambda z$, $x^2 + y^2 - z^2 = 0$. Observe that $z \neq 0$; otherwise $x = y = z = 0$, which is not a solution to the Lagrange multiplier equations. Therefore, the third equation gives $\lambda = -1$, the first two equations give $x = \frac{1}{2}$, $y = 1$ and the constraint gives $z = \pm \frac{\sqrt{5}}{2}$. Therefore, the points on the cone closest to $(1, 2, 0)$ is $\left(\frac{1}{2}, 1, \pm \frac{\sqrt{5}}{2}\right)$.

15.8.35 It suffices to find the extreme values the function $f(x, y, z) = (x - 2)^2 + (y - 3)^2 + (z - 4)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$. We have $\nabla f = \langle 2x - 4, 2y - 6, 2z - 8 \rangle$, $\nabla g = \langle 2x, 2y, 2z \rangle$ so the Lagrange multiplier conditions are $2x - 4 = 2\lambda x$, $2y - 6 = 2\lambda y$, $2z - 8 = 2\lambda z$, $x^2 + y^2 + z^2 - 9 = 0$. We can write the first three equations in the form $(1 - \lambda)\langle x, y, z \rangle = \langle 2, 3, 4 \rangle$ so $\langle x, y, z \rangle = c\langle 2, 3, 4 \rangle$ for some scalar c ; using the constraint, we find that $c = \pm \frac{3}{\sqrt{29}}$, and hence $\langle x, y, z \rangle = \pm \frac{3}{\sqrt{29}}\langle 2, 3, 4 \rangle$; the corresponding values of f are $f\left(\pm \left(\frac{6}{\sqrt{29}}, \frac{9}{\sqrt{29}}, \frac{12}{\sqrt{29}}\right)\right) = 38 \mp 6\sqrt{29} = (\sqrt{29} \mp 3)^2$, so the minimum distance is $\sqrt{29} - 3$ and the maximum distance is $\sqrt{29} + 3$.

15.8.36 Assume the cylinder is the region $x^2 + y^2 \leq r^2$, $-\frac{h}{2} \leq z \leq \frac{h}{2}$; then r and h are the radius and height respectively of the cylinder, and the points on the cylinder furthest from the origin have distance $\left(r^2 + \left(\frac{h}{2}\right)^2\right)^{1/2}$; therefore, we must maximize the function $V = f(r, h) = \pi r^2 h$ subject to the constrain $g(r, h) = r^2 + \left(\frac{h}{2}\right)^2 - 16^2 = 0$. We have $\nabla f = \pi\langle 2rh, r^2 \rangle$, $\nabla g = \left\langle 2r, \frac{h}{2} \right\rangle$, so the Lagrange multiplier conditions are equivalent to $2rh = 2\lambda r$, $r^2 = \lambda \frac{h}{2}$, $r^2 + \left(\frac{h}{2}\right)^2 - 16^2 = 0$. Because we are interested in maximizing the volume we may assume that $r, h > 0$; then the first equation gives $\lambda = h$ and the second equation gives $h^2 = 2r^2$; substituting in the constraint then gives dimensions $r = \frac{16\sqrt{6}}{3}$, $h = \frac{32\sqrt{3}}{3}$.

15.8.37 Notice that the constraint is equivalent to $\ell + 2g = 6$. We have $\nabla U = 5\langle \ell^{-1/2}g^{1/2}, \ell^{1/2}g^{-1/2} \rangle$ so the Lagrange multiplier conditions are equivalent to $\ell^{-1/2}g^{1/2} = \lambda$, $\ell^{1/2}g^{-1/2} = 2\lambda$, $\ell + 2g = 6$. Eliminating λ from the first two equations gives $\ell^{1/2}g^{-1/2} = 2\ell^{-1/2}g^{1/2}$, which simplifies to $g = \frac{\ell}{2}$; substituting in the constraint then gives $\ell = 3$ and $g = \frac{3}{2}$. The value of the utility function at this point is $U = 15\sqrt{2}$.

15.8.38 Notice that the constraint is equivalent to $2\ell + g = 6$. We have $\nabla U = \frac{32}{3}\langle 2\ell^{-1/3}g^{1/3}, \ell^{2/3}g^{-2/3} \rangle$, so the Lagrange multiplier conditions are equivalent to $2\ell^{-1/3}g^{1/3} = 2\lambda$, $\ell^{2/3}g^{-2/3} = \lambda$, $2\ell + g = 6$. Eliminating λ from the first two equations gives $\ell^{-1/3}g^{1/3} = \ell^{2/3}g^{-2/3}$, which simplifies to $g = \ell$; substituting in the constraint then gives $\ell = g = 2$. The value of the utility function at this point is $U = 64$.

15.8.39 Notice that the constraint is equivalent to $5\ell + 4g = 20$. We have $\nabla U = \frac{8}{5}\langle 4\ell^{-1/5}g^{1/5}, \ell^{4/5}g^{-4/5} \rangle$, so the Lagrange multiplier conditions are equivalent to $4\ell^{-1/5}g^{1/5} = 5\lambda$, $\ell^{4/5}g^{-4/5} = 4\lambda$, $5\ell + 4g = 20$. Eliminating λ from the first two equations gives $16\ell^{-1/5}g^{1/5} = 5\ell^{4/5}g^{-4/5}$, which simplifies to $g = \frac{5\ell}{16}$;

substituting in the constraint then gives $\ell = \frac{16}{5}$, $g = 1$. The value of the utility function at this point is

$$U = 8 \cdot \left(\frac{16}{5}\right)^{4/5} \approx 20.287.$$

15.8.40 We have $\nabla U = \frac{1}{6} \langle \ell^{-5/6} g^{5/6}, 5\ell^{1/6} g^{-1/6} \rangle$, so the Lagrange multiplier conditions are equivalent to $\ell^{-5/6} g^{5/6} = 4\lambda$, $5\ell^{1/6} g^{-1/6} = 5\lambda$, $4\ell + 5g = 20$. Eliminating λ from the first two equations gives $\ell^{-5/6} g^{5/6} = 4\ell^{1/6} g^{-1/6}$, which simplifies to $g = 4\ell$; substituting in the constraint then gives $\ell = \frac{5}{6}$, $g = \frac{10}{3}$. The value of the utility function at this point is $U = \left(\frac{5}{6}\right)^{1/6} \left(\frac{10}{3}\right)^{5/6} \approx 2.646$.

15.8.41

- True. This is because the tangent plane to a sphere at any point has normal vector in the direction of the line joining the point to the center of the sphere.
- False in general. In fact, the two vectors ∇f and ∇g are in the same direction, so $\nabla f \cdot \nabla g = 0$ only if one of these vectors is zero.

15.8.42 Let $x, y \geq 0$ be the dimensions of the base of the box and h be the height; then x, y, h satisfy the constraint $2x + 2y + h = 96$. The volume of the box is $V = xyh$, so the Lagrange multiplier conditions are $yh = 2\lambda$, $xh = 2\lambda$, $xy = \lambda$, $2x + 2y + h = 96$. Eliminating λ from the first three equations gives $y = x$ and $h = 2x$; substituting in the constraint then gives $x = 16$, $y = 16$, $h = 32$ in.

15.8.43 Let $x, y > 0$ be the dimensions of the base and $h > 0$ be the height of the box; then the four sides of the box plus the base have total area $2xh + 2yh + xy = 2$, which is our constraint. The volume of the box is $V = xyh$, so the Lagrange multiplier conditions are $yh = \lambda(2h + y)$, $xh = \lambda(2h + x)$, $xy = 2\lambda(x + y)$, $2xh + 2yh + xy = 2$. The first two equations give $\frac{2h + y}{yh} = \frac{2h + x}{xh} \implies \frac{2}{y} + \frac{1}{h} = \frac{2}{x} + \frac{1}{h}$, so $y = x$. The second and third equations give $\frac{2h + x}{xh} = \frac{2x + 2y}{xy} \implies \frac{2}{x} + \frac{1}{h} = \frac{2}{x} + \frac{2}{y} = \frac{4}{x}$, so $h = \frac{x}{2}$. Then substituting in the constraint gives $3x^2 = 2$, so $x = \frac{\sqrt{6}}{3}$. Therefore, the box with largest volume has height $\frac{\sqrt{6}}{6}$ m and base $\frac{\sqrt{6}}{3} \times \frac{\sqrt{6}}{3}$ m.

15.8.44 Let $x, y > 0$ be the dimensions of the base and $h > 0$ be the height of the box; then the volume of the box is $xyh = 4$, which is our constraint. The four sides of the box plus the base have total area $A = 2xh + 2yh + xy$, so the Lagrange multiplier conditions are $2h + y = \lambda y h$, $2h + x = \lambda x h$, $2x + 2y = \lambda x y$, $xyh = 4$. The first two equations give $\frac{2h + y}{yh} = \frac{2h + x}{xh} \implies \frac{2}{y} + \frac{1}{h} = \frac{2}{x} + \frac{1}{h}$, so $y = x$. The second and third equations give $\frac{2h + x}{xh} = \frac{2x + 2y}{xy} \implies \frac{2}{x} + \frac{1}{h} = \frac{2}{x} + \frac{2}{y} = \frac{4}{x}$, so $h = \frac{x}{2}$. Then substituting in the constraint gives $x^3 = 8$, so $x = 2$. Therefore, the box with smallest area has dimensions 2 m $\times$ 2 m $\times$ 1 m.

15.8.45 Let $x, y \geq 0$ be the dimensions of the base and $z \geq 0$ be the height of the box; then $x + 2y + 3z = 6$ is our constraint. The box has volume $V = xyz$, so the Lagrange multiplier conditions are $yz = \lambda$, $xz = 2\lambda$, $xy = 3\lambda$, $x + 2y + 3z = 6$. The first two equations give $y = \frac{x}{2}$, the first and third equations give $z = \frac{x}{3}$, and substituting in the constraint gives $x = 2$. Therefore, the box with largest volume has dimensions $x = 2$, $y = 1$, $z = \frac{2}{3}$.

15.8.46 Let (x, y, z) be the point on the ellipsoid; then $x, y, z > 0$ and the box has volume $V = xyz$; the relation $36x^2 + 4y^2 + 9z^2 = 36$ is our constraint. The Lagrange multiplier conditions are $yz = 72\lambda x$, $xz = 8\lambda y$, $xy = 18\lambda z$, $36x^2 + 4y^2 + 9z^2 = 36$. The first two equations give $\lambda = \frac{yz}{72x} = \frac{xz}{8y}$, which implies

$y^2 = 9x^2$ or $y = 3x$. Similarly the first and third equations give $z = 2x$ and then the constraint gives $3x^2 = 1$, so $x = \frac{\sqrt{3}}{3}$, and the box with maximum volume has dimensions $\frac{\sqrt{3}}{3} \times \sqrt{3} \times \frac{2\sqrt{3}}{3}$.

15.8.47 The constraint is the equation of the plane $x - y + z = 2$; the distance from a point (x, y, z) to the point $(1, 1, 1)$ is given by $d^2 = (x - 1)^2 + (y - 1)^2 + (z - 1)^2$, so it suffices to minimize the function $f(x, y, z) = (x - 1)^2 + (y - 1)^2 + (z - 1)^2$ subject to the constraint $x - y + z = 2$. The Lagrange multiplier conditions are $2x - 2 = \lambda$, $2y - 2 = -\lambda$, $2z - 2 = \lambda$, $x - y + z = 2$. The first two equations give $2(x + y) = 4$, so $y = 2 - x$, the first and third equations give $z = x$, and then substituting in the constraint gives $x = \frac{4}{3}$, so the closest point on the plane to $(1, 1, 1)$ is $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$.

15.8.48 The function $f(x, y) = x^2 + 4y^2 + 1$ has a unique critical point at $(0, 0)$, and $f(0, 0) = 1$. On the boundary $\{x^2 + 4y^2 = 1\}$ of R , $f(x, y) = 2$ at all points. Hence, the absolute minimum and maximum values of f on R are 1 and 2.

15.8.49 $\nabla f = \langle 2x, 2y - 2 \rangle$ and $\nabla g = \langle 2x, 2y \rangle$. Note that $\nabla f = \langle 0, 0 \rangle$ at $(0, 1)$, and $f(0, 1) = 0$. Consider $\nabla f = \lambda \nabla g$. If $x = 0$ then $y = \pm 2$. If $x \neq 0$, then because $2x = 2\lambda x$, we have $\lambda = 1$ which leads to a contradiction with the equation $2y - 2 = 2\lambda y$. So we need to check $(0, \pm 2)$. The absolute minimum is 0 at $(0, 1)$. At $(0, -2)$ we have the absolute maximum of 9.

15.8.50 $\nabla f = \langle 4x, 2y \rangle$ and $\nabla g = \langle 2x, 2y \rangle$. If $x = 0$ then $y = \pm 4$. If $x \neq 0$, then the equation $4x = 2\lambda x$ tells us that $\lambda = 2$, in which case $2y = 4y$ so $y = 0$ and $x = \pm 4$. So we must check $(0, \pm 4)$ and $(\pm 4, 0)$ on the boundary. Also, $\nabla f = \langle 0, 0 \rangle$ at $(0, 0)$. There is an absolute minimum of 0 at $(0, 0)$ and an absolute maximum of 32 at $(\pm 4, 0)$.

15.8.51 $\nabla f = \langle 4x - 4, 6y \rangle$, which is equal to $\langle 0, 0 \rangle$ at $(1, 0)$. $\nabla g = \langle 2x - 2, 2y \rangle$. We have $6y = 2\lambda y$, so either $y = 0$ (which means either $x = 0$ or $x = 2$), or $\lambda = 3$. If $\lambda = 3$ then $4x - 4 = 6(x - 1)$, so $2 = 2x$ and $x = 1$ (and therefore $y = \pm 1$.) We must check $(1, 0)$, $(2, 0)$, $(0, 0)$, $(1, 1)$ and $(1, -1)$. There is an absolute minimum of 0 at $(1, 0)$ and an absolute maximum of 3 at $(1, \pm 1)$.

15.8.52

- It suffices to maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^4 + y^4 + z^4 = 1$. The Lagrange multiplier conditions give $2x = 4\lambda x^3$, $2y = 4\lambda y^3$, $2z = 4\lambda z^3$, $x^4 + y^4 + z^4 = 1$. Assume first that $x, y, z \neq 0$; then we can eliminate λ and conclude that $x^2 = y^2 = z^2$; the constraint gives $3x^4 = 1$ so $x^2 = \frac{1}{\sqrt{3}}$ and $f(x, y, z) = \sqrt{3}$. There are also solutions to the Lagrange conditions with x, y or $z = 0$, but one can check that these do not give larger values of f . Therefore, the extreme points have coordinates $x, y, z = \pm 3^{-1/4}$.
- It suffices to maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^{2n} + y^{2n} + z^{2n} = 1$. The Lagrange multiplier conditions give $2x = 2\lambda n x^{2n-1}$, $2y = 2\lambda n y^{2n-1}$, $2z = 2\lambda n z^{2n-1}$, $x^{2n} + y^{2n} + z^{2n} = 1$. As above, assume that $x, y, z \neq 0$; then we can eliminate λ and conclude that $x^{2n-2} = y^{2n-2} = z^{2n-2}$, which implies $x^{2n} = y^{2n} = z^{2n}$; the constraint gives $3x^{2n} = 1$ so $x^2 = 3^{-1/n} \implies f(x, y, z) = 3^{-\frac{n-1}{n}}$. There are also solutions to the Lagrange conditions with x, y or $z = 0$, but one can check that these do not give larger values of f . Therefore, the extreme points have coordinates $x, y, z = \pm 3^{-\frac{1}{2n}}$.
- As $n \rightarrow \infty$, the coordinates of the extreme points $x, y, z \rightarrow \pm 1$, and the limiting distance of these points to the origin is $\sqrt{3}$.

15.8.53 Notice that the constraint is equivalent to $2K + 3L = 30$. We have

$$\nabla f = \frac{1}{2} \langle K^{-1/2} L^{1/2}, K^{1/2} L^{-1/2} \rangle,$$

so the Lagrange multiplier conditions are equivalent to $K^{-1/2}L^{1/2} = 2\lambda$, $K^{1/2}L^{-1/2} = 3\lambda$, $2K + 3L = 30$. Eliminating λ from the first two equations gives $3K^{-1/2}L^{1/2} = 2K^{1/2}L^{-1/2}$, which simplifies to $3L = 2K$; substituting in the constraint then gives $K = 7.5$ and $L = 5$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f . Therefore, the values of K and L found above must maximize f .

15.8.54 Notice that the constraint is equivalent to $K + 2L = 12$. We have

$$\nabla f = \frac{10}{3} \langle K^{-2/3}L^{2/3}, 2K^{1/3}L^{-1/3} \rangle,$$

so the Lagrange multiplier conditions are equivalent to $K^{-2/3}L^{2/3} = \lambda$, $2K^{1/3}L^{-1/3} = 2\lambda$, $K + 2L = 12$. Eliminating λ from the first two equations gives $K^{-2/3}L^{2/3} = K^{1/3}L^{-1/3}$, which simplifies to $K = L$; substituting in the constraint then gives $K = L = 4$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f . Therefore, the values of K and L found above must maximize f .

15.8.55 We have $\nabla f = \langle aK^{a-1}L^{1-a}, (1-a)K^aL^{-a} \rangle$, so the Lagrange multiplier conditions are equivalent to $aK^{a-1}L^{1-a} = \lambda p$, $(1-a)K^aL^{-a} = \lambda q$, $pK + qL = B$. Eliminating λ from the first two equations gives $aqK^{a-1}L^{1-a} = (1-a)pK^aL^{-a}$, which simplifies to $aqL = (1-a)pK$; substituting in the constraint then gives $K = \frac{aB}{p}$ and $L = \frac{(1-a)B}{q}$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f . Therefore, the values of K and L found above must maximize f .

15.8.56 We have $\nabla T = \langle 50x, 50y \rangle$ and $\nabla g = \langle 2x + y, 2y + x \rangle$ where $g(x) = x^2 + y^2 + xy = 1$. The Lagrange multiplier conditions are $50x = \lambda(2x + y)$, $50y = \lambda(2y + x)$, and $x^2 + y^2 + xy = 1$. Multiplying the first equation by y and the second by x and subtracting gives $0 = y^2 - x^2$, so $x = \pm y$. If $x = y$, then the constraint gives $3x^2 = 1$, so $x = \pm \frac{1}{\sqrt{3}}$. Note that $T(\pm(1/\sqrt{3}), \pm(1/\sqrt{3})) = 50/3$. If $x = -y$, the constraint gives $x = \pm 1$, and note that $T(1, -1) = T(-1, 1) = 50$. The hottest temperature on the edge of the plate is 50 and the coldest is $\frac{50}{3}$.

15.8.57 The function to be maximized is $f(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$, subject to the constraint $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$. The Lagrange multiplier conditions are $1 = 2\lambda x_1$, $1 = 2\lambda x_2$, $1 = 2\lambda x_3$, $1 = 2\lambda x_4$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$. The first four equations give $x_1 = x_2 = x_3 = x_4$, and then the constraint gives $4x_1^2 = 16$, so $x_1 = \pm 2$. Therefore, the maximum of f on the closed, bounded set given by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$ is $f(2, 2, 2, 2) = 8$ (and the minimum is $f(-2, -2, -2, -2) = -8$).

15.8.58 The function to be maximized is $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$, subject to the constraint $x_1^2 + x_2^2 + \dots + x_n^2 = c^2$. The Lagrange multiplier conditions are $1 = 2\lambda x_1$, $1 = 2\lambda x_2$, $\dots$, $1 = 2\lambda x_n$, $x_1^2 + x_2^2 + \dots + x_n^2 = c^2$. The first n equations give $x_1 = x_2 = \dots = x_n$, and then the constraint gives $nx_1^2 = c^2$, so $x_1 = \pm \frac{c}{\sqrt{n}}$. Therefore, the maximum of f on the closed, bounded set given by $x_1^2 + x_2^2 + \dots + x_n^2 = c^2$ is

$$f\left(\frac{c}{\sqrt{n}}, \dots, \frac{c}{\sqrt{n}}\right) = c\sqrt{n} \text{ (and the minimum is } -c\sqrt{n}\text{)}.$$

15.8.59 The function to be maximized is $f(x_1, x_2, \dots, x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n$, subject to the constraint $x_1^2 + x_2^2 + \dots + x_n^2 = 1$. The Lagrange multiplier conditions are $a_1 = 2\lambda x_1$, $a_2 = 2\lambda x_2$, $\dots$, $a_n = 2\lambda x_n$, $x_1^2 + x_2^2 + \dots + x_n^2 = 1$. The first n equations are equivalent to $c(x_1, x_2, \dots, x_n) = (a_1, a_2, \dots, a_n)$ for some c , and then the constraint gives $c^2 = a_1^2 + a_2^2 + \dots + a_n^2$. Therefore, the maximum of f on the closed, bounded set given by $f\left(\frac{a_1}{c}, \frac{a_2}{c}, \dots, \frac{a_n}{c}\right) = \frac{1}{c}(a_1^2 + a_2^2 + \dots + a_n^2) = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$ (and the minimum is $-\sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$).

15.8.60

- a. The function to be maximized is $f(x, y, z) = xyz$, subject to the constraints $x + y + z = k$ and $x, y, z > 0$. The Lagrange multiplier conditions are $yz = \lambda$, $xz = \lambda$, $xy = \lambda$, $x + y + z = k$; the first three equations imply $\frac{xyz}{\lambda} = x = y = z$, and the constraint gives $x = y = z = \frac{k}{3}$. The set determined by the constraints $x + y + z = k$ and $x, y, z \geq 0$ is a triangle, and $f(x, y, z) = 0$ along its edges. Therefore, the maximum of f must occur at the point we found. This means that if $x, y, z > 0$ and $x + y + z = k$, then $xyz \leq \left(\frac{k}{3}\right)^3$ if and only if $(xyz)^{1/3} \leq \frac{x + y + z}{3}$.
- b. Generalizing the case above, the function to be maximized is $f(x_1, \dots, x_n) = x_1 \cdots x_n$, subject to the constraints $x_1 + \cdots + x_n = k$ and $x_1, \dots, x_n > 0$. The Lagrange multiplier conditions are $x_2 \cdots x_n = \lambda$, $x_1 x_3 \cdots x_n = \lambda, \dots, x_2 \cdots x_{n-1} = \lambda$, $x_1 + \cdots + x_n = k$; the first n equations imply $\frac{x_1 \cdots x_n}{\lambda} = x_1 = x_2 = \cdots = x_n$, and the constraint gives $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$. The set determined by the constraints $x_1 + \cdots + x_n = k$ and $x_1, \dots, x_n \geq 0$ is closed and bounded, and $f(x_1, \dots, x_n) = 0$ along its boundary. Therefore, the maximum of f must occur at the point we found. This means that if $x_1, \dots, x_n > 0$ and $x_1 + \cdots + x_n = k$, then $x_1 \cdots x_n \leq \left(\frac{k}{n}\right)^n \iff (x_1 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n}$.

15.8.61

- a. Gradients are perpendicular to level surfaces.
- b. If ∇f was not in the plane spanned by ∇g and ∇h , then f could be increased or decreased by moving the point P slightly along the curve C .
- c. Because ∇f is in the plane spanned by ∇g and ∇h , we can express ∇f as a linear combination of ∇g and ∇h .
- d. The gradient condition from part (c), as well as the constraints, must be satisfied.

15.8.62 It suffices to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g(x, y, z) = x + 2z - 12 = 0$ and $h(x, y, z) = x + y - 6 = 0$. Using the method described in problem 61 above, we solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$, together with the constraints. This gives the conditions $2x = \lambda + \mu$, $2y = \mu$, $2z = 2\lambda$, $x + 2z = 12$, $x + y = 6$. The first three equations imply $2x = 2y + z$, and the constraints give $y = 6 - x$, $z = 6 - \frac{1}{2}x$; therefore $2x = 2(6 - x) + 6 - \frac{1}{2}x$, so $x = 4$, which gives $y = 2$, $z = 4$. There is a point on the line closest to the origin, so this point must be $(4, 2, 4)$.

15.8.63 We wish to find the extreme values of the function $f(x, y, z) = xyz$ subject to the constraints $g(x, y, z) = x^2 + y^2 - 4 = 0$ and $h(x, y, z) = x + y + z - 1 = 0$. Using the method described in problem 61 above, we solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$, together with the constraints. This gives the conditions $yz = 2\lambda x + \mu$, $xz = 2\lambda y + \mu$, $xy = \mu$, $x^2 + y^2 = 4$, $x + y + z = 1$. The first and second equations together give $z(y - x) = 2\lambda(x - y)$, which implies either $x = y$ or $z = -2\lambda$. In the former case the first constraint gives $2x^2 = 4$, so $x = y = \pm\sqrt{2}$, solving for z gives $z = 1 \mp 2\sqrt{2}$ and $f(x, y, z) = 2 \pm 4\sqrt{2}$. In the latter case we obtain $(x + y)z = \mu = xy$ from the first and third equations, and then solving for $z = 1 - x - y$ gives $(x + y)(1 - x - y) - xy = 0 \iff x + y - 3xy - 4 = 0$, using the relation $x^2 + y^2 = 4$. Therefore, $x - 4 = (3x - 1)y(x - 4)^2 = (3x - 1)(4 - x^2)x^2 - 8x + 16 = (9x^2 - 6x + 1)(4 - x^2)$ which simplifies to $9x^4 - 6x^3 - 34x^2 + 16x + 12 = 0$. This equation has roots $x \approx -0.42, -1.78, 1.96, 0.91$ in the interval $(-2, 2)$; one can check that the corresponding solutions (x, y, z) to the Lagrange conditions do not give values larger or smaller resp. than $2 + 4\sqrt{2}$, $2 - 4\sqrt{2}$, so these are in fact the maximum and minimum values of f subject to the constraints.

15.8.64 We wish to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g(x, y, z) = 4x^2 + 4y^2 - z^2 = 0$ and $h(x, y, z) = 2x + 4z - 5 = 0$. Using the method described in problem 61 above, we solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$, together with the constraints. This gives the conditions

$2x = 8\lambda x + 2\mu$, $2y = 8\lambda y$, $2z = -2\lambda z + 4\mu$, $z^2 = 4x^2 + 4y^2$, $2x + 4z = 5$. The second equation gives $y(1 - 4\lambda) = 0$, so either $y = 0$ or $\lambda = \frac{1}{4}$. Consider first the case $y = 0$; then the first constraint equation gives $z = \pm 2x$. If $z = 2x$ then the second constraint equation gives $x = \frac{1}{2}$, $z = 1$ and we obtain the point $\left(\frac{1}{2}, 0, 1\right)$; similarly if $z = -2x$ then we obtain the point $\left(-\frac{5}{6}, 0, \frac{5}{3}\right)$. In the case $\lambda = \frac{1}{4}$ the first equation gives $\mu = 0$ and then the third equation gives $z = 0$; but then the first of the constraints implies that $x = y = z = 0$, which violates the second constraint. Hence there are no solutions to the Lagrange conditions in this case, and the minimum and maximum values of the function f along this curve are $f\left(\frac{1}{2}, 0, 1\right) = \frac{5}{4}$ and $f\left(-\frac{5}{6}, 0, \frac{5}{3}\right) = \frac{125}{36}$.

15.8.65

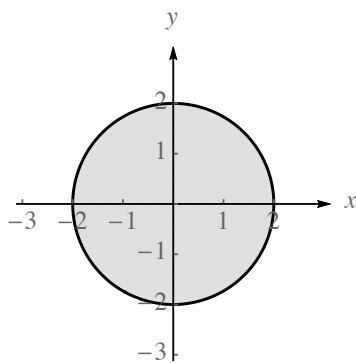
- $\nabla f = \langle y + 1, x + 1 \rangle$ and $\nabla g = \langle y, x \rangle$, so the system of equations is $y + 1 = \lambda y$, $x + 1 = \lambda x$, and $xy = 4$.
- Subtracting the first equation from the second equation, we obtain $x - y = \lambda(x - y)$, which implies that $x - y - \lambda(x - y) = 0$, or $(x - y)(1 - \lambda) = 0$. So either $x = y$ or $\lambda = 1$. Substituting $\lambda = 1$ into either of the first two equations in part (a) results in the equality $1 = 0$, which means that there is no solution to the system of equations in this case. So it must be the case that $x = y$. Replacing y with x in the third equation in the solution to part (a), we find that $x^2 = 4$, or $x = \pm 2$. Therefore the method of Lagrange multipliers gives us the points $(x, y) = (\pm 2, \pm 2)$.
- We have $f(2, 2) = 108$. Replacing y with $4/x$ in the function f , we find that $h_1(x) = x + \frac{4}{x} + 104$. The first and second derivatives of h_1 are $h_1'(x) = 1 - \frac{4}{x^2}$ and $h_1''(x) = \frac{8}{x^3}$. Because $h_1'(2) = 0$ and $h_1''(x) > 0$ for $x > 0$, it follows that f has an absolute minimum of $h_1(2) = f(2, 2) = 108$ along the curve C_1 and it does not have an absolute maximum value along C_1 .
- We have $f(-2, -2) = 100$. By part (c), $h_2(x) = x + 4/x + 104$, $h_2'(x) = 1 - 4/x^2$ and $h_2''(x) = 8/x^3$, for $x < 0$. Because $h_2'(-2) = 0$ and $h_2''(x) < 0$ for $x < 0$, it follows that f has an absolute maximum of $h_2(-2) = f(-2, -2) = 100$ along the curve C_2 and it does not have an absolute minimum value along C_2 .
- By parts (c) and (d), it follows that f has a local minimum of 108 at $(2, 2)$ and a local maximum of 100 at $(-2, -2)$ over the constraint curve $xy = 4$. Because f has just two local extrema, with the local minimum being larger than the local maximum, f does not have an absolute maximum or an absolute minimum over C . Because the constraint curve is unbounded, the values of f get arbitrarily large as we move away from $(2, 2)$ along the curve C_1 and values of f get arbitrarily small as we move away from $(-2, -2)$ along the curve C_2 .

Chapter Fifteen Review

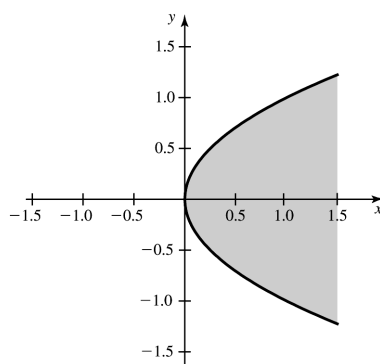
1

- True. If $e^{x+y} = k$, then $x + y = \ln k$, which is the equation of a line.
- False. If $2x^2 - 6y^2 > 0$ then $z = \sqrt{2x^2 - 6y^2}$ or $z = -\sqrt{2x^2 - 6y^2}$.
- False. For example $f(x, y) = x^2y$ has $f_{xxy} = 2$, $f_{xyy} = 0$.
- False. ∇f lies in the xy -plane.

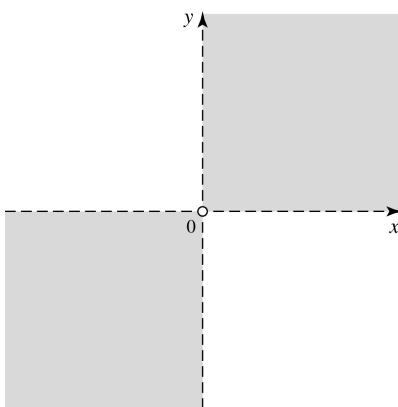
2 The domain is $D = \{(x, y) : x^2 + y^2 \leq 4\}$.



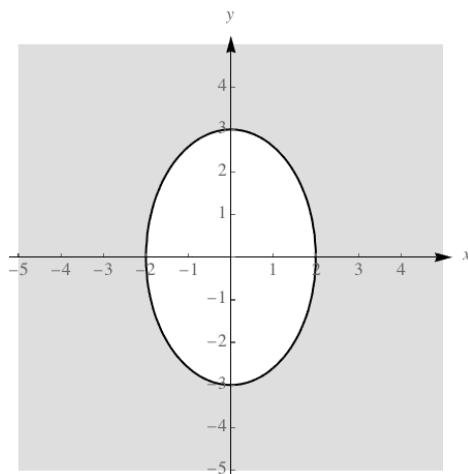
3 The domain is $D = \{(x, y) : x \geq y^2\}$.



4 The domain is $D = \{(x, y) : xy > 0\}$.



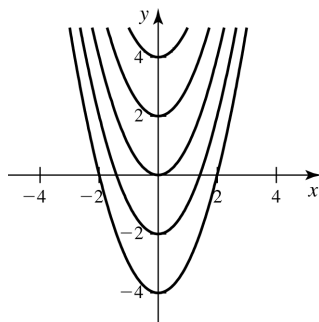
5 The domain is $D = \{(x, y) : \frac{x^2}{4} + \frac{y^2}{9} \geq 1\}$.



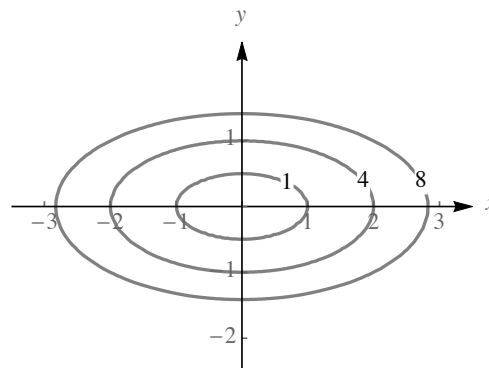
6 The domain is $\mathbb{R}$ and the range is $(-\infty, 0]$. This is the lower half of the cone $z^2 = x^2 + y^2$.

7 The domain is $D = \{(x, y) : x^2 + y^2 \geq 1\}$. The range is $(-\infty, 0]$. This is the lower half of the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$.

8



9



10

- a. This matches A.
- b. This matches C.
- c. This matches D.
- d. This matches B.

11 This limit may be evaluated directly by substitution: $\lim_{(x,y) \rightarrow (4,-2)} (10x - 5y + 6xy) = 10 \cdot 4 - 5(-2) + 6 \cdot 4(-2) = 2$.

12 This limit may be evaluated directly by substitution: $\lim_{(x,y) \rightarrow (1,1)} \frac{xy}{x+y} = \frac{1 \cdot 1}{1+1} = \frac{1}{2}$.

13 Along the path $y = -x$ we have $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{0}{-x^2} = 0$, but along the path $y = x$ we have $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{2x}{x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{2}{x} \neq 0$, so the limit does not exist.

14 Along the path $y = x$ we have $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2)}{2x^2} = \frac{1}{2}$, while along the line $y = -x$ we have $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(-x^2)}{2x^2} = -\frac{1}{2}$, $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2)}{x^2} = \frac{-1}{2}$, so the limit does not exist.

15 This limit may be evaluated by factoring the numerator and denominator and canceling the common factor $x + y$: $\lim_{(x,y) \rightarrow (-1,1)} \frac{x^2 - y^2}{x^2 - xy - 2y^2} = \lim_{(x,y) \rightarrow (-1,1)} \frac{(x-y)(x+y)}{(x-2y)(x+y)} = \lim_{(x,y) \rightarrow (-1,1)} \frac{x-y}{x-2y} = \frac{2}{3}$.

$$\mathbf{16} \quad \lim_{(x,y) \rightarrow (0,0)} 25 \cdot \frac{x^3 y^2}{\sin x^3 y^2} \cdot \frac{x^3 y^2}{\sin x^3 y^2} = 25 \cdot \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^2}{\sin x^3 y^2} \cdot \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^2}{\sin x^3 y^2} = 25 \cdot 1 \cdot 1 = 25.$$

$$\mathbf{17} \quad \lim_{(x,y,z) \rightarrow (2,2,3)} \frac{x^2(z-3) - y^2(z-3)}{x(z-3) - y(z-3)} = \lim_{(x,y,z) \rightarrow (2,2,3)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y,z) \rightarrow (2,2,3)} x + y = 2 + 2 = 4.$$

$$\mathbf{18} \quad \lim_{(x,y,z) \rightarrow (3,4,7)} \frac{(\sqrt{x+y} - \sqrt{z})}{(x+y-z)} \cdot \frac{(\sqrt{x+y} + \sqrt{z})}{(\sqrt{x+y} + \sqrt{z})} = \lim_{(x,y,z) \rightarrow (3,4,7)} \frac{x+y-z}{(x+y-z)(\sqrt{x+y} + \sqrt{z})} = \lim_{(x,y,z) \rightarrow (3,4,7)} \frac{1}{\sqrt{x+y} + \sqrt{z}} = \frac{1}{2\sqrt{7}}.$$

19 This is continuous where $y - x^2 - 1 > 0$, so for $\{(x,y) : y > x^2 + 1\}$.

20 This is defined and continuous for $\{(x,y) : (x,y) \neq (0,0)\}$.

$$\mathbf{21} \quad f_x = 6xy^5, f_y = 15x^2y^4.$$

$$\mathbf{22} \quad g_x = 4yz^2 - 3/y, g_y = 4xz^2 + 3x/y^2, g_z = 8xyz.$$

$$\mathbf{23} \quad f_x = \frac{(x^2+y^2)(2x) - x^2(2x)}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2}, f_y = \frac{0 - x^2(2y)}{(x^2+y^2)^2} = \frac{-2x^2y}{(x^2+y^2)^2}.$$

$$\mathbf{24} \quad g_x = \frac{(x+y)yz - (xyz)}{(x+y)^2} = \frac{y^2z}{(x+y)^2}, g_y = \frac{(x+y)xz - (xyz)}{(x+y)^2} = \frac{x^2z}{(x+y)^2}, g_z = \frac{xy}{x+y}.$$

$$\mathbf{25} \quad \frac{\partial}{\partial x} [xye^{xy}] = ye^{xy} + xy^2e^{xy} = y(1+xy)e^{xy}, \frac{\partial}{\partial y} [xye^{xy}] = xe^{xy} + x^2ye^{xy} = x(1+xy)e^{xy}.$$

$$\mathbf{26} \quad \frac{\partial}{\partial u} [u \cos v - v \sin u] = \cos v - v \cos u, \frac{\partial}{\partial v} [u \cos v - v \sin u] = -u \sin v - \sin u.$$

$$\mathbf{27} \quad f_x = 2ye^{2xy} \text{ and } f_y = 2xe^{2xy}. \text{ Then } f_{xx} = 4y^2e^{2xy}, f_{yy} = 4x^2e^{2xy}, \text{ and } f_{xy} = f_{yx} = 2e^{2xy} + 2y(2xe^{2xy}) = 2e^{2xy}(1+2xy).$$

$$\mathbf{28} \quad \text{Note that } H = (p^5 + 2p^4r)^{1/2}. \text{ Then } H_p = \frac{5p^4 + 8p^3r}{2p^2\sqrt{p+2r}} = \frac{5p^2 + 8pr}{2\sqrt{p+2r}}, \text{ and } H_r = \frac{p^2}{\sqrt{p+2r}}. \text{ Then } H_{pp} = \frac{2\sqrt{p+2r}(10p+84) - (5p^2+8pr)(p+2r)^{-1/2}}{4(p+2r)} = \frac{2(p+2r)(10p+8r) - (5p^2+8pr)}{4(p+2r)^{3/2}} = \frac{15p^2+48pr+32r^2}{4(p+2r)^{3/2}},$$

$$H_{pr} = H_{rp} = \frac{\sqrt{p+2r}(2p) - p^2 \cdot \frac{1}{2} \cdot (p+2r)^{-1/2}}{p+2r} = \frac{(p+2r)(4p) - p^2}{2(p+2r)^{3/2}} = \frac{2p^2+8pr}{2(p+2r)^{3/2}}, \text{ and}$$

$$H_{rr} = \frac{0 - p^2(p+2r)^{-1/2}}{p+2r} = \frac{-p^2}{(p+2r)^{3/2}}.$$

$$\mathbf{29} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0.$$

$$\mathbf{30} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = 0.$$

31

- a. If r is held fixed and R increases then V increases, so $V_R > 0$, whereas, if R is held fixed and r increases then V decreases, so $V_r < 0$.
- b. We have $V_R = 4\pi R^2 > 0$ and $V_r = -4\pi r^2 < 0$, consistent with the predictions in part (a).
- c. If $R = 3$, $r = 1$ and R is increased by $\Delta R = 0.1$, then $\Delta V \approx 4\pi \cdot 3^2 \cdot 0.1 = 3.6\pi$; if r is decreased by 0.1 then $\Delta V \approx -4\pi \cdot 1^2 \cdot (-0.1) = 0.4\pi$. Therefore, the volume changes more if R is increased.

32 The chain rule gives $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = 2t \cos yz - xz \sin yz - xy(3t^2) \sin yz = 2t \cos t^4 - t^3(t^2 + 1) \sin t^4 - 3t^3(t^2 + 1) \sin t^4 = 2t \cos t^4 - 4t^3(t^2 + 1) \sin t^4$.

33 The chain rule gives $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = \frac{2xz}{x^2 + y^2}(3e^t) + \frac{2yz}{x^2 + y^2}(4e^t) + \ln(x^2 + y^2) = \frac{18te^{2t}}{25e^{2t}} + \frac{32te^{2t}}{25e^{2t}} + \ln 25e^{2t} = \frac{18t}{25} + \frac{32t}{25} + \ln 25 + \ln e^{2t} = 2t + \ln 25 + 2t = 4t + 5 \ln 2$.

34 The chain rule gives $w_s = w_x x_s + w_y y_s + w_z z_s = yz \cdot 2t + xz \cdot t^2 + xy \cdot 2st = st^2 \cdot s^2 t \cdot 2t + 2st \cdot s^2 t \cdot t^2 + 2st \cdot st^2 \cdot 2st = 8s^3 t^4$ and similarly $w_t = w_x x_t + w_y y_t + w_z z_t = yz \cdot 2s + xz \cdot 2st + xy \cdot s^2 = st^2 \cdot s^2 t \cdot 2s + 2st \cdot s^2 t \cdot 2st + 2st \cdot st^2 \cdot s^2 = 8s^4 t^3$.

35 The chain rule gives $w_r = w_x x_r + w_y y_r = \frac{1}{x}(st) + \frac{2}{y}(1) = \frac{1}{r} + \frac{2}{r+s} = \frac{3r+s}{r(r+s)}$. $w_s = w_x x_s + w_y y_s = \frac{1}{x}(rt) + \frac{2}{y}(1) = \frac{1}{s} + \frac{2}{r+s} = \frac{r+3s}{s(r+s)}$. $w_t = w_x x_t + w_y y_t = \frac{1}{x}(rs) + \frac{2}{y} \cdot 0 = \frac{1}{t}$.

36 Let $F(x, y) = 2x^2 + 3xy - 3y^4 - 2$; then if y is determined by $F(x, y) = 0$, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x+3y}{3x-12y^3} = \frac{4x+3y}{12y^3-3x}$.

37 Let $F(x, y) = y \ln(x^2 + y^2) - 4$; then if y is determined by $F(x, y) = 0$, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{2xy}{x^2 + y^2}}{\ln(x^2 + y^2) + \frac{2y^2}{x^2 + y^2}} = -\frac{2xy}{2y^2 + (x^2 + y^2) \ln(x^2 + y^2)}$.

38

- a. The chain rule gives $z'(t) = 8xx'(t) + 2yy'(t) = -6 \cos t \sin t = -3 \sin 2t$.
- b. Walking uphill corresponds to $z'(t) > 0$, which occurs when $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$.

39

- a. The chain rule gives $z'(t) = 2xx'(t) - 4yy'(t) = -24 \cos t \sin t = -12 \sin 2t$.
- b. Walking uphill corresponds to $z'(t) > 0$, which occurs when $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$.

40 Observe that $r'(t) = \frac{ar}{t}$ and $h'(t) = -\frac{bh}{t}$; hence, by the chain rule, $V'(t) = \frac{\pi}{3} (2rhr'(t) + r^2 h'(t)) = \frac{\pi r^2 h}{3t} (2a - b)$. Therefore, the volume of the cone remains constant if and only if $2a = b$. (This can also be seen by substituting the formulas for $r(t)$ and $h(t)$ in the formula for the volume of a cone to obtain $V(t) = \frac{\pi}{3} t^{2a-b}$.)

41

	$(a, b) = (0, 0)$	$(a, b) = (2, 0)$	$(a, b) = (1, 1)$
a. $\mathbf{u} = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$	0	$4\sqrt{2}$	$-2\sqrt{2}$
$\mathbf{v} = \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle$	0	$-4\sqrt{2}$	$-6\sqrt{2}$
$\mathbf{w} = \langle -\sqrt{2}/2, -\sqrt{2}/2 \rangle$	0	$-4\sqrt{2}$	$2\sqrt{2}$

- b. The function is increasing at $(2, 0)$ in the direction of $\mathbf{u}$ and decreasing at $(2, 0)$ in the directions of $\mathbf{v}$ and $\mathbf{w}$.

42 $\nabla f = \langle 2x, 0 \rangle$, so $\nabla f(1, 2) = \langle 2, 0 \rangle$. $D_{\mathbf{u}}f(1, 2) = \langle 2, 0 \rangle \cdot \mathbf{u} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

43 $\nabla g = \langle 2xy^3, 3x^2y^2 \rangle$, so $\nabla g(-1, 1) = \langle -2, 3 \rangle$. $D_{\mathbf{u}}g(-1, 1) = \langle -2, 3 \rangle \cdot \mathbf{u} = \frac{-10}{13} + \frac{36}{13} = 2$.

44 $\nabla f = \langle 1/y^2, -2x/y^3 \rangle$, so $\nabla f(0, 3) = \langle 1/9, 0 \rangle$. $D_{\mathbf{u}}f(0, 3) = \langle 0, 3 \rangle \cdot \mathbf{u} = \sqrt{3}/18$.

45 The gradient of h is given by $\nabla h(x, y) = h_x \mathbf{i} + h_y \mathbf{j} = \frac{1}{\sqrt{2+x^2+2y^2}}(x\mathbf{i} + 2y\mathbf{j})$; therefore, $\nabla h(2, 1) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$ and the directional derivative in the direction of $\mathbf{u}$ is given by $\nabla h(2, 1) \cdot \mathbf{u} = \frac{\sqrt{2}}{10}(\mathbf{i} + \mathbf{j}) \cdot (3\mathbf{i} + 4\mathbf{j}) = \frac{7\sqrt{2}}{10} \approx 0.9899$.

46 The gradient of f is given by $\nabla f(x, y, z) = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k} = \langle e^{1+y^2+z^2}, 2xye^{1+y^2+z^2}, 2xze^{1+y^2+z^2} \rangle$; therefore, $\nabla f(0, 1, -2) = e^6 \mathbf{i}$ and the directional derivative in the direction of $\mathbf{u}$ is given by $\frac{4e^6}{9} + 0 + 0 = \frac{4e^6}{9}$.

47 The gradient of f is given by $\nabla f(x, y, z) = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k} = \langle y \cos xy, x \cos xy, -\sin z \rangle$, so $\nabla f(1, \pi, 0) = \langle -\pi, -1, 0 \rangle$. The directional derivative is therefore

$$\langle -\pi, -1, 0 \rangle \cdot \langle 2/7, 3/7, 6/7 \rangle = -2\pi/7 - 3/7 = -\frac{1}{7}(2\pi + 3).$$

48

a. The gradient of f is given by $\nabla f(x, y) = f_x \mathbf{i} + f_y \mathbf{j} = \frac{1}{1+xy}(y\mathbf{i} + x\mathbf{j})$, so $\nabla f(2, 3) = \frac{1}{7}(3\mathbf{i} + 2\mathbf{j})$. The direction of steepest ascent is the unit vector in this direction, $\mathbf{u} = \frac{3}{\sqrt{13}}\mathbf{i} + \frac{2}{\sqrt{13}}\mathbf{j}$, and the direction of steepest descent is $-\mathbf{u}$.

b. The unit vectors that point in the direction of no change are $\mathbf{v} = \pm \left(\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j} \right)$, because $\mathbf{u} \cdot \mathbf{v} = 0$.

49

a. The gradient of f is given by $\nabla f(x, y) = f_x \mathbf{i} + f_y \mathbf{j} = -\frac{1}{\sqrt{4-x^2-y^2}}(x\mathbf{i} + y\mathbf{j})$, so $\nabla f(-1, 1) = -\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$. The direction of steepest ascent is the unit vector in this direction, $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$, and the direction of steepest descent is $-\mathbf{u}$.

b. The unit vectors that point in the direction of no change are $\mathbf{v} = \pm \left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} \right)$, because $\mathbf{u} \cdot \mathbf{v} = 0$.

50 If y is determined by $f(x, y) = C$ we have $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{-4x}{-2y} = -\frac{2x}{y}$. Therefore, the level curve $f(x, y) = 5$ has slope $m = -2$ at the point $(1, 1)$, so the tangent line has direction $\mathbf{i} - 2\mathbf{j}$. The gradient of f at this point is $\nabla f(1, 1) = (-4x\mathbf{i} - 2y\mathbf{j})\big|_{(1,1)} = -4\mathbf{i} - 2\mathbf{j}$, which is perpendicular to the tangent direction.

51 If x is determined by $f(x, y) = C$ we have $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{-4x}{-2y} = -\frac{2x}{y}$. Therefore, the level curve $f(x, y) = 0$ has a vertical tangent at the point $(2, 0)$, so the tangent line has direction $\mathbf{j}$. The gradient of f at this point is $\nabla f(2, 0) = (-4x\mathbf{i} - 2y\mathbf{j})\big|_{(2,0)} = -8\mathbf{i}$, which is perpendicular to the tangent direction.

52 The gradient of f at $(1, 1)$ is given by $\nabla f(1, 1) = (8x\mathbf{i} - 2y\mathbf{j})\big|_{(1,1)} = 8\mathbf{i} - 2\mathbf{j} = 2(4\mathbf{i} - \mathbf{j})$, so the unit vectors in the direction of no change are $\mathbf{u} = \pm \frac{1}{\sqrt{17}}(4\mathbf{i} + \mathbf{j})$.

53 Observe that $V = -\frac{k}{2}(\ln(x^2 + y^2) - \ln R^2)$; therefore $\mathbf{E} = -\nabla V = \frac{k}{2}\left(\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right) = \frac{kx}{x^2 + y^2}\mathbf{i} + \frac{ky}{x^2 + y^2}\mathbf{j}$.

54 Let $f(x, y, z) = 2x^2 + y^2 - z$. $\nabla f = \langle 4x, 2y, -1 \rangle$ and $\nabla f(1, 1, 3) = \langle 4, 2, -1 \rangle$. The equation of the tangent plane at $(1, 1, 3)$ is $4(x - 1) + 2(y - 1) - (z - 3) = 0$, or $4x + 2y - z = 3$. At the other given point we have $\nabla f(0, 2, 4) = \langle 0, 4, -1 \rangle$, so the equation of the tangent plane is $4(y - 2) - (z - 4) = 0$, or $4y - z = 4$.

55 Let $f(x, y, z) = x^2 + y^2/4 - z^2/9$. $\nabla f = \langle 2x, y/2, -2z/9 \rangle$, so $\nabla f(0, 2, 0) = \langle 0, 1, 0 \rangle$. The equation of the tangent plane at $(0, 2, 0)$ is given by $1(y - 2) = 0$, or $y = 2$. At the point $(1, 1, 3/2)$ we have $\nabla f = \langle 2, 1/2, -1/3 \rangle$, so the equation of the tangent plane is $2(x - 1) + (1/2)(y - 1) + (-1/3)(z - 3/2) = 0$, or $12x + 3y - 2z = 12$.

56 Let $F(x, y, z) = x^2 - 2x_y^2 + 4y + 3z^2$; then $\nabla F = \langle 2x - 2, 2y + 4, 6z \rangle$ and $\nabla F(3, -1, 1) = \langle 4, 0, 6 \rangle$. The tangent plane is given by $4(x - 3) + 6(z - 1) = 0$, or $2x + 3z = 9$. At the point $(-1, -2, 1)$ we have $\nabla F = \langle -4, 0, 6 \rangle$, so the tangent plane there is given by $-4(x + 1) + 6(z - 1) = 0$, or $-2x + 3z = 5$.

57 Let $F(x, y, z) = e^{xy^2z^3-1}$. Then

$$\nabla F(x, y, z) = \left\langle y^2z^3e^{xy^2z^3-1}, 2xyz^3e^{xy^2z^3-1}, 3xy^2z^3e^{xy^2z^3-1} \right\rangle.$$

$\nabla F(1, 1, 1) = \langle 1, 2, 3 \rangle$, so the plane is $x - 1 + 2(y - 1) + 3(z - 1) = 0$, or $x + 2y + 3z = 6$. Also, $\nabla F(1, -1, 1) = \langle 1, -2, 3 \rangle$, so the plane is $x - 1 - 2(y + 1) + 3(z - 1) = 0$, or $x - 2y + 3z = 6$.

58 Let $F(x, y, z) = z - \tan^{-1}xy$. Then $\nabla F = \left\langle \frac{-y}{1+(xy)^2}, \frac{-x}{1+(xy)^2}, 1 \right\rangle$. $\nabla F(1, 1, \pi/4) = \langle -1/2, -1/2, 1 \rangle$, so the plane is $-(x - 1) - (y - 1) + 2(z - \pi/4) = 0$, or $x + y - 2z = 2 - \pi/2$. Also, $\nabla F(1, \sqrt{3}, \pi/3) = \langle -\sqrt{3}/4, -1/4, 1 \rangle$, so the plane is $-(\sqrt{3}/4)(x - 1) - (1/4)(y - \sqrt{3}) + (z - \pi/3) = 0$, or $-\sqrt{3}x - y + 4z = 4\pi/3 - 2\sqrt{3}$.

59 Let $F(x, y) = \sqrt{\frac{x+y}{z}} - 1$. Then $\nabla F = \left\langle \frac{\sqrt{z}}{2z\sqrt{x+y}}, \frac{\sqrt{z}}{2z\sqrt{x+y}}, -\frac{\sqrt{x+y}}{2z^{3/2}} \right\rangle$. At the point $(2, 2, 4)$ we have $\nabla F = \langle 1/8, 1/8, -1/8 \rangle = (1/8)\langle 1, 1, -1 \rangle$. So the equation of the tangent plane is $x - 2 + y - 2 - (z - 4) = 0$, or $x + y - z = 0$. At the point $(10, -1, 9)$ we have $\nabla F = \langle 1/18, 1/18, -1/18 \rangle = (1/18)\langle 1, 1, -1 \rangle$, so the equation of the tangent plane is $x - 10 + y + 1 - (z - 9) = 0$, or $x + y - z = 0$.

60

a. The linear approximation is given by $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) = 2\sqrt{2} - 8\sin\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + 4\sin\left(\frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) = -4\sqrt{2}x + 2\sqrt{2}y + 2\sqrt{2} + \frac{\sqrt{2}\pi}{2}$.

- b. This gives the estimate $f(0.8, 0.8) \approx -3.2 + 2\sqrt{2} + \pi \approx 2.787$ (the actual answer is 2.787 to three decimal places).

61

- a. The linear approximation is given by $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) = 2 + (x - 2) + 5y = x + 5y$.

- b. This gives the estimate $f(1.95, 0.05) \approx 2.2$ (the actual answer is 2.205 to three decimal places).

62 We have $\Delta f \approx f_x(1, -2) \Delta x + f_y(1, -2) \Delta y = 4 \cdot 0.05 + 9 \cdot 0.1 = 1.1$ (the actual change is 1.093 to three decimal places).

63 We have $dV = \pi(2rhd r + r^2 dh)$, so $\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$. Therefore, if the radius decreases by 3% and the height increases by 2%, the approximate change in volume is -4%.

64 We have $dV = \pi(bcda + acdb + abdc)$, so $\frac{dV}{V} = \frac{da}{a} + \frac{db}{b} + \frac{dc}{c}$. Therefore, if a, b, c change by 2%, 1.5%, -2.5% respectively, the approximate change in the volume is 1%.

65

- a. We have $dV = \pi(2rh - h^2) dh = 2\pi(-0.05) = -0.1\pi \text{ m}^3$ (notice that $r = 1.50 \text{ m}$ is constant, so there is no contribution from the dr term in dV).

- b. The surface of the water is a disc with radius $s = \sqrt{2rh - h^2}$, so the surface area is $S = \pi(2rh - h^2)$. Therefore $dS = 2\pi(r - h) dh = -0.05\pi \text{ m}^2$.

66 We have $f_x = 4x^3 - 16y$, $f_y = 4y^3 - 16x$; therefore, the critical points satisfy the equations $y = \frac{x^3}{4}$ and $x = \frac{y^3}{4}$. Eliminating y gives $x^9 = 2^8 x$, so $x = 0, \pm 2$, and the critical points are $(0, 0), \pm(2, 2)$. We also have $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = -16$; hence $D(x, y) = 16(9x^2y^2 - 1)$. We see that $D(0, 0) < 0$ so $(0, 0)$ is a saddle. We have $D(2, 2) = D(-2, -2) > 0$; $f_{xx}(2, 2) = f_{xx}(-2, -2) > 0$, which by the Second Derivative Test implies that f has local minima at $\pm(2, 2)$.

67 We have $f_x = x^2 + 2y$, $f_y = -y^2 + 2x$; therefore, the critical points satisfy the equations $y = -\frac{x^2}{2}$ and $x = \frac{y^2}{2}$. Eliminating y gives $x^4 = 8x$ so $x = 0, 2$, and the critical points are $(0, 0), (2, -2)$. We also have $f_{xx} = 2x$, $f_{yy} = -2y$ and $f_{xy} = 2$; hence $D(x, y) = -4(1 + xy)$. We see that $D(0, 0) < 0$ so $(0, 0)$ is a saddle. We have $D(2, -2) > 0$, $f_{xx}(2, -2) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(2, -2)$.

68 We have $f_x = 2xy^2 - 6xy + 2y^2 - 6y$, $f_y = 2x^2y - 3x^2 + 4xy - 6x$; therefore, the critical points satisfy the equations $y(2xy - 6x + 2y - 6) = 0$, $x(2xy - 3x + 4y - 6) = 0$. If $y = 0$ then the second equation gives $x = 0, -2$, and if $x = 0$ then the first equation gives $y = 0, 3$. If both $x, y \neq 0$ then we can divide the equations by x, y respectively and subtract to obtain $3x + 2y = 0$, so $y = -\frac{3x}{2}$; substituting this in the first equation gives $x^2 + 3x + 2 = 0$, which has roots $x = -1, -2$. Therefore, the critical points are $(0, 0), (0, 3), (-2, 0), \left(-1, \frac{3}{2}\right)$ and $(-2, 3)$. We also have $f_{xx} = 2y^2 - 6y$, $f_{yy} = 2x^2 + 4x$ and $f_{xy} = 4xy - 6x + 4y - 6$. We see that $D(0, 0), D(0, 3), D(-2, 0)$, and $D(-2, 3) < 0$ so $(0, 0), (0, 3), (-2, 0)$ and $(-2, 3)$ are saddles. We have $D\left(-1, \frac{3}{2}\right) > 0$, $f_{xx}\left(-1, \frac{3}{2}\right) < 0$, which by the Second Derivative Test implies that f has a local maximum at $\left(-1, \frac{3}{2}\right)$.

69 We have $f_x = -3x^2 - 6x$, $f_y = -3y^2 + 6y$; therefore the critical points must have $x = 0, -2$ and $y = 0, 2$. We also have $f_{xx} = -6x - 6$, $f_{yy} = -6y + 6$ and $f_{xy} = 0$; hence $D(x, y) = 36(x+1)(y-1)$. We see that $D(0, 0)$, $D(-2, 2) < 0$ so $(0, 0)$ and $(-2, 2)$ are saddles. We have $D(0, 2) > 0$, $f_{xx}(0, 2) < 0$, which by the Second Derivative Test implies that f has a local maximum at $(0, 2)$; $D(-2, 0) > 0$, $f_{xx}(-2, 0) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(-2, 0)$.

70 First we find the critical points of f inside the rectangle R : we have $f_x = x^2 + 2y$, $f_y = -y^2 + 2x$; the equation $f_x = 0$ gives $y = -\frac{x^2}{2}$, and then substituting this in the equation $f_y = 0$ gives $x^4 = 8x$, or $x = 0, 2$. Hence the critical points are $(0, 0)$ and $(2, -2)$, neither of which is in the interior of R . Therefore, we must find the maximum and minimum values of f on the boundary of R , which consists of four segments. On the segment $0 \leq x \leq 3$, $y = -1$, let $g(x) = f(x, -1) = \frac{x^3}{3} - 2x + \frac{1}{3}$: then g has a critical point at $x = \sqrt{2}$, and find that g has extreme values $\frac{1-4\sqrt{2}}{3}$ and $\frac{10}{3}$ on $[0, 3]$. Similarly, we find that on the segment $0 \leq x \leq 3$, $y = 1$, $g(x) = f(x, 1) = \frac{x^3}{3} + 2x - \frac{1}{3}$ has extreme values $-\frac{1}{3}$ and $\frac{44}{3}$. On the segment $-1 \leq y \leq 1$, $x = 0$, let $h(y) = f(0, y) = -\frac{y^3}{3}$: then h has extreme values $\pm\frac{1}{3}$ on this segment. Similarly, we find that on the segment $-1 \leq y \leq 1$, $x = 3$, $h(y) = f(3, y) = -\frac{y^3}{3} + 6y + 9$ has a critical point at $y = \sqrt{6}$, and h has extreme values $\frac{10}{3}$ and $4\sqrt{6} + 9$. Therefore the absolute minimum and maximum values of f on R are $f(\sqrt{2}, 1) = \frac{1-4\sqrt{2}}{3} \approx -0.886$ and $f(3, \sqrt{6}) = 4\sqrt{6} + 9 \approx 18.798$.

71 First we find the critical points of f inside the square R : we have $f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$; the equation $f_x = 0$ gives $y = x^3$, and then substituting this in the equation $f_y = 0$ gives $x^9 = x$, or $x = 0, \pm 1$. Hence, the critical points are $(0, 0)$ and $\pm(1, 1)$, which are all in the interior of R . We observe that the values of f at these points are $f(0, 0) = 1$, $f(1, 1) = f(-1, -1) = -1$. Next we must find the maximum and minimum values of f on the boundary of R , which consists of four segments. On the segment $-2 \leq x \leq 2$, $y = 2$, let $g(x) = f(x, 2) = x^4 - 8x + 17$: then g has a critical point at $x = \sqrt[3]{2}$, and find that g has extreme values $17 - 6\sqrt[3]{2}$ and 49 on $[0, 3]$. We also note that $f(y, x) = f(-x, -y) = f(x, y)$; therefore f takes the same values on all four segments of the square. Hence, the absolute minimum and maximum values of f on R are $f(1, 1) = f(-1, -1) = -1$ and $f(2, -2) = f(-2, 2) = 49$.

72 First we find the critical points of f inside the triangle. We have $f_x = 2xy$ and $f_y = x^2 - 3y^2$, so there are no critical points inside the triangle. Now consider the boundary triangle. On the vertical side $x = 0$, we have $f(x, y) = -y^3$ which has a maximum for $0 \leq y \leq 2$ of 0 and a minimum of -8 . On the horizontal side $y = 0$, we have $f(x, y) = 0$, so the minimum and maximum are both 0 . On the side $y = 2 - x$, we have $f(x, y) = g(x) = 2x^2 - x^3 - (2 - x)^3$. Note that $g'(x) = 4x - 3x^2 + 3(2 - x)^2 = 12 - 8x$, which is zero for $x = 3/2$. The value of g at $x = 3/2$ is $g(3/2) = 1$. Thus, the absolute maximum of f is 1 at $(3/2, 1/2)$ and the absolute minimum is -8 at $(0, 2)$.

73 First we find the critical points of f inside the semicircle. We have $f_x = y$ and $f_y = x$, so the only critical point is $(0, 0)$, where the value of f is 0 . On the flat part of the semicircle where $y = 0$, the value of f is also 0 . Now we parametrize the circular part of the boundary by letting $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$. Then $f(x, y) = g(t) = \cos t \sin t$, and $g'(t) = -\sin^2 t + \cos^2 t$. This is zero for $t = \pi/4$ and $t = 3\pi/4$, where the value of f is $1/2$ and $-1/2$, respectively. Thus, the absolute maximum of f is $1/2$ at $(\sqrt{2}/2, \sqrt{2}/2)$ and the absolute minimum is $-1/2$ at $(-\sqrt{2}/2, \sqrt{2}/2)$.

74 First, observe that there is a unique point on any plane that is closest to the origin. To find this point, it suffices to minimize the function $f(y, z) = x^2 + y^2 + z^2 = (8 - y - 4z)^2 + y^2 + z^2$ over all points (y, z) . We have $f_y = 2(y + 4z - 8) + 2y$, $f_z = 2 \cdot 4(4z + y - 8) + 2z$ and setting $f_y = f_z = 0$ gives the equations $y + 2z = 4$, $4y + 17z = 32$; solving these simultaneously gives $y = \frac{4}{9}$, $z = \frac{16}{9}$ and hence $x = \frac{4}{9}$; therefore, the closest point is $\left(\frac{4}{9}, \frac{4}{9}, \frac{16}{9}\right)$.

75 We have $\nabla f = \langle 2, 1 \rangle$ and $\nabla g = \langle 4(x-1), 8(y-1) \rangle$, where $g(x, y) = 2(x-1)^2 + 4(y-1)^2$. The Lagrange equations are thus $2 = 4\lambda(x-1)$, $1 = 8\lambda(y-1)$, and $2(x-1)^2 + 4(y-1)^2 = 1$. Solving gives $\lambda = 3/4$, $x = 5/3$, and $y = 7/6$, or $\lambda = -3/4$, $x = 1/3$, and $y = 5/6$. The absolute maximum for f is $f(5/3, 7/6) = 29/2$ and the absolute minimum for f is $f(1/3, 5/6) = 23/2$.

76 We have $\nabla f = \langle y, x \rangle$ and $\nabla g = \langle 6x - 2y, -2x + 6y \rangle$ where $g(x, y) = 3x^2 - 2xy + 3y^2$. Lagrange equations are therefore $y = \lambda(6x - 2y)$ and $x = \lambda(2x + 6y)$, and $3x^2 - 2xy + 3y^2 = 4$. Solving for λ , we have

$$\lambda = \frac{y}{6x - 2y} = \frac{x}{6y - 2x},$$

and from this we can deduce that $x = \pm y$. Then using the constraint equation, we have that $x = \pm 1$ or $x = \pm \frac{1}{\sqrt{2}}$. Checking these, we find an absolute maximum of 1 at $\pm(1, 1)$ and an absolute minimum of -1 at $\pm(1, -1)$.

77 The Lagrange multiplier conditions are $1 = 2\lambda x$, $2 = 2\lambda y$, $-1 = 2\lambda z$, $x^2 + y^2 + z^2 = 1$. Hence $2\lambda = \frac{1}{x} = \frac{2}{y} = -\frac{1}{z} \implies y = 2x$, $z = -x$; substituting these in the constraint gives $6x^2 = 1$ so $x = \pm \frac{\sqrt{6}}{6}$ and

we obtain solutions $\pm \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6} \right)$. Therefore the extreme values of f on the closed bounded set given

by $x^2 + y^2 + z^2 = 1$ are an absolute maximum of $f \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6} \right) = \sqrt{6}$, and an absolute minimum of

$$f \left(-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right) = -\sqrt{6}.$$

78 The Lagrange multiplier conditions are $2xy^2z = 4\lambda x$, $2x^2yz = 2\lambda y$, $x^2y^2 = 2\lambda z$, $2x^2 + y^2 + z^2 = 25$. If x , y or $z = 0$ then $f(x, y, z) = 0$, which is neither the maximum or minimum of f on the ellipsoid given by the constraint (f takes both positive and negative values on this set). Hence, we can assume x , y , $z \neq 0$ in the Lagrange conditions, and eliminating λ in the first three equations gives $\lambda = \frac{y^2z}{2} = x^2z = \frac{x^2y^2}{2z} \implies y^2 = 2z^2 = 2x^2$; substituting this in the constraint gives $5x^2 = 25$, so $x = \pm\sqrt{5}$, $y = \pm\sqrt{10}$ and $z = \pm\sqrt{5}$. Therefore the maximum value of f on the closed bounded set given by $2x^2 + y^2 + z^2 = 25$ is $f(\pm\sqrt{5}, \pm\sqrt{10}, \sqrt{5}) = 50\sqrt{5}$, and the minimum value is $f(\pm\sqrt{5}, \pm\sqrt{10}, -\sqrt{5}) = -50\sqrt{5}$.

79 Let (x, y) be the corner of the rectangle in the first quadrant; then the perimeter of the rectangle is $4(x + y)$, so it suffices to find the maximum value of $x + y$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The Lagrange multiplier conditions are $1 = \frac{2\lambda x}{a^2}$, $1 = \frac{2\lambda y}{b^2}$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence, $2\lambda = \frac{a^2}{x} = \frac{b^2}{y} \implies y = \frac{b^2x}{a^2}$; substituting in the constraint gives $x^2(a^2 + b^2) = a^4$ so $x = \frac{a^2}{\sqrt{a^2 + b^2}}$, $y = \frac{b^2}{\sqrt{a^2 + b^2}}$, and the dimensions of the rectangle with greatest perimeter are $\frac{2a^2}{\sqrt{a^2 + b^2}}$ by $\frac{2b^2}{\sqrt{a^2 + b^2}}$.

80 Let r be the radius and h the height of the cylinder; then the surface area is $A = 2\pi r^2 + 2\pi rh$ and the volume is $\pi r^2 h$, so it suffices to minimize $r^2 + rh$ subject to the constraint $\pi r^2 h = 32$. The Lagrange multiplier conditions are $2r + h = 2\lambda rh$, $r = \lambda r^2$, $\pi r^2 h = 32$. We must have $r, h > 0$, so the second equation gives $\lambda = \frac{1}{r}$, and then the first equation reduces to $2r = h$. Substituting this in the constraint gives $r^3 = \frac{16}{\pi}$, so $r = 2\sqrt[3]{\frac{2}{\pi}}$ in. and $h = 4\sqrt[3]{\frac{2}{\pi}}$ in.

81 It suffices to minimize the function $f(x, y, z) = (x-1)^2 + (y-3)^2 + (z-1)^2$ subject to the constraint $x^2 + y^2 - z^2 = 0$. The Lagrange multiplier conditions are equivalent to $x-1 = \lambda x$, $y-3 = \lambda y$, $z-1 = -\lambda z$,

$x^2 + y^2 - z^2 = 0$. The first two equations give $\lambda xy = (x-1)y = (y-3)x \implies y = 3x$ and similarly, the first and third equations give $\lambda xz = (x-1)z = -x(z-1) \implies (2x-1)z = x \implies z = \sqrt{10}x$. Substituting these equations in the constraint gives $(2x-1)^2 \cdot 10x^2 = x^2$, so either $x = 0$ (and hence $y = z = 0$ as well) or $10(2x-1)^2 = 1$, which has solutions $x = \frac{1}{2} \pm \frac{\sqrt{10}}{20}$. Therefore, there are three solutions to the Lagrange conditions: $(0, 0, 0)$, $\left(\frac{1}{2} \pm \frac{\sqrt{10}}{20}, \frac{3}{2} \pm \frac{3\sqrt{10}}{20}, \frac{1}{2} \pm \frac{\sqrt{10}}{2}\right)$. We see that $f(0, 0, 0) = 11$, $f\left(\frac{1}{2} \pm \frac{\sqrt{10}}{20}, \frac{3}{2} \pm \frac{3\sqrt{10}}{20}, \frac{1}{2} \pm \frac{\sqrt{10}}{2}\right) = \frac{11}{2} \mp \sqrt{10}$, so the closest point is $\left(\frac{1}{2} + \frac{\sqrt{10}}{20}, \frac{3}{2} + \frac{3\sqrt{10}}{20}, \frac{1}{2} + \frac{\sqrt{10}}{2}\right)$. (The function $f(x, y, z) \rightarrow \infty$ as either x , y or $z \rightarrow \infty$ on the cone; therefore, f must have minimum somewhere on the cone, which corresponds to the point we found.)

82

a. We have $d(x, y, z) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, so

$$\nabla d(x, y, z) = \left\langle \frac{x-a}{d(x, y, z)}, \frac{y-b}{d(x, y, z)}, \frac{z-c}{d(x, y, z)} \right\rangle.$$

b. Observe that $\nabla d(x, y, z) = \frac{1}{d(x, y, z)} \overrightarrow{PP_0}$ and $|\overrightarrow{PP_0}| = d(x, y, z)$; therefore, $\nabla d(x, y, z)$ is a unit vector.

c. The level surfaces of d are spheres centered at a, b, c , and $\nabla d(x, y, z)$ is perpendicular to these spheres (pointing outwards).

d. If $P \rightarrow P_0$ in the direction of a unit vector $\mathbf{u}$, we have $\nabla d(x, y, z) = \pm \mathbf{u}$; but because the limit must be the same in all directions, we conclude that this limit does not exist.

83 It suffices to minimize the function $f(x, y, z) = (x-5)^2 + (y-10)^2 + (z-3)^2$ subject to the constraint $x^2 + y^2 - z = 0$. The Lagrange multiplier conditions are equivalent to $x-5 = \lambda x$, $y-10 = \lambda y$, $2(z-3) = -\lambda$, and $z = x^2 + y^2$. Combining the first two equations gives that $y = 2x$, and this combined with the fourth equation gives that $z = 5x^2$. Then the first and third equations give that $\lambda = \frac{x-5}{x} = 2(3-5x^2)$, and solving this equation gives $x = 1$. Therefore we have $x = 1$, $y = 2$, and $z = 5$. (The function $f(x, y, z) \rightarrow \infty$ as either x , y or $z \rightarrow \infty$ on the paraboloid; therefore, f must have minimum somewhere on the paraboloid, which corresponds to the point we found.)